Optimal Control Algorithm for Complex Heat Transfer Model

Alexander Chebotarev, Gleb Grenkin, and Andrey Kovtanyuk *

[1] Far Eastern Federal University,
Sukhanova st. 8, 690950 Vladivostok, Russia,
[2] Institute for Applied Mathematics,
Radio st. 7, 690041 Vladivostok, Russia
{cheb@iam.dvo.ru,glebgrenkin@gmail.com,kovtanyuk.ae@dvfu.ru}

Abstract. An optimal control problem for a nonlinear steady-state heat transfer model accounting for heat radiation effects is considered. The problem consists in minimization of a given cost functional by controlling the sources in the heat equation. The solvability of this control problem is proved, optimality conditions are derived, and an iterative algorithm for solving the optimal control problem is constructed.

Keywords: optimal control; radiative heat transfer; conductive heat transfer; gradient descent method

1 Introduction

The interest in studying problems of complex heat transfer (where the radiative, convective, and conductive contributions are simultaneously taken into account) is motivated by their importance for many engineering applications. Here, the following examples can be mentioned: modeling and predicting the heat transfer in molten glass [1–3], nanofluids [4, 5], etc.

A considerable number of works devoted to optimal control problems of complex heat transfer models consider the evolutionary systems (see, e.g., [1–3, 6–10]). In the mentioned works, the radiation transfer is described by steady-state radiative transfer equation. The temperature field is simulated by the conventional evolutionary heat transfer equation with additional source terms describing the contribution of the radiative heat transfer.

Theoretical analysis of optimal control problems for steady-state systems of complex heat transfer with source terms in the heat equation is an open question. It is worth to mention the work [11], where the problem of optimal boundary multiplicative control for a steady-state complex heat transfer model was considered. The problem was formulated as the maximization of the energy outflow from the model domain by

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controlling reflection properties of the boundary. On the basis of new a priori estimates of solutions of the control system, the solvability of the optimal control problem was proved. The main result there was the proof of an analogue of the bang-bang principle arising in control theory for ordinary differential equations.

In this paper, an optimal control problem of obtaining a desired temperature and(or) radiative intensity distributions in a part of the model domain by controlling the sources in the heat equation is considered. Analogous problems appear in many engineering applications and draw attention of many researchers. For example, similar optimal control problems for non-stationary complex heat transfer models were studied in [1–3, 8] in context of glass manufacturing. In the current work, the optimal control problem for a steady-state model is studied. The solvability of this problem is proved, and an optimality system is derived. Moreover, an iterative algorithm based on the gradient descent method is constructed, and results of numerical experiments are presented.

2 Problem formulation

The following steady-state normalized diffusion (P_1) model (see [12–15]) describing radiative and conductive heat transfer in a bounded domain $\Omega \subset \mathbb{R}^3$ is under consideration:

$$-a\Delta\theta + b\kappa_a(|\theta|\theta^3 - \varphi) = u, \quad -\alpha\Delta\varphi + \kappa_a(\varphi - |\theta|\theta^3) = 0, \tag{1}$$

$$a\partial_n\theta + \beta(\theta - \theta_b)|_\Gamma = 0, \quad \alpha\partial_n\varphi + \gamma(\varphi - |\theta_b|\theta_b^3)|_\Gamma = 0. \tag{2}$$

Here, θ is the normalized temperature, φ the normalized radiation intensity averaged over all directions, and κ_a the absorption coefficient. The physical sense of the parameters a, b, α, β, γ can be found in [13–15]. The control function u describes the internal sources of heat. The symbol ∂_n denotes the derivative in the outward normal direction $\mathbf{n}$ on the boundary $\Gamma := \partial\Omega$.

The problem of optimal control consists in the determination of functions u, θ, and φ which satisfy the conditions (1), (2) and minimize a cost functional $J_\mu(\theta, \varphi, u)$, i.e.

$$J_\mu(\theta, \varphi, u) = J(\theta, \varphi) + \frac{\mu}{2}\|u\|^2_{L^2(\Omega)} \to \inf, \; u \in U_{ad}. \tag{3}$$

Here, $U_{ad} \subset L^2(\Omega)$ is the set of admissible controls, $\mu \geq 0$ is a given cost parameter. In particular, the functional J can describe the L^2-deviation of the temperature and radiation fields from prescribed distributions, say θ_d and φ_d. Thus, e.g.

$$J(\theta, \varphi) = a_\theta \, \|\theta - \theta_d\|^2_{L^2(\Omega)} + a_\varphi \, \|\varphi - \varphi_d\|^2_{L^2(\Omega)},$$

where a_θ and a_φ are nonnegative weights.

