The cognitive contribution of spatial representation to arithmetical skills

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Abstract

The aim of the present paper is to show that, contrary to the main current approach to numerical cognition, the link between space and numbers cannot be reduced to the concept of a mental number line (MLN). A distinction between low-level numerical skills, that involve MLN processing, and higher-level arithmetical skills, which are related to algorithmic processing, is needed in order to understand the different contribution given by visuo-spatial skills.
I suggest that cognitive skills related to symbolic manipulation should be analyzed on their own in order to better understand the importance of spatial representation for arithmetical processing and, to this purpose, I propose a model of algorithmic skills that could be useful to study some typical features of symbolic manipulation, such as the influence of spatial schemes and the role for external resources in working memory offloading.

Keywords: Numerical cognition; Extended cognition.

Introduction

An important line of research in the field of numerical cognition is focused on the link between numbers and space. Many empirical data suggest that spatial representation and visuo-spatial skills are involved in various number related cognitive activities, as magnitude representation, magnitude comparison, simple mental arithmetic and multidigit calculations.
A very influential explanation of space/number interaction, which focuses on the concept of a mental number line (MLN), is based on the fact that numerical and spatial representation are mediated by common parietal circuits (Dehaene, Bossini, & Giraux, 1993; Hubbard, Piazza, Pinel, & Dehaene, 2005). However, this kind of explanation does not seem to fit with other types of space/number correlations, like those implied in the performance of multidigit arithmetical procedures (Szucs, Devine, Soltesz, Nobes, & Gabriel, 2013).
The aim of the present paper is to show that the link between space and numbers cannot be reduced to the concept of a MLN. We should recognize different kinds of cognitive contribution of space representation above numerical skills, according to the level of cognitive activity considered. A distinction between low-level numerical skills, that involve MLN processing, and higher-level arithmetical skills, which are related to algorithmic processing, is needed in order to understand the different contribution given by space-related skills.

Space and magnitude representation

A set of robust empirical data, mostly connected to the study of the Spatial Numerical Association of Response Codes (SNARC) effect, suggests that numerical representations trigger spatial ones, with smaller numbers connected to the left side, and bigger numbers to the right side of the space.
Verifications of the SNARC effect are based on simple numerical tasks such as parity judgement (Dehaene et al., 1993) and magnitude comparison (Brysbaert, 1995). In parity judgement tasks, where subjects are asked to classify a number as odd or equal by pressing either a right or a left-hand positioned key, left-hand responses are faster for smaller numbers and right-hand responses are faster for larger numbers. Magnitude comparison consists of judging whether a number is smaller or larger than a given one. Here, when left-hand and right-hand keys stand for, respectively, “smaller than” and “larger than”, responses are faster than the inverse response-key configuration.
These cognitive effects led to the hypothesis of a MLN, where numbers are ordered from smaller to larger according to a left-to-right or right-to-left orientation, depending on the writing direction (Restle, 1970; Seron, Pesenti, Noël, Deloche, & Cornet, 1992; Dehaene et al., 1993). A recent animal cognition study (Rugani, Vallortigara, Priftis, & Regolin, 2015) brings a strong evidence toward the existence of a left-to-right oriented MLN in newborn chicks, suggesting that experiential factors or even, in humans, cultural conventions (as the writing direction) may intervene in modulating or modifying the innately left-to right oriented MLN.
Evidence of the existence of a MLN come also from cognitive neuropsychology. In line bisection tasks, in which subjects are asked to indicate the midpoint of a line, hemi-neglect patients have the tendency to move the midpoint towards the portion of the line opposite to the controlesional (usually left) portion of the space (Driver & Vuilleumier, 2001). Interestingly, when tested on number bisection tasks, where subjects are asked to specify the midpoint number of various numerical intervals, these patients show a bias toward larger numbers, which suggests that spatial neglect affects also the MLN (Vuilleumier, Ortigue, & Brugger, 2004; Zorzi, Priftis, & Umiltà, 2002).[1]
Some researchers argue that several cognitive strategies for number processing have been developed to take advantage from the interaction between numerical and spatial representations. Strategies based on the use of the MLN are involved in some mental operations, like subtractions, more than in others, like multiplications, which rely mostly on verbal facts retrieval (Dehaene & Cohen, 1997; Ward, Sagiv, & Butterworth, 2009). Also, performances on visuo-spatial tasks in preschool children are positively related to the development

[1] This topic is currently debated. A thorough study involving 16 right brain-damaged subjects shows a dissociation between deviations in physical and number line-bisection tasks, suggesting that the navigation along physical space and number lines is governed by different brain networks (Doricchi, Guariglia, Gasparini, & Tomaiuolo, 2005).

of more advanced arithmetical skills (LeFevre et al., 2010). This fact is a further sign of the importance of space-numbers interaction. However, some aspects of the cognitive mechanisms underlying this interaction remain unexplained.

