Preservation of Modules

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Abstract. Within the Common Logic Ontology Repository (COLORE), relationships among ontologies such as the notions of faithful interpretability, logical synonymy, and reducibility have been used for ontology verification. Earlier work has shown how to use these relationships to find modules of theories, so a natural question is to determine how we can use the decomposition of one theory into modules to find the modules of another theory in the repository. In this paper, we examine a number of ontologies for which faithful interpretability and logical synonymy do not preserve their modules. Nevertheless, we identify a class of interpretations among theories which guarantees that the modules of a theory are preserved. We also show that the modules of reducible theories are preserved by logical synonymy.

Keywords. modules, first-order theories, logical synonymy, faithful interpretation, reducibility, translation definitions

1. Introduction

The task of decomposing a first-order ontology into modules – subtheories which are conservatively extended by the ontology – remains a significant challenge for the field of ontological engineering [1]. Understanding the structure of an ontology's modules gives us insight into a wide range of properties, from the characterization of the ontology's models to the relationship among other ontologies in the same domain. Given the computational intractability of automatically identifying the modules of an ontology [2], [3], one approach is to reuse the modularizations of existing theories. This idea leads to a key research question — under what conditions can we use the decomposition of one theory into a set of modules to find the modularization of another theory?

The notion of reusing modularizations arises in particular with an approach to ontology verification that is based on repositories. Verification is concerned with the relationship between the intended models of an ontology and the models of the axiomatization of the ontology. In particular, we want to characterize the models of an ontology up to isomorphism and determine whether or not these models are equivalent to the intended models of the ontology. In practice, the verification of an ontology is achieved by demonstrating that it is synonymous with a theory in the repository whose models have already been characterized up to isomorphism. Given two theories which are logically synonymous, we would ideally like to be able identify the modules of the new theory based on the modules of the synonymous theory in the repository.

A key observation of this paper is that there exist logically synonymous theories which have very different modular decompositions. In Section 2, we survey two different sets of ontologies – for lattices and time intervals – and find ontologies which have no proper modules yet which are synonymous with ontologies that have multiple modules. In the remainder of the paper, we explore two possible approaches to deal with this problem. In Section 3 we impose conditions on the interpretations to specify those which do preserve the modules of synonymous theories. In Section 4, we take an alternative approach in which we define a special class of modules that are preserved by logical synonymy. This new class of modules is based on the notion of reducible theories from [4].

2. Modules and Logically Synonymous Theories

A key objective of this paper focuses on the reuse of modularizations between theories. After reviewing the different metalogical relationships among theories that play a role in ontology verification, we find sets of theories for which none of these relationships preserve the sets of modules of the theories. We then consider properties of specific mappings between theories that provide sufficient conditions for the preservation of modules.

2.1. Modules and Conservative Extensions

In this paper, we consider a theory to be a set of first-order sentences closed under logical consequence. We follow previous work in terminology and notation [4] treating ontologies and their modules as logical theories. We do not distinguish between logically equivalent theories. For every theory T, $\Sigma(T)$ denotes its signature, which includes all the constant, function, and relation symbols used in T, and $\mathscr{L}(T)$ denotes the language of T, which is the set of first-order formulae that only use the symbols in $\Sigma(T)$. We remind the reader of the notion of conservative extension [1]:

Definition 1 *Let T_1, T_2 be two first-order theories such that $\Sigma(T_1) \subseteq \Sigma(T_2)$.*
T_2 is a proof-theoretic conservative extension of T_1 iff for any sentence $\sigma \in \mathscr{L}(T_1)$, $T_2 \models \sigma \Leftrightarrow T_1 \models \sigma$.
T_2 is a model-theoretic conservative extension of T_1 iff every model of T_1 can be expanded to a model of T_2.

In this paper, we will focus on model-theoretic conservative extension. [5] establishes the close relationship between the two notions – T_2 is a proof-theoretic conservative extension of T_1 iff every model of T_1 is elementarily equivalent to a structure that can be expanded to a model of T_2.