3 Formalization of the optimal control problem

Suppose that the model data satisfy the following conditions:
(i) $\beta, \gamma \in L^\infty(\Gamma)$, $\beta \geq \beta_0 > 0$, $\gamma \geq \gamma_0 > 0$, $\beta_0, \gamma_0 = \text{const}$, $\theta_b \in L^\infty(\Gamma)$;

(ii) U_{ad} is a closed convex set; U_{ad} is a bounded set, if $\mu = 0$;
(iii) The cost functional $J : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is weakly lower semicontinuous and bounded from below.
Here and further, the Sobolev space $W_2^s(\Omega)$ is denoted by $H^s(\Omega)$, $s \geq 0$, and (f, g) and $\|f\|$ denote respectively the inner product and the norm of the space $L^2(\Omega)$.

Denote $H = L^2(\Omega)$, $V = H^1(\Omega)$, $Y = V \times V$. Identifying H with the dual space H' yields the Gelfand triple $V \subset H = H' \subset V'$. Let the value of a functional $f \in V'$ on an element $v \in V$ be denoted by (f, v). Notice that (f, v) is the inner product in H if f and v are elements of H.

Assuming that θ, φ, v are arbitrary elements of V, define operators and functionals $A_1, A_2 \colon V \to V'$, $f, g \in V'$ by the following relations:

$$(A_1\theta, v) = a(\nabla\theta, \nabla v) + \int_\Gamma \beta\theta v d\Gamma, \quad (A_2\varphi, v) = \alpha(\nabla\varphi, \nabla v) + \int_\Gamma \gamma\varphi v d\Gamma,$$

$$(f, v) = \int_\Gamma \beta\theta_b v d\Gamma, \quad (g, v) = \int_\Gamma \gamma|\theta_b|\theta_b^3 v d\Gamma.$$

A pair $\{\theta, \varphi\} \in V$ is called weak solution of the problem (1), (2) if

$$A_1\theta + b\kappa_a(|\theta|\theta^3 - \varphi) = f + u, \quad A_2\varphi + \kappa_a(\varphi - |\theta|\theta^3) = g. \tag{4}$$

The optimal control problem consists in the minimization of a functional J_μ defined on solutions of system (4) provided that $u \in U_{ad}$. That is,

$$J_\mu(\theta, \varphi, u) \to \inf, \quad \{\theta, \varphi\} \text{ are solutions of (4) yielded by } u \in U_{ad}. \tag{5}$$

A pair $\{\widehat{\theta}, \widehat{\varphi}\}$ minimizing J_μ and corresponding to a function $\widehat{u}$ is called optimal state, and $\widehat{u}$ is called optimal control.

4 Solvability of the optimal control problem

To prove the solvability of the problem (5), establish some properties of the boundary value problem (1), (2).

Lemma 1. *If the conditions (i) hold and $u \in U_{ad}$, then for a weak solution, $\{\theta, \varphi\}$, of the problem (1), (2) the following estimate is true:*

$$\|\theta\|_V^2 + \|\varphi\|_V^2 \leq C. \tag{6}$$

Here, a positive constant C depends only on a, b, α, κ_a, β, γ, $\|u\|$, and Ω.

Proof. Let $h_p(s) := |s|^p \operatorname{sign} s$, $p > 0$, $s \in \mathbb{R}$. Denote $\varphi_1 = h_{1/4}(\varphi)$ and, for $\varepsilon > 0$, define

$$w_\varepsilon = \begin{cases} \varphi_1 - \varepsilon, & \varphi_1 > \varepsilon, \\ 0, & |\varphi_1| \leq \varepsilon, \\ \varphi_1 + \varepsilon, & \varphi_1 < -\varepsilon. \end{cases}$$

Notice that if $\varphi \in V$, then $\varphi_1 \in L^{24}(\Omega)$, $\varphi_1|_\Gamma \in L^{16}(\Gamma)$, $w_\varepsilon \in V$, and

$$\nabla w_\varepsilon = \frac{1}{4}\begin{cases} |\varphi|^{-3/4}\nabla\varphi, & |\varphi_1| > \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

