On Strong Accessibility of the Core of TU Cooperative Game

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Abstract. In the paper, a strengthening of the core-accessibility theorem by the author is proposed. It is shown that for any imputation outside of the nonempty core of TU cooperative game a strongly monotonic trajectory originating from this imputation exists, which converges to some element of the core. Here, strong monotonicity means that each imputation from the trajectory dominates several preceding elements and, besides, the number of these dominated imputations tends to infinity. To show that transferable utility assumption is relevant for strong accessibility of the core, we give an example of NTU cooperative game with a "black hole" being a nonempty closed subset of dominated imputations that contains all the sequential improvement trajectories originating from its points.

Keywords: Domination, core, strong monotonicity, strong accessibility, dynamic system, endpoint, generalized Lyapunov function

1 Introduction

Let $N = \{1, \dots, n\}$ be a set of players. Recall (see, e.g., [2], [4]), that an n-person cooperative game with transferable utility (TU cooperative game, for short) on N is a function $v : 2^N \to \mathbb{R}$ that associates with each subset $S \subseteq N$ (called a coalition, if nonempty), a real number $v(S)$, the worth of S (it is required that $v(\emptyset) = 0$). If "big coalition" N forms, then it can divide its worth, $v(N)$, in any possible way among the players $i \in N$. In case each player i is endowed with amount not less than her worth, $v(\{i\})$, we get so-called imputation (individually-rational and efficient payoff) of the game v. Collection of imputations of the game v is denoted by $I(v)$ (below, we use a standard shortening $v(i) = v(\{i\})$):

$$I(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N),\ x_i \geq v(i),\ i \in N\} .$$

Further, we apply one more shortening: for any $x = (x_1, \dots, x_n) \in \mathbb{R}^N$ and $S \in 2^N$ we put

$$x(S) = \sum_{i \in S} x_i .$$

In: A. Kononov et al. (eds.): DOOR 2016, Vladivostok, Russia, published at http://ceur-ws.org

In order to introduce classic domination relation α_v on $I(v)$, some additional terms are required. First, recall that imputation x is said to be dominated by imputation y via a coalition S, if $x_i < y_i$, $i \in S$, and $y(S) \leq v(S)$. Further, we say, for short, that $x \in I(v)$ is dominated by $y \in I(v)$, if there exists a coalition S such that x is dominated by y via S. Finally, we say that $x \in I(v)$ is dominated, if there exists an imputation y that dominates x. Respectively, an imputation x is said to be un-dominated, if there is no imputation that dominates it.

To summarize, we get formal descriptions of the classic domination relation α_v on $I(v)$, given by the formula

$$x \ \alpha_v \ y \Leftrightarrow \exists S \in 2^N \setminus \{\emptyset\} \ [(x_i < y_i, \ i \in S) \ \& \ (y(S) \leq v(S))] \ ,$$

and the core $C(\alpha_v)$ of v being the set of un-dominated imputations of v :

$$C(\alpha_v) = \{x \in I(v) \ \big| \ \nexists y \in I(v)[\ x \ \alpha_v \ y \ ]\} \ .$$

Recall (see, e.g., [3], [4]), that the core $C(\alpha_v)$ is one of the main solutions of cooperative game theory with numerous applications to social, economic, and political problems. One of the most interesting problems relating to the core $C(\alpha_v)$ seems to be an asymptotic behavior of the processes of sequential improvement of dominated alternatives (does there exist a non-dominated limit, or we deal with chaotic movement, etc.). Speaking formally, we have to study so-called α_v-monotonic trajectories $\{x_r\}_{r=0}^{\infty}$ with $x_r \in I(v)$, $r \geq 0$, originated from imputations outside the core $C(\alpha_v)$.

Definition 1. *A sequence $\{x_r\}_{r=0}^{\infty}$ with $x_r \in I(v)$, $r \geq 0$, is said to be α_v-monotonic (monotonic, for short), if for each natural number $r \geq 1$ it holds $x_{r-1} \ \alpha_v \ x_r$.*

Speaking differently, a sequence $\{x_r\}_{r=0}^{\infty}$ of imputations is called α_v-monotonic, if any element x_{r+1} dominates preceding element x_r.

In the paper, we consider one of the problem relating to the asymptotic behavior of monotonic sequences, so-called core-accessibility problem. Namely, we are interesting if there exists, for any imputation outside the core, at least one monotonic sequence starting at that imputation and converging to some un-dominated imputation. And if so, what additional improvements of this sequence can be made. To isolate the cores satisfying accessibility property mentioned, we present one of the key definitions of the paper.

Definition 2 (Vasil'ev, 1987). *The core $C(\alpha_v)$ of TU cooperative game v is called accessible, if $C(\alpha_v) \neq \emptyset$, and for any imputation $x \notin C(\alpha_v)$ there exists a convergent α_v-monotonic sequence $\{x_r\}_{r=0}^{\infty}$ of imputations such that $x_0 = x$ and $\lim_{r\to\infty} x_r$ belongs to the core $C(\alpha_v)$ of the game v.*

It turned out that the core-accessibility takes place for any TU cooperative game with non-empty core.

