Understanding Reasoning Using Utility Proportional Beliefs

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Abstract. Traditionally very little attention has been paid to the reasoning process that underlies a game theoretic solution concept. When modeling bounded rationality in one-shot games, however, the reasoning process can be a great source of insight. The reasoning process itself can provide testable assertions, which provide more insight than the fit to experimental data. Based on Bach and Perea's [1] concept of utility proportional beliefs, we analyze the players' reasoning process and find three testable implications: (1) players form an initial belief that is the basis for further reasoning; (2) players reason by alternatingly considering their own and their opponent's incentives; (3) players perform only several rounds of deliberate reasoning.

Keywords: Epistemic game theory, interactive epistemology, solution concepts, bounded rationality, utility proportional beliefs, reasoning.

1 Introduction

Most of the ongoing research in game theory focuses on the prediction of players' choices. In this paper, we want to take another perspective and look at the players' reasoning process rather than their decisions. This perspective can be especially helpful to understand experimental data from one-shot games without opportunities for learning or coordinating. Existing bounded rationality concepts (e.g. Quantal Response Equilibrium (QRE) [5] or Cognitive Hierarchy Models (CHM) [3]) focus mainly on the prediction of empirical frequencies rather than accurately mimicking players' reasoning process. Therefore, these concepts do not provide a clear rationale for players selecting certain choices. The underlying idea of a concept might give a hint but no a clear insight. Most often, these models are evaluated by comparing their fit to the data. This method, however, tells us little about the validity of certain features of the models. A deeper insight into the reasoning process could help to better understand a concept's characteristics and therefore which features work well and which do not. Therefore, a concept that makes clear assertions about the reasoning process can be tested much more rigorously. In fact, a good concept should present clear assertions about the reasoning process that can be tested individually and in their interaction with each other. In this paper, we discuss a solution concept that is based on a general idea and provides detailed assertions about the reasoning process that can be tested individually.

2 Utility Proportional Beliefs

Bach and Perea [1], henceforth BP, suggest a concept for bounded rationality that builds up on a simple idea: the differences in probabilities a player assigns to his opponent's choices should be equal to the differences in the opponent's utilities for these choices. BP, formalize the solution concept using the type-based approach to epistemic game theory. Here we will only introduce the main definition and focus on the two player case. For a more formal treatment consult BP. However, before stating the definition of utility proportional beliefs we need to introduce some further notation.

By $I = \{1, 2\}$ we denote the set of players, by C_j we denote player j's finite choice set, by T_j we denote player j's set of types, and by $U_i : C_i \times C_j \to \mathbb{R}$ we denote player i's utility function. The best and the worst possible utilities of player j are denoted as $\bar{u}_j := \max_{c \in C} u_j(c)$ and $\underline{u}_j := \min_{c \in C} u_j(c)$. $(b_i(t_i))(c_j|t_j)$ gives the probability that player i's type t_i assigns to j's choice c_j given that j is of type t_j, where $t_i \in T_i$ and $t_j \in T_j$.

Definition 1. *Let $i, j \in I$ be the two players, and $\lambda_j \in \mathbb{R}$ such that $\lambda_j \geq 0$. A type $t_i \in T_i$ of player i expresses λ_j-utility-proportional-beliefs, if*

$$(b_i(t_i))(c_j|t_j) - (b_i(t_i))(c'_j|t_j) = \frac{\lambda_j}{\bar{u}_j - \underline{u}_j}(u_j(c_j, t_j) - u_j(c'_j, t_j)) \tag{2.1}$$

for all $t_j \in T_j(t_i)$, for all c_j, $c'_j \in C_j$.

The definition directly corresponds to the idea of utility proportional beliefs: the difference in probabilities player i assigns to the opponents' choices is equal to the difference of the utilities times the proportionality factor $\lambda_j/(\bar{u}_j - \underline{u}_j)$. BP give an intuitive interpretation of λ_j as measure of the sensitivity of a player's beliefs to differences in the opponents utilities. Note that there exists an upper bound for the λ_j called λ_j^{max}. It is the maximum value of λ_j for which equation (2.1) yields well-defined probability measures. The lower limit of λ_j is 0.