Most work on modularity in ontologies considers a module to be a subset of axioms in a theory; In this paper, we adopt a more general notion for a module of a theory, by considering a module to be a subtheory of the theory:

Definition 2 *T_1 is a module of T_2 iff T_2 is a conservative extension of T_1.*

2.2. Relationships Among Theories

There are a range of fundamental metalogical relationships among first-order theories. All of them consider mappings between the signatures of the theories which preserve entailment and satisfiability. In this section, we review the three different metalogical relationships that play key roles in ontology verification.

2.2.1. *Interpretability*

The notion of interpretability between theories is widely used within mathematical logic and applications of ontologies for semantic integration. We adopt the following definition from [6]:

Definition 3 *An interpretation π of the theory T_1 with signature $\Sigma(T_1)$ into a theory T_2 with signature $\Sigma(T_2)$ is a function on the set of non-logical symbols of $\Sigma(T_1)$ and formulae in $\mathscr{L}(T_1)$ such that*

1. *π assigns to $\forall$ a formula $\pi_\forall$ of $\Sigma(T_2)$ in which at most the variable v_1 occurs free, such that $T_2 \models (\exists v_1)\, \pi_\forall$*
2. *π assigns to each n-place relation symbol P a formula π_P of $\Sigma(T_2)$ in which at most the variables $v_1, ..., v_n$ occur free.*
3. *for any atomic sentence $\sigma \in \Sigma(T_1)$ with relation symbol P, $\pi(\sigma) = \pi(P)$;*
4. *for any sentence $\sigma \in \Sigma(T_1)$, $\pi(\neg\sigma) = \neg\pi(\sigma)$;*
5. *for any sentences $\sigma, \tau \in \Sigma(T_1)$, $\pi(\sigma \supset \tau) = \pi(\sigma) \supset \pi(\tau)$;*
6. *for any sentence $\sigma \in \Sigma(T_1)$, $\pi(\forall x\, \sigma) = \forall x\, \pi_\forall \supset \pi(\sigma)$;*
7. *For any sentence $\sigma \in \Sigma(T_1)$,*

$$T_1 \models \sigma \Rightarrow T_2 \models \pi(\sigma)$$

Thus, the mapping π is an interpretation of T_1 if it preserves the theorems of T_1 and we say T_1 is interpretable in T_2.

Definition 4 *An interpretation π of a theory T_1 into a theory T_2 is faithful iff for any sentence $\sigma \in \Sigma(T_1)$,*

$$T_1 \not\models \sigma \Rightarrow T_2 \not\models \pi(\sigma)$$

Thus, the mapping π is a faithful interpretation of T_1 if it preserves satisfiability with respect to T_1. We will also refer to this by saying that T_1 is faithfully interpretable in T_2.

Definition 5 *Let T_0 be a theory with signature $\Sigma(T_0)$ and let T_1 be a theory with signature $\Sigma(T_1)$ such that $\Sigma(T_0) \cap \Sigma(T_1) = \emptyset$.*
Translation definitions for T_0 into T_1 are conservative definitions in $\Sigma(T_0) \cup \Sigma(T_1)$ of the form

$$\forall \overline{x}\, p_i(\overline{x}) \equiv \Phi(\overline{x})$$

where $p_i(\overline{x})$ is a relation symbol in $\Sigma(T_0)$ and $\Phi(\overline{x})$ is a formula in $\Sigma(T_1)$.

Following [7], translation definitions can be considered to be an axiomatization of the interpretation of T_0 into T_1.