Theorem 1 (Accessibility Theorem). *The core $C(\alpha_v)$ of any TU cooperative game v is accessible whenever it is nonempty.*

This theorem was established in [5] (for more details, see [6]). It is worth to stress, that both articles mentioned exploit only classic settings, including classic definitions of imputation and domination (not like in [8], for instance, where the author, instead of $I(v)$, deals with a greater set of payoffs, thus essentially enlarging possibilities of reconstruction for the elements outside the core).

A strengthening of Accessibility Theorem, proposed in the paper, is motivated by the lack of transitivity of domination α_v. Sequential improvement process $\{x_r\}_{r=0}^{\infty}$, based on non-complete and non-transitive binary relation α_v, may contain several fragments of type $x_{r-1}\ \alpha_v\ x_r\ \alpha_v\ x_{r+1}$ with x_{r+1} that doesn't dominate x_{r-1}. Even more, we may get a cycle $x_{r-1}\ \alpha_v\ x_r\ \alpha_v\ x_{r+1}\ \alpha_v\ x_{r-1}$ or more lengthy cycle

$$x_{r-1}\ \alpha_v\ x_r\ \alpha_v \ldots \alpha_v\ x_m\ \alpha_v\ x_{r-1}$$

with $m > r+1$. Unfortunately, we have no idea relating to the exclusion of this type of cycling in sequential improvement process. What can we do is just providing each imputation x_{r+1} in the process $\{x_r\}_{r=0}^{\infty}$ to be able dominate a wider collection of previous imputations than a singleton $\{x_r\}$. Certainly, it would be very nice to organize an improvement process in such a way that any imputation x_{r+1} dominates all the previous x_m, $m = 0, 1, \ldots, r$. But already very simple 3-person games demonstrate impossibility of such a total domination phenomenon. Nevertheless, it turned out that we can essentially refine improvement process provided that instead of α_v-monotonicity we apply its strengthening, given in the following definition.

Definition 3. *A sequence $\{x_r\}_{r=0}^{\infty}$ of imputations of TU cooperative game v is said to be strongly α_v-monotonic (strongly monotonic, for short), if there exists a sequence of natural numbers $k_r \to \infty$ such that $k_r \leq r$, $r \geq 1$, and $x_{r-m}\ \alpha_v\ x_r$ for any $r \geq 1$ and $m \in [1, k_r]$.*

Now, we are in position to present main definition of the paper.

Definition 4. *Let the core $C(\alpha_v)$ be non-empty. Then $C(\alpha_v)$ is said to be strongly accessible, if for any $x \in I(v) \backslash C(\alpha_v)$ there exists a convergent and strongly α_v-monotonic sequence of imputations $\{x_r\}_{r=0}^{\infty}$ such that $x_0 = x$ and $\lim\ x_r$ belongs to the core $C(\alpha_v)$.*

In the paper, we propose a strengthening of above mentioned Accessibility Theorem: for any TU cooperative game v the core $C(\alpha_v)$ is strongly accessible whenever it is nonempty. Thus, in case of TU cooperative games we can raise the quality of a monotonic improvement process to a rather high level. Note that another variant of raising quality of improvement process is given in [7]: the case of constant k_r ($k_r = k, r \geq k$) is considered instead of the case $k_r \to \infty$.

The paper is organized as follows. Besides Introduction, it contains two parts and appendix. First part is devoted to the proof of the strong accessibility theorem, second part contains an example of NTU cooperative game that has nonempty core, which is not accessible. Appendix, for the sake of completeness, is devoted to brief outline of the proof of Accessibility Theorem. In contrast to [6], this proof is a straightforward one. Moreover, it is not "immersed" into a more general setting, like in [5], [6].

2 Proof of the Strong Accessibility Result

In this section we give a proof of strong accessibility of the core of TU cooperative game. We show that any sequential improvement process that exists according to Theorem 1 admits a considerable improvement of its quality, suffering due to the non-transitivity of classic domination relation. This lack of transitivity in the process of sequential improvement of dominated alternatives may be inadmissible in many applications of game theory to economical, sociological and political problems. By reconstructing a monotonic process in appropriate way, we can get a strongly monotonic improvement process with each imputation (except the first and second ones) dominating several preceding alternatives. Even more, the number of this dominated alternatives tends to infinity. Passing on to the formal presentation of the strong accessibility result, we stress once more that its proof heavily relies upon Accessibility Theorem and some elementary topological properties of the domination relation α_v.

Theorem 2. *Let v be an n-person TU cooperative game with nonempty core $C(\alpha_v)$. Then for any imputation $x \in I(v) \setminus C(\alpha_v)$ there exist a sequence of natural numbers $k_r \to \infty$ with $k_r \leq r$, $r \geq 1$, and convergent sequence of imputations $\{x_r\}_0^\infty$ such that $x_0 = x$, $\lim_{r\to\infty} x_r$ belongs to the core $C(\alpha_v)$, and, besides, $x_{r-m}\ \alpha_v\ x_r$ for all natural numbers $r \geq 1$ and $m \in [1, k_r]$.*