Neurocognitive explanations

Since the last decade of the past century, many researchers have inquired into the neural basis of numerical cognition.

An influential model is Dehaene's triple-code model of number processing (Dehaene, 1992). According to Dehaene, number representation involves different neural substrates in which numerical information is encoded visually (as strings of arabic digits), verbally (as number words, sets of number facts etc.), and analogically (as magnitide). The latter kind of numerical representation is mediated by a neurocognitive system called "number module", or Approximate Number System (ANS), which is shared by different animal species and makes for the representation of approximate quantities (Dehaene, 2011; Landerl, Bevan, & Butterworth, 2004; Piazza, 2010).

The typical problem in which the ANS comes into play is the comparison between different sets of objects to decide which is the largest. Some researchers suppose that the ANS is the only preverbal representational system needed for the development of basic numerical concepts (Piazza, 2010). Spelke (2011) combines the work of the ANS with the information gained by the Object Tracking System (OTS), which provides the capacity to recognize at a glance the number of objects in sets less than or equal to 4 items (subitizing). In this account, the OTS supports the representation of integers, which is needed for symbolic arithmetic processing. On the contrary, Leslie, Gelman, and Gallistel (2008) propose that the representation of exact quantities does not involve the OTS. Instead, the natural number set is recursively constructed through the *successor* function on the base of some innate concept of integer number—the concept of *exactly one*, at least—and, then, is mapped on the approximate magnitude representations given by the ANS.

A Theory Of Magnitude (ATOM)

Despite the aforesaid differences, all theoretical accounts recognize a crucial role of the ANS. Moreover, space-numbers correlations, as those revealed by the SNARC effect, suggest that there should be some kind of interaction between neural circuitry involved in spatial and numerical processing. In fact, functional MRI (fMRI) studies reveal that non-symbolic number processing activates neurons of a bilateral parietal cortical area, the Intra-Parietal Sulci (IPS), which have a functional role also in visuo-spatial and manual tasks, such as grasping and pointing (Butterworth, 2000; Hubbard et al., 2005; Pinel, Dehaene, Rivire, & LeBihan, 2001; Simon, Mangin, Cohen, Bihan, & Dehaene, 2002).

In 2003 an influential proposal, called A Theory Of Magnitude (ATOM), was made in order to "bring together [...] disparate literatures on time, space and number, and to show similarities between these three domains that are indicative of common processing mechanisms, rooted in our need for information about the spatial and temporal structure of the external world" (Walsh, 2003). According to Walsh, the parietal cortex represents environmental information about different kinds of magnitudes—number, space, time, size, speed etc.—whose interaction are supposed to have a specific—and, to date, neglected—meaning for the guidance of action. In Walsh's view these cognitive interactions are due to overlapping sets of parietal neurons that share a common metric for magnitude representation, and this fact is at the base of the SNARC effect. ATOM predicts that the SNARC effect is an instance of a broader SQUARC (Spatial QUantity Association of Response Codes) effect, "in which any spatially or action-coded magnitude will yield a relationship between magnitude and space" (Walsh, 2003). In some cases this prediction has, indeed, been verified (Bueti & Walsh, 2009).

Space and arithmetical skills

So far, ATOM represents the most convincing explanatory account of the link between numerical and spatial representations. However, the question about the function of non-symbolic magnitude representation in the development of arithmetical skills remains opened.

Developmental dyscalculia and spatial skills

Developmental dyscalculia (DD) is a learning disability which specifically affects the acquisition of arithmetical skills in otherwise normal subjects. In its pure form (i.e., without co-morbidity with other learning problems, such as dyslexia or attention-deficit-hyperactivity disorder) it is estimated that it affects $3-6,5\%$ of school-age population. As other learning disabilities, it is thought to have a neural basis (Shalev & Gross-Tsur, 2001).