In [4], it was shown that T_1 is interpretable in T_2 iff there exist a set of translation definitions Δ for T_1 into T_2 such that

$$T_2 \cup \Delta \models T_1$$

2.2.2. Logical Synonymy

One notion of equivalence among theories is mutual faithful interpretability, that is, T_1 faithfully interprets T_2 and T_2 faithfully interprets T_1. An even stronger equivalence relation is that of logical synonymy:

Definition 6 *Two ontologies T_1 and T_2 are synonymous iff there exists an ontology T_3 with the signature $\Sigma(T_1) \cup \Sigma(T_2)$ that is a definitional extension of T_1 and T_2.*

It is easy to see that logical synonymy implies mutual faithful interpretability:

Lemma 1 *If T_1 and T_2 are synonymous, then T_1 faithfully interprets T_2, and T_2 faithfully interprets T_1.*

Logical synonymy is a powerful metalogical relationship that supports ontology verification; [8] shows how this relationship preserves the models of a theory, allowing us to characterize the models of a theory up to isomorphism.

2.2.3. Isomorphism of Categories

As powerful as logical synonymy is, it does not preserve the following model-theoretic properties of a theory – submodels/embedding, homomorphic images, direct products, existentially closed models, model completeness, and the amalgamation property. However, these properties are preserved by isomorphism of categories [9].

For any first-order theory T, $Mod(T)$ forms a category in which the models of T are the objects and homomorphisms on the models are the morphisms.

Definition 7 *Two categories C_1, C_2 are isomorphic iff there exists functors F and G such that*

- $F : C_1 \rightarrow C_2$ *and* $G : C_2 \rightarrow C_1$;
- $FG = 1_{C_2}$ *and* $GF = 1_{C_1}$.

In [10], this is referred to as definitional equivalence of algebraic theories.

Isomorphism of categories of models of theories is stronger than logical synonymy.

Lemma 2 *If there is an isomorphism of categories between $Mod(T_1)$ and $Mod(T_2)$ then T_1 is logically synonymous with T_2.*

In order to guarantee that we have an isomorphism of categories between the category of models of T_1 and the category of models of T_2, [9] imposes the following conditions on the translation definitions Δ for T_1 into T_2 and the Π for T_2 into T_1:

1. Δ is T_1-existential, T_2-universal, and T_2-equivalent to positive formulae;
2. Π is T_2-existential, T_1-universal, and T_1-equivalent to positive formulae;

Thus, the translation definitions Δ and Π must be quantifier-free formulae.

2.3. Modules Are Not Preserved by Synonymy

The following examples show that modules are not preserved by logical synonymy (and hence they are not preserved by interpretation or faithful interpretation). In fact, not even isomorphism of categories is strong enough to preserve modules. In some cases, there exists a theory with no nontrivial modules, yet it is logically synonymous with a theory that does have nontrivial modules. In other cases, there exist modules of one theory which are not logically synonymous to modules of the other theory.

2.3.1. Boolean Lattices

There is a variety of alternative axiomatizations for boolean lattices with different signatures. Within COLORE, there are axiomatizations of Boolean lattices in four different hierarchies: [1]

- $T_{boolean\_lattice}$[2] in $\mathbb{H}^{lattices}$, with signature $\{\mathbf{meet}, \mathbf{join}, \mathbf{zero}, \mathbf{one}\}$;
- $T_{boolean\_lattice\_ordering}$[3] in $\mathbb{H}^{ordering}$, with signature $\{\mathbf{leq}\}$;
- $T_{boolean\_ring}$[4] in $\mathbb{H}^{ringoids}$, with signature $\{\mathbf{sum}, \mathbf{prod}, \mathbf{zero}, \mathbf{one}\}$;
- $T_{boolean\_disjoint}$[5] in $\mathbb{H}^{disjointness}$, with signature $\{\mathbf{disjoint}\}$;

All of these theories are logically synonymous with each other. In addition, $Mod(T_{boolean\_ring})$ and $Mod(T_{boolean\_lattice})$ are category isomorphic.