Proof. Let v be an n-person TU cooperative game with nonempty core $C(\alpha_v)$, and x be some dominated imputation. Without loss of generality we may assume that v satisfies requirement: $v(\{i\}) = 0$ for each $i \in N$. Due to Accessibility Theorem there exists an α_v-monotonic convergent sequence of imputations $\{x_r\}_{r=0}^\infty$ such that $x_0 = x$ and $x_* = \lim_{r\to\infty} x_r \in C(\alpha_v)$. By Definition 1, for any $r \geq 1$ there exists a coalition S_r such that imputation $x_r = (x_{r,1}, \ldots, x_{r,n})$ dominates imputation $x_{r-1} = (x_{r-1,1}, \ldots, x_{r-1,n})$ via S_r :

$$x_{r-1,i} < x_{r,i},\ i \in S_r; \quad x_r(S_r) \leq v(S_r), \quad r = 1, 2, \ldots \ . \tag{1}$$

Below, we exploit max-metric $\rho_\infty(x, y) = \|x - y\|_\infty = \max\ \{|x_i - y_i| \big| i \in N\}$ (respectively, all the distances and neighborhoods we use in this section are given in this metric). By applying max-metric we define vicinities U_r of imputations x_r in order to modify the sequence $\{x_r\}_{r=0}^\infty$ in a way required by the definition of strong accessibility. Namely, for any $r > 1$ we put

$$\varepsilon_r = \min\ \{x_{r,i} - x_{r-1,i} \big|\ i \in S_r\}/2$$

and define $\delta_r = \min\ \{\varepsilon_r, \varepsilon_{r+1}\},\ r > 1$. Further, for any $r > 1$ denote by U_r the δ_r-neighborhood of the imputation x_r :

$$U_r = \{y \in I(v) \big| \rho_\infty(x_r, y) < \delta_r\}, \quad r = 2, 3, \ldots \ . \tag{2}$$

Fix some $r \geq 1$. Since x_{r-1} is dominated by x_r via the coalition S_r, directly from (1) and 0-normalization condition $v(\{i\}) = 0$, $i \in N$, it follows that

$$x_r(S_r) \leq v(S_r) \quad \text{and} \quad x_{r,i} > 0,\ i \in S_r\ .$$

Hence, we may construct a "bundle" $\{x_r^m\}_{m=1}^{r-1} \subseteq U_r$ satisfying α_v-monotonicity condition with respect to the coalition S_r :

$$x_{r,i}^1 \ < \ x_{r,i}^2 \ < \dots \ < \ x_{r,i}^{r-1} \ < x_{\ r,i} \ , \quad i \in S_r \ , \tag{3}$$

where $x_{r,i}^m$ is an i-th component of the imputation $x_r^m = (x_{r,1}^m, \dots, x_{r,n}^m)$ of the game under consideration. One can rather easily check that $x_r^1, \dots, x_r^{r-1}$ may be given by the formulae: $x_r^m = x_r + (m-r)c$ with

$$c_i = \begin{cases} \delta_r(n-s_r)/nr \ , i \in S_r \ , \\ -\delta_r s_r/nr \qquad , i \in N \setminus S_r \ , \end{cases}$$

where $s_r = |S_r|$, $r \geq 1$. In particular, inclusions $x_r + (m-r)c \in U_r,\ m \in [1, r-1],\ r \geq 1$, follow directly from the elementary relations[1]

$$\sum_{i \in S_r} c_i = \delta_r \frac{s_r(n-s_r)}{nr} = - \sum_{i \in N \setminus S_r} c_i, \quad r \geq 1 \ ,$$

$$(r-m)c_i = \delta_r \frac{r-m}{r} \frac{n-s_r}{n} < \delta_r,\ i \in S_r,\ m \in [1, r-1],\ r \geq 1 \ ,$$

$$(r-m)c_i = \delta_r \frac{r-m}{r} \frac{s_r}{n} < \delta_r,\ i \in N \setminus S_r,\ m \in [1, r-1],\ r \geq 1 \ .$$

Let us mention also, that monotonicity condition means (due to the inequality $x(S_r) \leq v(S_r)$ following from the assumption that x_r dominates x_{r-1} via the coalition S_r), that in (general) collection $\{x_r^m\}_{m=1}^{r-1}$ each imputation x_r^m dominates preceding x_r^{m-1} via coalition S_r. Besides, the last element in the collection is dominated by $x_r : x_r^{r-1}\ \alpha_v\ x_r$. Note, that relations mentioned

$$x_r^1\ \alpha_v\ x_r^2\ \alpha_v\ \dots\ \alpha_v\ x_r^{r-1}\ \alpha_v\ x_r$$

possess some "transitivity features": each element of the collection under consideration dominates all the preceding elements. As to x_r, it dominates (together with x_{r-1}) each imputation x_r^m, $m = 1, \dots, r-1$. Note also, that according to the choice of the vicinities U_r we get: the first imputation x_r^1 (together with the others x_r^m, $m \leq r-1$) dominates x_{r-1} and x_{r-1}^m, $m = 1, \dots, r-2$, via the coalition S_r.