It is important that

$$\int_\Gamma \gamma\varphi w_\varepsilon d\Gamma - \kappa_a(h_4(\theta) - \varphi, w_\varepsilon) - (g, w_\varepsilon) = \\ = \int_\Gamma \gamma(\varphi - h_4(\theta_b))\varphi_1 d\Gamma - \kappa_a(h_4(\theta) - \varphi, \varphi_1) + c_\varepsilon. \quad (7)$$

In this expression $|c_\varepsilon| \le C\varepsilon$, where $C > 0$ does not depend on ε.

Multiply, in the sense of the inner product of H, the first equation of (4) by θ, the second equation by bw_ε, and add the equalities. Then, taking into account monotonicity of $(h_4(\theta) - \varphi)(\theta - h_{1/4}(\varphi)) \ge 0$, we obtain the inequality

$$a\|\nabla\theta\|^2 + \int_\Gamma \beta\theta^2 d\Gamma + \frac{16}{25}\alpha b\int_{|\psi|>\varepsilon^{5/2}} |\nabla\psi|^2 dx + b\int_\Gamma \gamma\psi^2 d\Gamma \\ \le (f + u, \theta) + b\int_\Gamma \gamma h_4(\theta_b)\varphi_1 d\Gamma - bc_\varepsilon. \quad (8)$$

Here, $\psi = h_{5/8}(\varphi)$, $\varphi_1 = h_{2/5}(\psi)$. Passing to the limit in inequality (8) as $\varepsilon \to +0$, we obtain $\psi \in V$ and

$$k_1\|\theta\|_V^2 + k_2\|\psi\|_V^2 \le |(f + u, \theta)| + b\int_\Gamma \gamma|h_4(\theta_b)h_{2/5}(\psi)|d\Gamma. \quad (9)$$

Here, $k_1 = \min\{a, \beta_0\}$, $k_2 = b\min\{\frac{16}{25}\alpha, \gamma_0\}$. The norm in the space V is defined by the following equality:

$$\|v\|_V^2 = \|\nabla v\|^2 + \int_\Gamma v^2 d\Gamma.$$

Taking into account the continuity of the trace operator from V into $L^2(\Gamma)$, we obtain from (9):

$$\|\theta\|_V^2 + \|\psi\|_V^2 \le K_1\left(\|f + u\|_{V'}^2 + \|\theta_b\|_{L^5(\Gamma)}^4\right). \quad (10)$$

Here, K_1 depends only on a, α, b, β_0, γ_0, $\|\gamma\|_{L^\infty(\Gamma)}$, and the domain Ω.

The estimate of $\|\theta\|_V$ allows to obtain the estimate of $\|\varphi\|_V$. Multiplying the second equation of (4) by φ in the sense of the inner product of H, and denoting $k_3 = \min\{\alpha, \gamma_0\}$, we obtain the inequality

$$k_3\|\varphi\|_V^2 + \kappa_a\|\varphi\|^2 \le \kappa_a|(h_4(\theta), \varphi)| + \int_\Gamma \gamma|h_4(\theta_b)\varphi|d\Gamma.$$

Using Hölder and Young inequalities with parameter $\delta > 0$, we estimate:

$$|(h_4(\theta), \varphi)| \le \frac{\delta}{2}\|\varphi\|_{L^3(\Omega)}^2 + \frac{1}{2\delta}\|\theta\|_{L^6(\Omega)}^8,$$

$$\int_{\Gamma} \gamma |h_4(\theta_b)\varphi| d\Gamma \leq \|\gamma\|_{L^\infty(\Gamma)} \left(\frac{\delta}{2} \|\varphi\|^2_{L^4(\Gamma)} + \frac{1}{2\delta} \|\theta_b\|^8_{L^{16/3}(\Gamma)} \right).$$

Taking into account the continuity of the embedding of V into $L^6(\Omega)$, the continuity of the trace operator from V into $L^4(\Gamma)$, and a sufficiently small δ, we obtain the estimate of $\|\varphi\|_V$:

$$\|\varphi\|^2_V \leq K_2 \left(\|\theta_b\|^8_{L^{16/3}(\Gamma)} + \|\theta\|^8_V \right). \tag{11}$$

Here, K_2 depends only on α, γ_0, κ_a $\|\gamma\|_{L^\infty(\Gamma)}$, and the domain Ω. The estimates (10) and (11) prove the lemma. □

On the base of the estimate (6), similarly as in [11], the solvability of the problem (5) is proved.