Given the essential function in number processing accorded to the ANS—which is neurally located in the bilateral intraparietal sulci—it is natural to suppose that DD is related to impairments to this neurocognitive module (Dehaene, 2011; Piazza, 2010).

However, a recent paper by Szucs et al. (2013) reviews previous experiments directed at verifying the implication of ANS impairments for DD and denies that we have sufficient empirical evidence to prove this correlation. They conducted an extensive series of tests and experiments on a population of 1004 DD affected 9-10-year-old children, concluding that the main cognitive factors that cause DD are visuo-spatial memory and inhibition impairments which, crucially, are related to IPS activity as well as the ANS. Consequently, they propose a different approach to the explanation of DD in which findings about IPS morphological and functional differences between DD subjects and controls (Mussolin et al., 2010; Price, Holloway, Räsänen, Vesterinen, & Ansari, 2007; Rotzer et al., 2008) are linked to general purpose cognitive processes involving the IPS, rather than magnitude representation deficits. In particular, inhibition impairments "could lead to mathematical problems because Numerical Operations require the temporal and spatial (in imagination) coordination of several processes and the retrieval of several highly similar facts—impaired inhibition probably interferes with the organization of these processes" (Szucs et al., 2013). Also, inhibitory processes seem to have a crucial function for the central executive component of the working memory (WM) (Carretti, Cornoldi, De Beni, & Palladino, 2004). Then, for DD subjects, problem in visuo-spatial memory tasks may be secondary to the impairment of inhibition processes.

Neuropsychological studies

Some hints about the correlation of spatial and numerical skills also come from the field of neuropsychology. Here, some interesting cases have been described that seem consistent with the findings of Szucs et al. (2013) reported in the previous paragraph.

Semenza, Miceli, and Girelli (1997) report a case where arithmetical difficulties seem to be related to the lack of monitoring of arithmetical procedures. The patient, M.M., 17 years old, was affected by hydrocephalus with the posterior portion at the right dorso-frontal cortex and the right upper parietal lobe severely damaged. His mental calculation abilities were excellent and, in some cases, surprising (he could solve 2×2 digit multiplications where the two factors were the same, as, e.g., 24×24, with the same speed as for table problems). In written calculation, however, his performances were dramatically poorer. His problems, especially manifest in complex multidigit multiplications, were of different nature, but the mostly committed errors were connected to wrong factor selection. M.M. often repeated subsequently the same operation, and did not realized when he had reached the end of the procedure—the authors report that "[h]e kept asking the examiner whether the operation was finished" (Semenza et al., 1997). In some cases, M.M. made errors also in the spatial arrangements of numbers.

In another study, Granà, Hofer, and Semenza (2006) report a case of "spatial acalculia", i.e. a specific deficit in the spatial arrangement of numbers in written calculation. The patient, PN, had a vast parietal damage as a consequence of a brain haemorrage. In addition to spatial arrangements errors, PN committed errors in factor selection. According to the authors, "the best explanation for PNs problems is that he might have difficulties in relying on a visuo-spatial store containing a layout representation specific to multiplication. As a consequence, while knowing what, when and how to carry out the various steps, PN does not know where" (Granà et al., 2006).

The procedural problems described in these neurospychological reports seem consistent with the functional interpretation of inhibition processes in arithmetical procedures given by Szucs et al. (2013). M.M.'s problems with factor selection and operation ordering may be explained as a consequence of the interference of previously performed operational steps. On the other hand, visuo-spatial deficits like those manifested by PN may be also linked to inhibition impairments by a similar mechanism as that seen in the previous paragraph.

Different cognitive contributions

To sum up, we have seen at least two very different cognitive contribution of spatial skills to numerical abilities:

1. Numerical and spatial representations are linked by the existence of a MLN, where numbers are represented as ordered from smaller to larger according to a precise orientation. This hypothesis is supported by, e.g. , experiments on the SNARC effect and the number bisection task in hemineglect patients.

2. Visuo-spatial skills are implied in arithmetical processing for monitoring the temporal and spatial coordination of the many processes needed in order to carry out an arithmetical operation—selecting the right factor, keeping track of the partial results, arranging numbers correctly in the space, performing operation steps according to a given schema etc. In this case, the link between space and numbers is explained by resorting to general purpose cognitive capacities, such as inhibition processes, which are crucial for the central executive component of the working memory.

The two points sketched above may be put in correspondence with different skill levels in which a link between numbers and space is on hand.