Passing on to the formation of a sequence $\{y_s\}_{s=0}^{\infty}$, originating from the above mentioned imputation $x \notin C(\alpha_v)$ and satisfying all the requirements of strong accessibility, we introduce first the numbers $e_r = r(r+1)/2, r = 1, \dots$. Further, by exploiting above mentioned elements x_r and x_r^m we define the sought for sequence $\{y_s\}_{s=0}^{\infty}$ by the formula

$$y_s = \begin{cases} x_0 = x \ , & \text{if } s = 0; \\ x_r & , \ \text{if } s = e_r; \\ x_{r+1}^m & , \ \text{if } s = e_r + m \text{ and } m \in [1, r] \ . \end{cases} \tag{4}$$

[1] One may proceed by investigating these or some other appropriate concrete bundles. Below, we apply more general consideration, which admits extensions to a more abstract settings.

We start with the proof that sequence $\{y_s\}_{s=0}^\infty$ is convergent, and its limit is equal to $x_* = \lim_{r\to\infty} x_r$. Fix an arbitrary $\varepsilon > 0$ and show that there exists a number s_0 such that $\rho_\infty(y_s, x_*) < \varepsilon$ for any $s > s_0$. To this end let us mention that due the convergency $x_r \to x_*$ there exists r_0 such that $\rho_\infty(x_r, x_*) < \varepsilon/2$ for any $r > r_0$. Further, from $x_r \to x_*$ it follows also that $\lim_{r\to\infty} \delta_r = 0$. In fact, by definition of the numbers δ_r we get $\delta_r \le \rho_\infty(x_r, x_{r-1})$, $r \ge 1$. But the right-hand sides of these inequalities converge to zero, and, consequently, we get desired: $\delta_r \to 0$. Therefore, there exists r_1 such that $\delta_r < \varepsilon/2$ for any $r > r_1$. Put $s_0 = e_{\bar{r}} = \bar{r}(\bar{r}+1)/2$, where $\bar{r} = \max\ \{r_0, r_1\} + 1$. To estimate the distance $\rho_\infty(y_s, x_*)$ we consider two different cases: 1) $s = e_r$ for some $r \ge \bar{r}$, and 2) $s = e_r + m$ for some $r \ge \bar{r}$ and $m \in [1, r]$. In the first case, due to the inequality $s = e_{r(s)} > e_{\bar{r}}$ we get $r(s) > r_0$. Hence, by equality $y_{e_{r(s)}} = x_{r(s)}$ following from the definition of y_s we get

$$\|y_s - x_*\|_\infty = \|x_{r(s)} - x_*\|_\infty < \varepsilon/2\ .$$

As to the second case, when $s = e_{r(s)} + m > s_0$ with m belonging to the interval $[1, r(s)]$, according to the formula (4) we have $y_s = x_{r(s)+1}^{m(s)}$. Since $x_{r(s)+1}^{m(s)}$ belongs to the vicinity $U_{r(s)+1}$ of the imputation $x_{r(s)+1}$ we get (due to the definition of $U_{r(s)+1}$ and obvious inequality $r(s) + 1 > r_1$)

$$\|y_s - x_*\|_\infty \le \|x_{r(s)+1}^{m(s)} - x_{r(s)+1}\|_\infty + \|x_{r(s)+1} - x_*\|_\infty < \delta_{r(s)+1} + \varepsilon/2 < \varepsilon.$$

Thus, in both cases we get estimations desired, which complete the proof of convergency $y_s \to x_* \in C(\alpha_v)$.

To finalize the proof of Theorem 2 we have to show that for any $s > 1$ imputation y_s dominates together with the preceding element y_{s-1} some more distant previous imputations y_{s-m}, $m \in [2, k_s]$ with $k_s \to \infty$. Put $k_s = r$ for any $s \in [e_r, e_{r+1})$, $r \ge 1$. It is clear that k_s chosen tends to infinity. To prove that for any $s \in [e_r, e_{r+1})$ imputation y_s dominates at least r nearest preceding imputations of the sequence $\{y_s\}_{s=0}^\infty$ we consider two possible cases: 1)$s = e_r$, and 2) $s = e_r + m$ with $m \in [1, r]$. In the first case according to the formulae (3), (4), and α_v-monotonicity of the sequence $\{x_r\}_{r=0}^\infty$ we get: $y_{e_{r-1}} = x_{r-1}\ \alpha_v\ x_r = y_{e_r} = y_s$ and $y_{e_{r-1}+p} = x_r^p\ \alpha_v\ x_r = y_{e_r} = y_s$ for all $p \in [1, r-1]$. In the second case without loss of generality we consider situation $m = 1$ only. Taking account our choice of vicinities U_r (see formula (2)), we obtain (below, domination is realized via coalition S_{r+1}): $y_{s-1} = x_r\ \alpha_v\ x_{r+1}^1 = y_s$ and, besides, $y_{c_{r-1}+p} = x_r^p\ \alpha_v\ x_{r+1}^1 = y_s$ for all $p \in [1, r-1]$.

Thus, the sequence $\{y_s\}_{s=0}^\infty$ meets all the requirements of Definition 3, and this fact completes the proof of Theorem 2. □

3 NTU Case: A Counterexample

To demonstrate that transferable utility assumption (namely, assumption that games under consideration are TU games) is essential even if we deal with the weakest form of accessibility, we give an example of NTU cooperative game with “black hole” being a nonempty closed subset of dominated imputations, which contains all the monotonic trajectories originating from its points[2].

[2] A bit more complicated counterexample can be found, also, in [6].