The theories $T_{boolean\_lattice\_ordering}$ and $T_{boolean\_disjoint}$ have no nontrivial modules. On the other hand, the theories $T_{boolean\_lattice}$ and $T_{boolean\_ring}$ do have nontrivial modules, but there is not a one-to-one correspondence between the sets of modules for these two theories. For example, the maximal module of $T_{boolean\_lattice}$

$$\{(\forall x)\,(join(x,x) = x),\ (\forall x,y)\,(join(x,y) = join(y,x))\}$$

is not synonymous with any maximal module $T_{boolean\_ring}$.

The relevance of these theories to the broader ontology community lies in their relationship to mereologies. The ontology T_{cem_c} for first-order complete extensional mereology is closely related to boolean lattices, being logically synonymous with the theory of boolean semilattices (i.e. boolean lattices with the 0 element removed).

2.3.2. Ontologies for Time Intervals

Within COLORE, there are three hierarchies of time interval ontologies:

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[1] The basic organizational principle in COLORE is the notion of a hierarchy [4], which is a set of ontologies with the same signature. In particular, a hierarchy $\mathbb{H} = \langle \mathscr{H}, \leq \rangle$ is a partially ordered, finite set of theories $\mathscr{H} = T_1, ..., T_n$ such that

1. $\Sigma(T_i) = \Sigma(T_j)$, for all i, j;
2. $T_1 \leq T_2$ iff T_2 is an extension of T_1;
3. $T_1 < T_2$ iff T_2 is a non-conservative extension of T_1.

[2] `http://colore.oor.net/lattices/boolean_lattice.clif`

[3] `http://colore.oor.net/orderings/boolean_lattice_ordering.clif`

[4] `http://colore.oor.net/ringoids/boolean_ring.clif`

[5] `http://colore.oor.net/disjointness/boolean_disjoint.clif`

- the hierarchy $\mathbb{H}_{Periods}$, whose theories were introduced in [11],
- the hierarchy $\mathbb{H}_{Approximate-Point}$ presented in [12],
- the hierarchy $\mathbb{H}_{Interval-Meeting}$, which has been explored in [12].

[13] gives an overview of the metalogical relationships among theories in these three hierarchies. T_{ap_root}[6] and $T_{periods\_root}$[7] are logically synonymous, and there is a one-to-one correspondence between the modules of these two theories. On the other hand, T_{meets_root}[8] and $T_{periods\_root}$ are logically synonymous, yet the former theory has no proper modules.

2.3.3. Summary

We have shown that metalogical relationships as strong as logical synonymy and isomorphism of categories fail to preserve the modules of the theories. This is particularly noteworthy because logical synonymy is typically considered to be a demonstration that the theories are merely notational variants of each other, essentially providing alternative axiomatizations of a theory using different signatures.

2.4. Preservation of Modules by Translation Definitions

Although logical synonymy alone does not preserve modules of a theory, it is important to remember that all of the metalogical relationships such as interpretation, faithful interpretation, and synonymy are based on the existence of translation definitions. Furthermore, we have seen that the even stronger notion of isomorphism of categories required additional conditions be imposed on the translation definitions. If we want to preserve the modules of a theory, we also need to look at properties of the translation definitions themselves.

Theorem 1 *Suppose that T_1 and T_2 are logically synonymous with translation definitions Δ and Π such that*

$$T_1 \cup \Delta \models T_2$$
$$T_2 \cup \Pi \models T_1$$

If $T_{1i} \cup \Delta \cup \Pi$ is a definitional extension of T_{1i} for each module T_{1i} of T_1, then there is a one-to-one correspondence between the set of modules of T_1 and the set of modules of T_2.

Proof: Let π_1 be the (faithful) interpretation of T_1 into T_2 and let π_2 be the (faithful) interpretation of T_2 into T_1.