Point 1 corresponds to low level skills, based on innate cognitive systems of magnitude representation. These skills are needed to perform tasks such as non-symbolic magnitude comparison, number comparison and parity judgments. Also, strategy based on the representation of numbers on a MLN are very likely involved in some types of mental operations, like simple mental subtractions and, to a lesser extent, mental additions.

Point 2, on the other hand, corresponds to higher level skills, mostly based on learned abilities. These capacities, which may be included in the specific concept of *algorithmic skills*, are needed in order to apply computing strategies, i.e. to execute set of rules for symbolic transformation, and often rely on the use of paper and pen or equivalent external resources.

Algorithmic skills

With the word "algorithm" I refer to any finite set of rules for symbolic transformation which can be performed by a subject with the only aid of paper and pen (or equivalent external resources) and without resorting to any particular insight or ingenuity.

This informal definition may be dated back to the work of Alan Turing (Turing, 1936). Only in recent past (Giunti, 2009; Wells, 2005) the cognitive importance of this work has been fully recognized.

Algorithmic skills consist on all those cognitive skills that employ algorithms. The possibility of using external tools such as paper and pen, which is included in the concept of algorithm, seems particularly relevant for the issue about numbers/space interaction. Algorithmic strategies (e.g., the standard multiplication algorithm), indeed, make extensive use of spatial schemes in order to subdivide arithmetical operations into simple computational steps and for working memory offloading. Rumelhart, Smolensky, McClelland, and Hinton (1986) consider such kind of cognitive tasks as cases of *online symbolic transformation*. Here, visuo-spatial skills are involved in order to "maintain on-line the spatial layout and digits of an ongoing multi-digit calculation" (Dehaene & Cohen, 1995).

The spatial layout may in itself facilitate carrying out computational steps, e.g., by simplifying the individuation of each single step or by making easier to recognize the temporal order of single operations.

Landy and Goldstone (2007) experimentally tested the hypothesis that symbolic spatial layout is organized to reflect syntactic relations among symbols. They start from suggesting that "formal notations are diagrammatic as well as sentential and that the property conventionally described as syntactic structure is cognitively mediated, in part, by spatial information. Elements of expressions are bound together through perceptual grouping, often induced by simple spatial proximity." (Landy & Goldstone, 2007). To test this hypothesis, they designed two experiments.

In the first, subjects had to write down with paper and pen simple equations they found written on a computer screen. In one side of the equation were three single-digit numbers with two operands (addition and/or multiplication sign); the other side contained the same expression, but with one operation completed (e.g, $x \times y + z = w + z$). The experimenters found that subjects tended to leave a tighter space between numbers and the operand in the operations that, according to syntactic rules, had to be completed earlier—e.g, the space between numbers and sign in multiplications was tighter than in additions—even if they need not perform any operation.

The second experiment consisted in the production and writing (with paper and pen) of formal propositional logic expressions. Even in this case, the result was that the spacing around logical connectives reflected syntactic rules (e.g., the blank space around principal connectives were wider than that around secondary connectives).

These results seem to point toward a confirmation of the central function earlier ascribed to spatial representations in algorithmic skills. In particular, the fact that similar spatial features are found in the production of both numerical and logical expressions is consistent with the definition of algorithmic skills given above, for this definition does not limit them to numerical transformation but refers to all kinds of symbolic manipulation. This, interestingly, suggests that we are facing some kind of broad set of cognitive skills, which deserves to be investigated in its own right.

Sketch of a model[2]

A model of algorithmic skills[3] should include different parts in order to reflect a set of relevant features that I previously highlighted, i.e.:

(i) a *central unit*, in which rules of execution and knowledge of arithmetic facts are embedded;

(ii) a *temporary store*, where, at each step of a computation, the relevant data are hold;

(iii) an *external store*, which contains written symbols;

(iv) a *perception/action mechanism*, which connects the central unit to the stores and makes for rules execution.

If we take the cue from algorithmic execution with paper and pen, part (i) and (ii) may be specified as the *internal* side of the model— respectively, the long-term memory and the WM's *slave systems* (Baddeley & Hitch, 1974; Baddeley, 2000)—, while part (iii) is the *external* memory (the paper and its content). However, the model should be flexible enough to include examples where, e.g., rules to be executed are found written or symbolic transformation is made mentally. In the latter case, part (iii) may be omitted.