Recall (see, e.g. [4]), that cooperative game with nontransferable utility (an NTU cooperative game, for short) is a pair (N, G), where $N = \{1, \ldots, n\}$ is a set of players, and G is a function that associates with each nonempty $S \subseteq N$ a nonempty subset $G(S)$ of $\mathbb{R}^S$ such that $G(S)$ is (i) comprehensive, (ii) closed, and (iii) $G(S) \bigcap (x^S + \mathbb{R}^S_+)$ is bounded for every $x^S \in \mathbb{R}^S$. Here comprehensiveness of $G(S)$ means that for any $x \in G(S)$ and $y \leq x$ it holds $y \in G(S)$. Besides, as usually, we put $G(\emptyset) = \emptyset$.

We say that $x \in G(N)$ is an efficient payoff of G if there are no $y \in G(N)$ such that $y_i > x_i$, $i \in N$. To introduce an analog of the set $I(v)$ we deal with in TU case, we propose the set $I(G)$ defined below. Put $g_i = \max \; \{x_i \in \mathbb{R}^{\{i\}} \big| \; x_i \in G(\{i\})\}$, $i \in N$. An element $x = (x_1, \ldots, x_n) \in G(N)$ is said to be individually rational payoff of a game G if $x_i \geq g_i$, $i \in N$. We are now in position to define the following analog of imputation set in case of of NTU game G

$$I(G) = \{x \in G(N) \big| \; x \text{ is efficient and individually rational}\}.$$

Elements of $I(G)$ are said to be imputations of NTU game G, and $I(G)$ is said to be imputation set of G. For the next step we introduce domination relation α_G, an analog of binary relation α_v

$$x \; \alpha_G \; y \Leftrightarrow \exists S \in 2^N \setminus \{\emptyset\}[\forall i \in S(x_i < y_i) \& (y_S \in G(S))], \quad x, y \in I(G) \;,$$

where $y_S \in \mathbb{R}^S$ is ordinary restriction of $y = (y_i)_{i \in N}$ to $S : (y_S)_i = y_i$, $i \in S$.

We introduce α_G-core $C(\alpha_G)$ (the core $C(\alpha_G)$, for short) of NTU game G as the set of un-dominated (maximal with respect to the binary relation α_G) elements of $I(G)$

$$C(\alpha_G) = \{x \in I(G) \big| \; \nexists y \in I(G)[\; x \; \alpha_G \; y \;]\}$$

Finally, we present an analog of accessibility of the core in NTU case.

Definition 5. *The core $C(\alpha_G)$ of NTU cooperative game G is called accessible if $C(\alpha_G) \neq \emptyset$, and for any $x \in I(G) \setminus C(\alpha_G)$ there exists a convergent sequence of imputations $\{x_r\}_{r=0}^{\infty}$ of this game such that* $\lim_{r \to \infty} x_r$ *belongs to the core $C(\alpha_G)$, $x_0 = x$, and $x_r \; \alpha_G \; x_{r+1}$ for any $r \geq 0$.*

Passing on to the example of NTU cooperative game with the above-mentioned "black hole" we consider a three-person NTU cooperative game G with imputation set $I(G)$ containing some closed subset B that doesn't intersect the core $C(\alpha_G)$ and meets the saturation condition $(x \in B) \& (x \; \alpha_G \; y) \Rightarrow y \in B$. So, let $N = \{1, 2, 3\}$ and function G is defined by the formulae

$$\begin{aligned}
G(\{1,2,3\}) &= \{x \in \mathbb{R}^{\{1,2,3\}} \big| \; \sum_{i=1}^{3} x_i \leq 7\}, \\
G(\{1,2\}) &= \{x \in \mathbb{R}^{\{1,2\}} \big| \; x_1 + x_2 \leq 5\}, \\
G(\{1,3\}) &= \{x \in \mathbb{R}^{\{1,3\}} \big| \; x_1 + x_3 \leq 5, \; x_1 \leq 3\}, \\
G(\{2,3\}) &= \{x \in \mathbb{R}^{\{2,3\}} \big| \; x_2 + x_3 \leq 6, \; x_2 \leq 4\}, \\
G(\{i\}) &= \{x_i \in \mathbb{R}^{\{i\}} \big| \; x_i \leq 0\}, \quad i = 1, 2, 3 \;.
\end{aligned}$$

First of all we show that the core $C(\alpha_G)$ of this game is a singleton. In fact, we prove the formula

$$C(\alpha_G) = \{u\} ,$$

where $u = (3, 4, 0)$. It is clear that imputation u is un-dominated. To prove that any imputation $x \neq u$ is dominated we note first that imputation set $I(G)$ of the game G under consideration is given by the formula