For any module T_{1i} of T_1, suppose

$$T_{2i} = \{\sigma : T_{1i} \cup \Delta \models \sigma\}$$
$$T_{1j} = \{\sigma : \sigma \in \mathscr{L}(T_{1i}), T_{1i} \cup \Delta \cup \Pi \models \sigma\}$$

[6]`http://colore.oor.net/approximate_point/ap_root.clif`
[7]`http://colore.oor.net/periods/period_root.clif`
[8]`http://colore.oor.net/interval_meeting/meets_root.clif`

Since $T_{1i} \cup \Delta \cup \Pi$ is a definitional extension of T_{1i}, we can see that T_{1j} is logically equivalent to T_{1i} , and hence $\pi_2\pi_1(T_{1i}) = T_{1i}$ for each module T_{1i} of T_1. Therefore, the interpretation π_1 is a bijection from the set of modules of T_1 to subtheories of T_2.

Now suppose that T_{2i} is not a module of T_2, so that there exists a sentence σ with the same signature of T_{2i} such that $T_{2i} \not\models \sigma, \ T_2 \models \sigma$

By definition, $T_{1j} \not\models \pi_2(\sigma)$ and hence $T_{1i} \not\models \pi_2(\sigma)$

However, since T_1 and T_2 are synonymous, we must also have $T_1 \models \pi_2(\sigma)$ which contradicts the assumption that T_{1i} is a module of T_1.

A similar argument shows that the interpretation π_2 is a bijection from the set of modules of T_2 to subtheories of T_1, and that these subtheories are modules of T_2. Thus, we have a bijection between the set of modules of T_1 and the set of modules of T_2. □

We can revisit the examples from the preceding section to see which interpretations saisfy the sufficient conditions for preserving the modules of the theories.

2.4.1. Boolean Lattices

Consider the translation definitions Δ:

$$(\forall x,y,z)\,(sum(x,y) = z) \equiv (\,join(meet(x, comp(y)), meet(comp(x), y)) = z)$$
$$(\forall x,y,z)\,(prod(x,y) = z) \equiv (meet(x,y) = z)$$

and the translation definitions Π

$$(\forall x,y,z)\,(\,join(x,y) = z) \equiv (sum(x, sum(y, prod(x,y))) = z)$$
$$(\forall x,y,z)\,(meet(x,y) = z) \equiv (prod(x,y) = z)$$
$$(\forall x,y)\,(comp(x) = y) \equiv (sum(x,y) = one)$$

$T_{boolean\_lattice} \cup \Delta \cup \Pi$ is not a definitional extension of $T_{boolean\_lattice}$ since it entails the sentence

$$(\forall x,y)\ join(x,y) = join(meet(x,y), join(meet(x, comp(y)), meet(comp(x), y)))$$

2.4.2. Time Interval Ontologies

The translation definitions Δ_{p_ap} for the interpretation of theories in $\mathbb{H}_{Approximate-Point}$ to theories in $\mathbb{H}_{Periods}$ is the set of sentences

$$(\forall x,y)\,precedence(x,y) \equiv precedes(x,y)$$
$$(\forall x,y)\,inclusion(x,y) \equiv finer(x,y)$$

and the translation definitions Π_{ap_p} for the interpretation of theories in $\mathbb{H}_{Periods}$ to theories in $\mathbb{H}_{Approximate-Point}$ are logically equivalent to these sentences, from which it is easy to see that $T_{periods} \cup \Delta_{p\_ap} \cup \Pi_{ap\_p}$ is a definitional extension of $T_{periods}$. Note that there is a one-to-one correspondence between the set of modules of T_{ap_root} and the set of modules of $T_{periods\_root}$.

3. Preservation of Modules in Reducible Theories

So far we have seen that in general, modules are not preserved by faithful interpretation and logical synonymy. In this section we introduce a special class of modules, based on the notion of reducible theories, and show that such modules are preserved by synonymy.

We start with the definition of reducible theories:

Definition 8 *A theory T is reducible to a set of theories $S_1,...,S_n$ ($n > 1$) iff*

1. *T faithfully interprets each theory S_i;*
2. *T is synonymous with $S_1 \cup ... \cup S_n$.*

We will refer to the set $S_1,...,S_n$ as a reduction of T.