The concept of common belief in λ-*utility-proportional-beliefs* requires that both players entertain utility proportional beliefs, that both players believe their opponent holds utility proportional beliefs, that both players believe their opponents believe that their opponents do so, and so on. BP introduce an algorithm to find exactly those beliefs that are possible under common belief in λ-*utility-proportional-beliefs*. The algorithm iteratively deletes beliefs so that only the beliefs, which are possible under common belief in λ-*utility-proportional-beliefs*, survive.

BP show in their Theorem 2 that beliefs are unique in the two player case. By using their Lemma 4 we find an explicit expression for the unique beliefs under common belief in λ-*utility-proportional-beliefs* instead of using their algorithm. This expression reveals clues about the reasoning process players might go through to obtain utility proportional beliefs.

3 Reasoning Process

To introduce the formula for the player's beliefs some more notation needs to be fixed. We denote the number of choices of player i by $n = |C_i|$ and the number of choices for player j by $m = |C_j|$. Moreover, let $N = \{1, ..., n\}$ and $M = \{1, ..., m\}$. The $n \times 1$ vector i_n with $i_n = (\frac{1}{n}, ..., \frac{1}{n})$. Let $C_i = \{c_i^1, \ldots, c_i^n\}$ and $C_j = \{c_j^1, \ldots, c_j^m\}$ so that we can denote player i's $n \times m$ utility matrix by

$$U_i^{norm} = \frac{1}{\overline{u}_i - \underline{u}_i} \begin{bmatrix} U_i(c_i^1, c_j^1) & \cdots & U_i(c_i^1, c_j^m) \\ \vdots & & \vdots \\ U_i(c_i^n, c_j^1) & \cdots & U_i(c_i^n, c_j^m) \end{bmatrix}.$$

The $m \times m$ matrix Z_m has $\frac{m-1}{m}$ on the diagonal and $-\frac{1}{m}$ off the diagonal. Intuitively, the centering matrix subtracts the mean from the columns of a matrix when left multiplied.

We define the matrix $G_j := \lambda_j Z_m U_j^{norm}$ since it will be useful to develop a more intuitive understanding. By left-multiplying the normalized utility matrix U_j^{norm} with the centering matrix Z_m, one obtains a matrix where for every element the average of its column has been subtracted. Note that the rows of U_j^{norm} correspond to i's choices and the columns to j's choices. The same holds for the matrix $Z_m U_j^{norm}$, only that now each element represents the relative goodness of a choice given an opponent's choice. Therefore, the matrix G_j gives the goodness of a choice given a belief about the opponent's choice, scaled by the sensitivity to the opponents differences in utility λ_j.

Now we can state the formula for i's beliefs about j's choices under common belief in *λ-utility-proportional-beliefs*:

$$\begin{aligned}\beta_i &= \sum_{k=0}^{\infty}(G_j G_i)^k (i_m + G_j i_n) \qquad (3.1)\\ &= (i_m + G_j i_n) + G_j G_i (i_m + G_j i_n)\\ &\quad + G_j G_i \left[G_j G_i (i_m + G_j i_n)\right] + \cdots ,\end{aligned}$$

where β_i is a $m \times 1$ vector with the probabilities that player i assigns to player j's choices.

We see that the expression $(i_m + G_j i_n)$ is repeated several times. In the second term, this expression is then adjusted by left multiplying the matrices $G_j G_i$. In the third term the second term is adjusted by left multiplying $G_j G_i$, and so on. Therefore, we call $(i_m + G_j i_n)$ the initial belief, $\beta_i^{initial}$. It shows how player i constructs her beliefs about player j without taking into account that player j reasons about her. Player i starts off by assigning equal probability to her opponent's choice combinations. Then she adjusts her belief by adding the term $G_j i_n$, which represents the goodness of j's choices when j assigns equal probability to all of i's choices.