Theorem 1. *If the conditions* (i)*–*(iii) *hold, then there exists at least one solution of the problem (5).*

5 The necessary conditions of optimality

To derive optimality relations, add to conditions (i)-(iii) the following assumption:
(iv) $J : Y \to \mathbb{R}$ is Fréchet differentiable.

Introduce a constraint operator $F : Y \times H \to Y'$ as follows:

$$F(y,u) = \{A_1\theta + b\kappa_a(|\theta|\theta^3 - \varphi) - f - u,\ A_2\varphi + \kappa_a(\varphi - |\theta|\theta^3) - g\},$$

where $y = \{\theta, \varphi\} \in Y,\ u \in H$.

Lemma 2. *For any* $y \in Y$ *the map* $F'_y : Y \to Y'$ *is epimorphic,* $Im\, F'_y = Y'$.

Proof. Equation $F'_y q = z = \{z_1, z_2\} \in Y'$ is equivalent to the following boundary value problem:

$$A_1 q_1 + b\kappa_a(4|\theta|^3 q_1 - q_2) = z_1,\ \ A_2 q_2 + \kappa_a(q_2 - 4|\theta|^3 q_1) = z_2,\ q = \{q_1, q_2\} \in Y. \tag{12}$$

To prove the solvability of a Fredholm problem (12), it suffices to prove the uniqueness of its solutions. Let $\operatorname{sign} s = s/|s|$, if $s \neq 0$, $\operatorname{sign} 0 = [-1, 1]$. Let us consider the function μ_δ which is regularization of multivalued function sign, $\mu_\delta(s) = s/|s|$, if $|s| \geq \delta$, $\mu_\delta(s) = s/\delta$, if $|s| < \delta$.

Let $z = 0$, $h = 4|\theta|^3$. Multiplying, in the sense of the inner product of H, the first equation of (12) by $\mu_\delta(q_1)$, the second equation by $b\mu_\delta(q_2)$, and adding these equalities, we obtain

$$(A_1 q_1, \mu_\delta(q_1)) + b(A_2 q_2, \mu_\delta(q_2)) + b\kappa_a(h q_1 - q_2, \mu_\delta(q_1) - \mu_\delta(q_2)) = 0.$$

Notice that

$$(A_1 q_1, \mu_\delta(q_1)) = a(\nabla q_1, \mu'_\delta(q_1)\nabla q_1) + \int_\Gamma \beta q_1 \mu_\delta(q_1) d\Gamma \geq \int_\Gamma \beta q_1 \mu_\delta(q_1) d\Gamma.$$

and

$$(A_2q_2, \mu_\delta(q_2)) = \alpha(\nabla q_2, \mu'_\delta(q_2)\nabla q_2) + \int_\Gamma \gamma q_2\mu_\delta(q_2)d\Gamma \ge \int_\Gamma \gamma q_2\mu_\delta(q_2)d\Gamma.$$

Therefore,

$$\int_\Gamma \beta q_1\mu_\delta(q_1)d\Gamma + b\int_\Gamma \gamma q_2\mu_\delta(q_2)d\Gamma + b\kappa_a(hq_1 - q_2, \mu_\delta(q_1) - \mu_\delta(q_2)) \le 0. \tag{13}$$

Passing to the limit as $\delta \to 0$, from inequality (13), we obtain

$$\int_\Gamma \beta|q_1|d\Gamma + b\int_\Gamma \gamma|q_2|d\Gamma + b\kappa_a(hq_1 - q_2, \operatorname{sign} q_1 - \operatorname{sign} q_2) \le 0.$$

From monotonicity of the function sign, it follows the conditions $q_1|_\Gamma = q_2|_\Gamma = 0$.

Further, notice that $A_1q_1 + bA_2q_2 = 0$. Scalarly multiplying this equation by $aq_1 + \alpha bq_2$, and taking into account zero boundary values of q_1, q_2, we obtain $\|\nabla(aq_1 + \alpha bq_2)\|^2 = 0$. Hence, $aq_1 + \alpha bq_2 = 0$. Therefore,

$$a(\nabla q_1, \nabla v) + b\kappa_a((h + a/\alpha b)q_1, v) = 0 \quad \forall v \in V. \tag{14}$$

Assuming $v = q_1$ in (14), we obtain $q_1 = 0$, and therefore $q_2 = 0$. □

Applying the principle of Lagrange for smooth convex extremal problems [16], we can prove the following result.