The relevant property of reducible theories is that reductions are preserved, up to logical equivalence, by synonymy, because both faithful interpretation and synonymy are transitive relations:

Theorem 2 *If the theory T is synonymous with the theory T', and T is reducible to $S_1,...,S_n$, then T' is reducible to $S_1,...,S_n$.*

Proof: Suppose T is reducible to $S_1,...,S_n$. By definition, T faithfully interprets each S_i.
By Lemma 7 of [4], T' also faithfully interprets each S_i.
By definition, we also know that $S_1 \cup ... \cup S_n$ faithfully interprets T; by Lemma 7 of [4], $S_1 \cup ... \cup S_n$ also faithfully interprets T'.
T' is synonymous with T. By definition, T is synonymous with $S_1 \cup ... \cup S_n$. By Theorem 2 in [8], synonymy is transitive. Therefore, T' is synonymous with $S_1 \cup ... \cup S_n$.
Together, these three conditions show that T' is reducible to $S_1,...,S_n$. □

Consider for example the theory $T_{endpoints}$ [9] (which is a time ontology for timepoints and time intervals) and the theory T_{psl_obj} (that axiomatizes object structures in the PSL ontology). $T_{endpoints}$ relates the time points and time intervals by defining the functions *beginof*, *endof*, and *between*. $beginof(i)$ and $endof(i)$ indicate the begin and the end point of an interval i respectively, while $between(p,q)$ denotes the interval between time points p and q. The theory includes a binary relation *before* over time points which is transitive and irreflexive. T_{psl_obj} includes all function and predicate symbols in $T_{endpoints}$ except *between* and the sort predicate *timeinterval*, but it includes another sort predicate *object*. Moreover, *beginof* and *endof* are defined over *object* elements. $T_{endpoints}$ and T_{psl_obj} are synonymous, and both are reducible to the theory $T_{linear\_ordering}$ [10] of linear ordering and the theory $T_{strict\_graphical}$ [11] of strict graphical incidence structures.

Faithful interpretation generalizes the notion of conservative extension, so each theory S_i in a reduction of a theory T should be related to a module T_i of T. The following theorem shows that S_i and T_i are related by synonymy.

Theorem 3 *Let $S_1,...,S_n$ be a reduction of a theory T.*
There exist theories $T_1,...,T_n$ such that

[9] `http://colore.oor.net/combined_time/endpoints.clif`
[10] `http://colore.oor.net/orderings/linear_ordering.clif`
[11] `http://colore.oor.net/bipartite_incidence/strict_graphical.clif`

1. *T_i is synonymous with S_i.*
2. *T_i is a module of T, for $1 \leq i \leq n$;*

Proof: Let $S_1, ..., S_n$ be a reduction of a theory T; by the definition of reducibility, T is synonymous with $S_1 \cup ... \cup S_n$.

There exists an interpretation π of $S_1 \cup ... \cup S_n$ into T, so that for any $\sigma \in \Sigma(S_1 \cup ... \cup S_n)$,

$$S_1 \cup ... \cup S_n \models \sigma \Leftrightarrow T \models \pi(\sigma)$$

Similarly, there exists an interpretation π_i of T into S_i. Let

$$T_i = \{\sigma : S_i \models \pi_i(\sigma), T \models \sigma\}$$

By the definition of reducibility, T faithfully interprets S_i, so that we have

$$S_i \models \sigma \Leftrightarrow T \models \pi(\sigma)$$

so that T_i is synonymous with S_i.

T_i is not synonymous with T_j, for any $i \neq j$, $1 \leq i, j \leq n$, because otherwise S_i is synonymous with S_j.

Suppose that T is a nonconservative extension of T_i. Then exists $\Phi \in \mathscr{L}(T_i)$ such that

$$T \models \Phi, T_i \not\models \Phi.$$

S_i and T_i are synonymous, so that

$$S_i \cup \Delta_i \not\models \Phi.$$

However, by the definition of reducibility, T is a conservative extension of $S_i \cup \Delta_i$. Since $\Phi \in \mathscr{L}(T_i)$, we have

$$T \not\models \Phi$$

which contradicts the assumption that T is a nonconservative extension of T_i. □

We will refer to the set of subtheories $T_1, ..., T_n$ as the reductive modularization of the theory T that corresponds to the reduction $S_1, ..., S_n$; the subtheories T_i will be referred to as reductive modules.