$$I(G) = \{x \in \mathbb{R}_+^{\{1,2,3\}} \mid \sum_{i \in N} x_i = 7\}.$$

Further, we consider separately two possible cases: 1) $x_3 = 0$, and 2) $x_3 > 0$. For the first case let us check two subcases 1a) $x_1 < 3$, and 1b) $x_1 > 3$ (note, that $x_1 = 3$ implies $x = u$). It is clear that in case 1a) imputation x is dominated by $y = (3, 2, 2)$ via the coalition $S = \{1, 3\}$. As to the case 1b) we have $x_2 < 4$. Hence, imputation $y = (1, 4, 2)$ dominates x via coalition $S = \{2, 3\}$. Further, in case 2) consider 3 subcases: 2a) $x_3 \in (0, 2],\ x_2 < 4$; 2b) $x_3 \in (0, 2],\ x_2 \geq 4$; and 2c) $x_3 > 2$. In the first subcase imputation x is, obviously, dominated by $y = (1, 4-\delta, 2+\delta)$ with $\delta > 0$ and $4-\delta > x_2$ via coalition $S = \{2, 3\}$. In the second subcase, due to the inequality $x_3 > 0$, we have $x_1 < 3$ and, since $x_1 + x_3 < 5$ we get: imputation $(x_1 + \delta, x_2 - 2\delta, x_3 + \delta)$ dominates x via coalition $S = \{1, 3\}$ provided that $\delta > 0$ and $\delta \leq \min\ \{2, 3 - x_1\}$. Finally, in the third subcase we obtain: x is dominated by $(x_1 + \delta, x_2 + \delta, 2)$ via $S = \{1, 2\}$, where $\delta = [5 - (x_1 + x_2)]/2$. Thus, summarizing all the cases considered we confirm the equality $C(\alpha_G) = \{(3, 4, 0)\}$.

To define so-called "black hole" put

$$B = \{x \in I(G) \big| \ x_1 \geq 1, x_2 \geq 2, x_3 \geq 2\}.$$

It is clear that $B \cap C(\alpha_G) = \emptyset$. As B is closed, the only thing we need to prove non-accessibility of the core $C(\alpha_G)$ is the following implication

$$(x\ \alpha_G\ y) \& (x \in B) \Rightarrow y \in B.$$

Fix an arbitrary $x \in B$ and some $y \in I(G)$ such that $x\ \alpha_G\ y$. To prove $y \in B$ let us look through three possible situations, which correspond to the two-person dominating coalitions $S = \{i, j\} : (y_i, y_j) \in G(\{i, j\}), x_i < y_i, x_j < y_j$.

1. $S = \{1, 2\}$. By definition of domination x by y via coalition $S = \{1, 2\}$ it holds: $y_1 > x_1 \geq 1,\ y_2 > x_2 \geq 2$. Further, due to relations $y_1 + y_2 \leq 5$ and $y_1 + y_2 + y_3 = 7$ we get $y_3 \geq 2$, which completes the proof of inclusion $y \in B$.

2. $S = \{1, 3\}$. Since directly by definition of domination via coalition $S = \{1, 3\}$ we get $y_1 > x_1 \geq 1,\ y_3 > x_3 \geq 2$, the only thing we have to prove is inequality $y_2 \geq 2$. But by definition of the set $G(\{1, 3\})$ we get $y_1 + y_3 \leq 5$, and, consequently, $y_2 = 7 - (y_1 + y_3) \geq 2$.

3. $S = \{2, 3\}$. By definition of domination via $S = \{2, 3\}$ we have that $y_2 > x_2 \geq 2,\ y_3 > x_3 \geq 2$. Further, since $y_2 + y_3 \leq 6$, we get $y_1 = 7 - (y_2 + y_3) \geq 1$, which completes the proof of inclusion $y \in B$.

Thus, in all possible situations $x\ \alpha_G\ y$ and $x \in B$ imply $y \in B$. Taking account that B is closed and $B \cap C(\alpha_G) = \emptyset$, we complete the proof of non-accessibility of the core $C(\alpha_G)$ of 3-player NTU cooperative game under consideration.

Acknowledgements The research was supported in part by Russian Fund for Basic Research (grant 16 - 06 - 00101). The author thanks Prof. P.Sudhölter for helpful comments and remarks. Also many thanks to anonymous referees for useful recommendations and suggestions.

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Appendix: Outline of the Proof of Accessibility Theorem

For the sake of completeness, we propose a brief outline of the straightforward proof of Accessibility Theorem, mentioned in the Introduction (for more details, see [6]). Fix an arbitrary n-person TU cooperative game v with nonempty core $C(\alpha_v)$. One of the basic ideas of the proof of core-accessibility is to consider some suitable subsystems of the set-valued dynamic system[3] φ^v generated by v

$$\varphi^v(x) = \alpha_v(x) \cup \{x\}, \quad x \in I(v)\ ,$$

with $\alpha_v(x)$ to be a collection of imputations dominating x

$$\alpha_v(x) = \{y \in I(v) \big|\ x\ \alpha_v\ y\}, \quad x \in I(v).$$

[3] Here and below we apply terms from [1],[4].

To construct the subsystems of φ^v required we exploit heavily lower semi-continuity of the correspondence φ^v, which follows directly from the definition of domination α_v, and apply so-called generalized Lyapunov functions [5]. To present their definition in a more general context useful in the further presentation, consider an arbitrary complete metric space X with metric d, and remind [1] that by *dynamic systems* (d.s.) on X we mean correspondences $\varphi : X \to 2^X$ for which $\varphi(x) \neq \emptyset$ for all $x \in X$. By trajectories of φ we mean sequences $\{x_r\}_{r=0}^{\infty}$ such that $x_{r+1} \in \varphi(x_r)$ for all $r = 0, 1, \ldots$.