Part (iv) is needed for making rules of execution effective by correctly connecting the other parts of the model, and incorporates the executive component of the WM. The cognitive work carried out by this part may help understanding the contribution of spatial representation to algorithmic skills. For instance, let us think of any case of algorithmic execution with paper and pen. A mechanism of perception/action, at each computational step, has to complete the following *cycle*:

- take inputs from the external store, i.e. draw subject's attention to relevant data among written symbols;
- choose from the central unit a rule to be executed, on the basis of the symbol or set of symbols on which subject's attention is drawn;
- apply the rule to the right symbol or set of symbols and elaborate the output or retrieve it from the central unit;

[2] Anderson, Lee, and Fincham (2014) have recently proposed a neuro-functional model of mathematical problem solving. Although relevant for this issue, it should not be seen in contrast with the model sketched here, for it is meant to explain a different set of cognitive activities, i.e., mathematical problems where the cognitive subject does not necessarily know in advance what is the best solving strategy. The model I propose, on the contrary, deals only with symbol transformation activities with clear-cut sets of rules of computation, such that the subject should be able to access them without particular cognitive effort.

[3] Giunti (2009) proposed a Turing machine-inspired model—the Bidimensional-Turing machine—of what he calls "Phenomena of human computation", a concept that roughly corresponds to the idea of algorithmic skills proposed above. An implementation of Giunti's approach to the study of human computational skills is in Pinna and Fumera (in press). Here, I will describe intuitively and without deepening into technicalities the main features that a model of algorithmic skills should include, in order to help explaining some issues of numerical cognition as the importance of the spatial component of the WM for arithmetic.

- hold in the temporary store, if necessary, some data relative to the output, e.g., a carry;
- transcribe the output of actual symbolic operation into the external store;
- hold in the temporary store the relevant information needed to start the next cycle.

Features of the spatial schema used in a paper and pen algorithm must be hold correctly to complete a cycle of computation. Spatial features seem to be particularly important, e.g., for choosing the right symbols to which apply a rule of transformation, and to transcribe in the right place of the schema the result of an operation. An incorrect execution of these functions, indeed, may be a source of arithmetical errors, as in cases of *spatial acalculia* (Granà et al., 2006).
Moreover, the possibility to include in the model external resources such as paper and pen may help clarifying the use of *externalized* strategies for working memory *offloading* and, broadly, the value of inner/outer interactions for cognition (Clark, 2008; Smith & Thelen, 1993; Tschacher & Dauwalder, 2003).

Conclusion

We can distinguish two different cognitive contributions of spatial skills to numerical knowledge:

(a) The concept of a MLN, on the one hand, seems to be implied in some low level numerical skills, based on innate cognitive systems of magnitude representation. These low level skills are at work in some tasks such as non-symbolic magnitude comparison, number comparison and parity judgments, and are likely involved in simple mental arithmetic, e.g., simple mental additions and subtractions.
(b) On the other hand, visuo-spatial skills are crucial in more complex arithmetical tasks—such as paper and pen strategies used in multidigit calculations—for monitoring the temporal and spatial coordination of the many processes needed in order to carry out symbolic transformations. These tasks seem to rely on higher level skills, mostly based on learned abilities, which are not specific to numerical knowledge but are implied in any kind of strategy for symbolic manipulation.
I proposed that type (b) of cognitive capacities may be included in what I called *algorithmic skills*, which are all cognitive skills employed in the execution of finite sets of rules for symbolic manipulation (algorithms).
The multi-part model of algorithmic skills sketched above could be useful to study some typical features of symbolic manipulation, such as the influence of spatial schemes and the use of external resources for working memory offloading. The formalization of specific algorithmic strategies through the model should, in particular, give a precise idea of the weight of the various cognitive components at each step of a computation. These theoretical hints may be used for making predictions and design experimental settings about, e.g., the level of WM resources needed to perform a certain symbol manipulation task, or the similarity/difference between mental and externalized strategies.
The empirical adequacy of the model may be tested, e.g, by inspecting to what extent different spatial schemes used for performing similar symbol manipulation tasks influence WM load. On the other hand, indirect indications may also come from behavioral experiments, e.g., by analyzing response times to the solution of similar tasks presented according to different spatial layouts.
Such kind of cognitive analysis may be useful to shed light on the relation between space and numbers and, by a broader perspective, to further investigate the issue of organism/environment interaction for cognition.

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