It is easy to see that not all modules of a theory are reductive. For example, the theory T_{wog} of weak ordered geometries contains two modules (one of which is synonymous with a betweenness relation and other which is synonymous with a bipartite incidence structure); however, since it is not reducible, T_{wog} has no reductive modules. Even if we restrict our attention to reducible theories, not all modules are reductive modules.

The reductive modules of a reducible theory are maximal modules in the following sense:

Theorem 4 *Each module of a reducible theory T is a module of a reductive module of T.*

Proof: If T is reducible, there exists a set of reductive modules of T by Theorem 3.
Suppose there exists a module T' of T which is not a module of any reductive module of T.
Case 1: T' is not a subtheory of any reductive module of T.
Since T' cannot be synonymous with a subtheory of any theory in the reduction of T, we must have

$$S_1 \cup ... \cup S_m \cup \Pi \models T', \;\; m < n$$

, in which case $S_1 \cup ... \cup S_m \cup \Pi$ is a nonconservative extension of T'. However, T is a nonconservative extension of $S_1 \cup ... \cup S_m \cup \Pi$, making T a nonconservative extension of T', which contradicts the assumption that T' is a module of T.
Case 2: T' is a subtheory of a reductive module T_i of T, but T' is not a module of T_i.
T_i must be a nonconservative extension of T', so that there exists a sentence $\Phi \in \mathscr{L}(T')$ such that $T' \not\models \Phi$ and $T_i \models \Phi$
Since T_i is a module of T, we must have $T \models \Phi$. which contradicts the assumption that T' is a module of T. □

Note that if there exist multiple reductions that contain different, but synonymous, sets of theories, each reduction leads to the same modularization, since the different theories in the reductions are synonymous.

On the other hand, since synonymy is transitive, reductive modules of different theories that are related to the same reduction are synonymous. We showed that synonymous theories have same reductions; reductive modules of a theory T are therefore preserved by synonymy:

Theorem 5 *If T_1, T_2 are synonymous reducible theories, then each reductive module of T_1 is synonymous with a reductive module of T_2.*

Proof: Let T_1, T_2 be synonymous reducible theories.
Let $S_1, ..., S_n$ be a reduction of T_1.
By Theorem 3 there exists reductive modularization $T_{11}, ..., T_{1n}$ of T_1 such that T_{1i} is synonymous with S_i.
By Theorem 2, $S_1, ..., S_n$ is also a reduction of T_2.
By Theorem 3 there exists reductive modularization $T_{21}, ..., T_{2n}$ of T_2 such that T_{2i} is synonymous with S_i.
By Theorem 2 in [8], synonymy is transitive. Therefore, T_{1i} is synonymous with T_{2i} for all $1 \leq i \leq n$. □

4. Preservation by Extension

To this point we have considered the preservation of the modules of a theory by logical synonymy, that is, whether there is a one-to-one correspondence between the modules of two theories. We can also consider whether the modules of a theory are preserved by different kinds of extensions, that is, what is the relationship between the modules of T and the modules of an extension of T? In this case, we are considering extensions of a theory which also expand the signature of the theory.

4.1. Preservation by Conservative Extensions

In general, the modules of a theory are preserved by conservative extension.