For correspondence $\psi : X \to 2^X$ let gr $\psi = \{(\mathrm{x}, \mathrm{y}) \in \mathrm{X} \times \mathrm{X} \big| \ \mathrm{y} \in \psi(\mathrm{x})\}$, $\psi_*(x) = \bigcup_{m=1}^{\infty} \psi^m(x)$ with $\psi^1(x) = \psi(x)$ and $\psi^{m+1}(x) = \bigcup_{y \in \psi^m(x)} \psi(y)$.

Definition 6. *We say that a d.s. φ admits a generalized Lyapunov function if there exists a bounded function $l : \mathrm{gr}\ \varphi_* \to \mathbb{R}$ such that*

(L1) $l(x, y) \geq d(x, y)$ *for all* $x \in X,\ y \in \varphi(x)$;

(L2) $l(x, z) \geq l(x, y) + l(y, z)$ *for all* $x \in X,\ y \in \varphi_*(x),\ z \in \varphi_*(y)$.

It is worth to note that the existence of a generalized Lyapunov function doesn't yet ensure the convergence of any trajectory of a d.s. φ to some element of its set of endpoints

$$E_\varphi = \{x \in X \big|\ \varphi(x) = \{x\}\} .$$

At the same time, there exists, for lower semi-continuous d.s. φ, a standard way of formation of subsystems ψ (gr $\psi \subseteq$ gr φ) possessing the property mentioned and satisfying equality $E_\psi = E_\varphi$

Proposition 1. *Let φ be a lower semi-continuous d.s. on X admitting a generalized Lyapunov function. Let*

$$\rho_\varphi(x) = \sup\{d(x, y) \mid y \in \varphi(x)\},\ x \in X .$$

Then for any $\delta \in (0, 1)$ all trajectories of the dynamic system

$$\varphi_\delta(x) = \begin{cases} \{x\} & , x \in E_\varphi , \\ \{y \in \varphi(x) \big|\ d(x, y) > \delta \rho_\varphi(x)\} & , x \in X \setminus E_\varphi , \end{cases}$$

converge to elements of E_φ.

So, for the proof of accessibility of the core $C(\alpha_v)$ on the basis of Proposition 1 it is enough to construct a lower semi-continuous subsystem ψ of d.s.φ^v that admits a generalized Lyapunov function and satisfies the relation $E_\psi = C(\alpha_v)$. Considering as the desired subsystem of d.s. φ^v a dynamic system ψ_l of the form

$$\psi_l(x) = \{y \in \alpha_v(x) \big|\ l(x, y) > d(x, y)\} \cup \{x\},\ x \in I(v) ,$$

with l to be some real-valued function on $I(v) \times I(v)$, one can obtain the following sufficient condition of accessibility (see, e.g., [6]).

Theorem 3. *Suppose there exists a bounded lower semi-continuous with respect to the first argument function $l : I(v) \times I(v) \to \mathbb{R}_+$ such that*

$$l(x, z) \geq l(x, y) + l(y, z), \quad x \in I(v),\ y \in \varphi_*^v(x),\ z \in \varphi_*^v(y)\ .$$

If, in addition, function l meets the requirement

$$\forall\ x \in I(v) \setminus C(\alpha_v)\ \exists\ y \in \alpha_v(x)\ [\ l(x, y) > d(x, y)]\ ,$$

then the core $C(\alpha_v)$ is accessible.

Below, we apply the following useful corollary of Theorem 3.

Corollary 1. *If there exists a continuous function $u : I(v) \to \mathbb{R}_+$ such that*

$$\forall\ x \in I(v) \setminus C(\alpha_v)\ \exists\ y \in \alpha_v(x)\ [\ u(x) - u(y) > d(x, y)]\ ,$$

then the core $C(\alpha_v)$ of TU cooperative game v is accessible.

In order to apply Corollary 1 we put $u = u_v$, where

$$u_v(x) = 4nd(x, C(\alpha_v)), \quad x \in I(v)\ ,$$

with $d(x, C(\alpha_v))$ to be the distance from x to the core $C(\alpha_v)$ in Euclidean norm $||x||_2 = \left(\sum\limits_N x_i^2\right)^{1/2}$. To complete an outline we will only sketch the proof of the main lemma, which plays a crucial role in justification of Accessibility Theorem.

Lemma 1. *Function u_v satisfies condition*

$$\forall x \in I(v) \setminus C(\alpha_v) \exists y \in \alpha_v(x) [u_v(x) - u_v(y) > ||x - y||_2]\ . \tag{5}$$

Sketch of the proof of Lemma 1. First of all we may assume w.l.g. that $v(\{i\}) = 0$ for all $i \in N$, and $0 \leq v(S) \leq v(N)$ for all $S \subseteq N$. It follows directly from the definition of $C(\alpha_v)$ that condition $v(S) \leq v(N)$, $S \subseteq N$, implies that the core of 0-normalized TU cooperative game v is given by the formula

$$C(\alpha_v) = \{x \in \mathbb{R}_+^N \big|\ x(N) = v(N),\ x(S) \geq v(S), \quad S \subseteq N\}\ .$$

In order to prove (5) let us fix an arbitrary imputation $x \in I(v) \setminus C(\alpha_v)$. We will show that there exists $z \in I(v)$ satisfying inequality

$$u_v(x) - u_v(z) > ||x - z||_2 \tag{6}$$

and the following weak domination requirement

$$x\ \tilde{\alpha}_v\ z\ , \tag{7}$$

where

$$x\ \tilde{\alpha}_v\ z \Leftrightarrow \exists S\ [(x(S) < z(S) \leq v(S))\&(x_i \leq z_i,\ i \in S)]\ .$$

If $x\ \tilde{\alpha}_v\ z$, then every neighborhood of z contains an element $y \in I(v)$, which dominates x w.r.t. α_v, and the inequality (6) , by continuity of u_v, holds in some neighborhood of z. Therefore the existence of z, satisfying (6) and (7) , proves the lemma .