To emphasize the reasoning process, we define β_i^k as the belief that player i holds after the kth reasoning step,

$$\begin{aligned}\beta_i^0 &:= \beta_i^{initial}\\ \beta_i^k &:= \beta_i^{initial} + G_j G_i (\beta_i^{k-1}),\end{aligned}$$

such that $\lim_{k\to\infty} \beta_i^k = \beta_i$ holds. To obtain a more intuitive understanding we rewrite β_i^k as follows

$$\beta_i^k = i_m + G_j (i_n + G_i(\beta_i^{k-1})).$$

We see that first player i takes j's perspective, which is reflected in the expression $i_n + G_i(\beta_i^{k-1})$. Here player j forms a belief about player i given i's belief about j from the previous reasoning step. First j assigns equal probability to all of i's choices. Then she corrects these beliefs by the goodness of i's choices given i's belief about j from the previous reasoning step. The result is a new belief of j about i. Then player i takes her own perspective and assigns equal probability to all of j's choices. These probabilities are then again corrected by the goodness of j's choices given the new belief of j about i. The process then continues in the same fashion for the subsequent reasoning steps.

It is also important to note that later reasoning steps will be less important for the final belief than earlier ones. Define $\lambda_i = \alpha_i \lambda^{max}$ with $\alpha_i \in [0, 1)$ and note that $G_j = \alpha_i \lambda_i^{max} Z_m U_j^{norm}$, so that (3.1) can be written as

$$\beta_i = \sum_{k=0}^{\infty}(\alpha_i \alpha_j \lambda_i^{max} \lambda_j^{max} Z_m U_j^{norm} Z_n U_i^{norm})^k \beta_i^{initial}.$$

Since $\alpha_i, \alpha_j \in [0, 1)$, later terms in $\sum_{k=0}^{\infty}(\alpha_i \alpha_j \lambda_i^{max} \lambda_j^{max} Z_m U_j Z_n U_i)^k$ will be smaller than earlier ones and therefore less important for the final belief β_i. This has also an important implication for the meaning of the proportionality factor λ_i: the lower the value of λ_i the fewer steps of reasoning a player will undergo to approximate the final belief within a reasonable bound. The same holds true for her opponent's proportionality factor.

4 Connections to Psychology

These features correspond closely to findings in the psychology literature. In his book "Thinking, Fast and Slow" Kahneman [4] advocates the idea of reasoning in two distinct ways. He calls the two modes of thinking System 1 and System 2, according to Stanovich [6]. Note that the word system should not indicate an actual system but only serves as label for different modes of thinking. System 1 is an automatic and mostly unconscious way of thinking that demands little computational capacity. System 2 describes the idea of deliberate reasoning. It comes into play when controlled analytical thinking is needed. Table 1 summarizes the properties of the two systems according to [6].

System 1	**System 2**
associative	rule-based
holistic	analytic
relatively undemanding of cognitive capacity	demanding of cognitive capacity
relatively fast	relatively slow
acquisition by biology, exposure, and personal experience	acquisition by cultural and formal tuition

Table 1. Properties of System 1 & 2

The concept of System 1 describes the unconscious first reaction to a situation, which happens almost immediately and without demanding a lot of cognitive resources. Moreover, Kahneman [4] argues that the beliefs formed by System 1 are the basis for conscious reasoning within System 2. This is consistent with our findings since the initial belief does not take into account any strategic interaction. This belief can be seen as an automatic initial reaction to the game. The deliberate reasoning process described above can be imagined as being executed by System 2 using the findings of System 1, or in this case the initial belief. Taking another player's perspective takes deliberate reasoning and can hardly be done automatically. Finally, we showed that the final belief can be approximated with finitely many steps of reasoning. This feature is closely related to the problem of limited working memory. Baddeley [2] defines working memory as "... [A] brain system that provides temporary storage and manipulation of the information necessary for such complex cognitive tasks as language comprehension, learning, and reasoning." Since working memory is critical for reasoning, however bounded, human beings can only perform a limited number of reasoning without the support of tools. Therefore, a model resembling human reasoning should not predict an infinite amount of reasoning steps.

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