Theorem 6 *If T_1 is a conservative extension of T_2, then each module of T_2 is also a module of T_1.*

Proof: Suppose T_3 is a module of T_2 and T_2 is a module of T_1. If every model of T_3 can be expanded to a model of T_2, and every model of T_2 can be expanded to a model of T_1, then every model of T_3 can be expanded to a model of T_1. Thus, T_1 is a conservative extension of T_3, and T_3 is a module of T_1. □

On the other hand, reductive modules of a theory are *not* preserved by conservative extension – there exist theories which have nontrivial reductive modules, yet a conservative extension of such a theory has no nontrivial reductive modules. For example, the theory[12] $T_{betweenness} \cup T_{weak\_bipartite}$ contains both $T_{betweenness}$ and $T_{weak\_bipartite}$ as reductive modules. However, the theory T_{wog}[13] is a conservative extension of $T_{betweenness} \cup T_{weak\_bipartite}$ yet it has no reductive modules.

4.2. Preservation by Definitional Extensions

Translation definitions are conservative definitions, so the failure of definitional extensions to preserve modules follows from the discussion of Section 2. Thus, there exists a theory T with definitional extension $T \cup \Delta$ such that $T \cup \Delta$ has modules which are not modules of T. For example, if Δ is the set of translation definitions

$$(\forall x,y,j)\, join(x,y) = j \equiv (leq(x,j) \wedge leq(y,j) \wedge ((\forall z)\,(leq(x,z) \wedge leq(y,z) \supset leq(j,z))$$
$$(\forall x,y,z)\, meet(x,y) = z \equiv (leq(z,x) \wedge leq(z,y) \wedge ((\forall z)\,(leq(z,x) \wedge leq(z,y) \supset leq(z,m))$$

then $T_{boolean\_lattice\_ordering} \cup \Delta$ is a definitional extension of $T_{boolean\_lattice\_ordering}$ and a conservative extension of $T_{boolean\_lattice}$, even though $T_{boolean\_lattice\_ordering}$ itself has no proper modules.

Theorem 7 *Let $T \cup \Sigma$ be a definitional extension of T.*
T has reductive modules $T_1, ..., T_n$ iff $T \cup \Sigma$ has reductive modules $T_1, ..., T_n$.

Proof: Let $T \cup \Sigma$ be a definitional extension of T. Then $T \cup \Sigma$ and T are synonymous.
Since synonymy is transitive, T is synonymous with $S_1 \cup ... \cup S_n$ iff $T \cup \Sigma$ is synonymous with $S_1 \cup ... \cup S_n$.
Considering that Σ is a set of conservative definitions of T, it is straightforward to see that T faithfully interprets S_i iff $T \cup \Sigma$ faithfully interprets S_i.
Hence T is reducible to $S_1, ..., S_n$ iff $T \cup \Sigma$ is reducible to $S_1, ..., s_n$.
By Theorem 3, there exist $T_1, ..., T_n$ such that $T_1, ..., T_n$ are reductive modules of T iff $T_1, ..., T_n$ are reductive modules of $T \cup \Sigma$. □

Thus, T and $T \cup \Sigma$ have the same reductive modules. Theorem 7 also implies that T is not a reductive module of $T \cup \Sigma$ whenever Σ is a set of conservative definitions, even though T is a module of $T \cup \Sigma$.

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[12] `http://colore.oor.net/between/betweenness.clif`
`http://colore.oor.net/bipartite_incidence/weak_bipartite.clif`

[13] `http://colore.oor.net/ordered_geometry/wog.clif`

5. Summary

In this paper, we have considered the problem of reusing the modularization of an ontology – use the decomposition of one ontology into modules to find the modules of another ontology. This problem is formulated as determining whether or not various metalogical relationships among ontologies (such as faithful interpretability, logical synonymy, isomorphism of categories, and reducibility) preserve the modules of an ontology. Interestingly, there are examples of theories whose modules are not preserved by logical synonymy or even by isomorphism of categories.

One observation is that all of these relationships are based on the existence of translation definitions, so one approach to characterizing the preservation of modules imposes additional conditions on these translation definitions.

An alternative approach is to show that restricted classes of modules are preserved by various metalogical relationships. In particular, we showed that the modules of a reducible theory that correspond to the theories in the reduction are indeed preserved by logical synonymy.

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