In order to construct z, mentioned above, consider the imputation $u \in C(\alpha_v)$, closest to x w.r.t. the norm $||\cdot||_2$. It is not very hard to verify that there exists a coalition $S \subseteq N$ such that $x(S) < u(S) = v(S)$. Put

$$\begin{aligned} S_1 &= \{i \in S \big|\ x_i < u_i\} \ , \\ S_2 &= \{i \in N \setminus S \big|\ x_i > u_i\} \ , \\ T_1 &= \{i \in S \big|\ x_i \geq u_i\} \ , \\ T_2 &= \{i \in N \setminus S \big|\ x_i \leq u_i\} \ . \end{aligned}$$

If $T_1^+ = \{i \in T_1 |\ x_i > u_i\} = \emptyset$, then u may be used as the sought for imputation. If $T_1^+ \neq \emptyset$, then setting

$$e(x,S) = v(S) - x(S) \ ,$$
$$q_j = e(x,S)/\big|u(S_j) - x(S_j)\big|,\ j = 1,2 \ ,$$

we define $z \in \mathbb{R}^N$ by the formula

$$z_i = \begin{cases} x_i + q_1(u_i - x_i) \ , i \in S_1 \ , \\ x_i + q_2(u_i - x_i) \ , i \in S_2 \ , \\ x_i \qquad\qquad\qquad\ , i \in T_1 \cup T_2 \ . \end{cases}$$

Since $q_1, q_2 \in (0,1]$, vector z clearly belongs to $I(v)$. Taking account that $S_1 \neq \emptyset$ and

$$z(S) = x(S) + e(x,S) = v(S),\ x_i = z_i,\ i \in T_1 \ ,$$

we have

$$x(S) < z(S) \leq v(S),\ x_i \leq z_i,\ i \in S \ .$$

By definition of the weak domination $\tilde{\alpha}_v$, these inequalities imply that imputation x is $\tilde{\alpha}_v$-dominated by z via coalition $S = S_1 \cup T_1$.

In order to check the inequality (6), note that

$$||x - z||_2 \leq \sqrt{2}\ e(x,S) \ , \tag{8}$$

and the lower bound of the difference $||x - u||_2 - ||z - u||_2$ has the following form

$$||x - u||_2 - ||z - u||_2 \geq (Q_1 + Q_2)/2||x - u||_2 \ , \tag{9}$$

where

$$Q_j = (2q_j - q_j^2) \cdot \sum_{i \in S_j} (x_i - u_i)^2,\ j = 1,2 \ .$$

Estimating denominator in the right hand side of (9) we have for the first step

$$\sum_{i\in T_j}(x_i-u_i)^2 \le (u(S_j)-x(S_j))^2,\ j=1,2\ . \tag{10}$$

By Cauchy-Schwartz inequality , from (10) we obtain

$$\sum_{i\in T_j}(x_i-u_i)^2 \le |S_j|\cdot\sum_{i\in S_j}(x_i-u_i)^2,\ j=1,2\ .$$

At last , by definition of Q_j, we have

$$||x-u||_2 \le \sqrt{n}\left(\sum_{j=1}^{2} Q_j/(2q_j-q_j^2)\right)^{1/2}\ . \tag{11}$$

So, applying definitions of q_1, q_2, and combining inequalities (8),(9) and (11), by lengthy, but elementary calculations we obtain the sought for relationship

$$u_v(x)-u_v(z) \ge 2e(x,S) > ||x-z||_2\ ,$$

which completes the sketch of the proof of Lemma 1. □

To conclude the outline of the proof of Theorem 1 we have to exploit some properties of the dynamic system

$$\varphi_\delta^v(x) = \begin{cases} \{x\} & ,\ x\in C(\alpha_v)\ ,\\ \{y\in\varphi^v(x)\big|\ u_v(x)-u_v(y) > ||x-y||_2 > \delta\rho^v(x)\} & ,\ x\notin C(\alpha_v)\ ,\end{cases}$$

where δ is an arbitrary number from the interval $(0,1)$ and

$$\rho^v(x) = \sup\{||x-y||_2\big|\ y\in\varphi^v(x),\ u_v(x)-u_v(y) > ||x-y||_2\},\ x\notin C(\alpha_v)\ .$$

Since by Lemma 1 the dynamic system φ_δ^v admits Lyapunov function, and φ^v is lower semi-continuous dynamic system, it is not very hard to verify (like in [6], for instance) that all the trajectories of the system φ_δ^v converge to the imputations from $C(\alpha_v)$. □