Theorem 2. *Let $\widehat{y} = \{\widehat{\theta}, \widehat{\varphi}\} \in Y$, $\widehat{u} \in U_{ad}$ be a solution of the control problem (5). Then there exists an adjoint state $p = \{p_1, p_2\} \in Y$ such that the triple $\{\widehat{y}, \widehat{u}, p\}$ satisfies the conditions*

$$A_1p_1 + 4|\widehat{\theta}|^3\kappa_a(bp_1 - p_2) = -J'_\theta(\widehat{y}),\ A_2p_2 + \kappa_a(p_2 - bp_1) = -J'_\varphi(\widehat{y}), \tag{15}$$

$$(\mu\widehat{u} - p_1, v - \widehat{u}) \ge 0, \quad \forall v \in U_{ad}. \tag{16}$$

6 Example of optimality system

Let $G_{1,2} \subset \Omega$ be subdomains of Ω. Consider the following cost functional:

$$J(\theta, \varphi) = \frac{1}{2}\int\limits_{G_1}(\theta - \theta_d)^2dx + \frac{1}{2}\int\limits_{G_2}(\varphi - \varphi_d)^2dx, \tag{17}$$

where $\theta_d \in L^2(G_1)$ and $\varphi_d \in L^2(G_2)$ are given functions. It is easy to see that

$$(J'_\theta(\theta, \varphi), \eta) = \int\limits_{G_1}(\theta - \theta_d)\eta dx, \quad (J'_\varphi(\theta, \varphi), \eta) = \int\limits_{G_2}(\varphi - \varphi_d)\eta dx, \quad \eta \in V.$$

In this case, if $U_{ad} = L^2(\Omega)$ and $\mu > 0$, the optimality system assumes the form

$$
\begin{aligned}
&- a\Delta\widehat{\theta} + b\kappa_a(|\widehat{\theta}|\widehat{\theta}^3 - \widehat{\varphi}) = \widehat{u}, \quad -\alpha\Delta\widehat{\varphi} + \kappa_a(\widehat{\varphi} - |\widehat{\theta}|\widehat{\theta}^3) = 0,\\
&\qquad a\partial_n\widehat{\theta} + \beta(\widehat{\theta} - \theta_b)|_\Gamma = 0, \quad \alpha\partial_n\widehat{\varphi} + \gamma(\widehat{\varphi} - |\theta_b|\theta_b^3)|_\Gamma = 0,
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&- a\Delta p_1 + 4\kappa_a|\widehat{\theta}|^3(bp_1 - p_2) = -\chi_{G_1}(\widehat{\theta} - \theta_d),\\
&\qquad - \alpha\Delta p_2 + \kappa_a(p_2 - bp_1) = -\chi_{G_2}(\widehat{\varphi} - \varphi_d),\\
&\qquad\qquad a\partial_n p_1 + \beta p_1|_\Gamma = 0, \quad \alpha\partial_n p_2 + \gamma p_2|_\Gamma = 0,
\end{aligned} \tag{19}
$$

and $\widehat{u} = p_1/\mu$.

Here, $\chi_{G_{1,2}}$ are the characteristic functions of the subdomains $G_{1,2}$, respectively.

7 Iterative algorithm

For the numerical solution of the optimality system (18), (19), we can apply the method of gradient descent:

$$
u_{k+1} = u_k - \lambda_k \left(\mu u_k - p_1^{(k)}\right), \quad k = 0, 1, 2, \ldots, \quad \text{where } \; u_0 \in H \text{ is given.}
$$

Here $\lambda_k > 0$ is a step size, $p^{(k)} = \{p_1^{(k)}, p_2^{(k)}\}$ is a pair satisfying the system (18), (19), where $\widehat{u} := u_k$.

If $U_{ad} \neq H$ we can apply the gradient projection method (see, e.g, [17]):

$$
u_{k+1} = P_{U_{ad}}\left(u_k - \lambda_k \left(\mu u_k - p_1^{(k)}\right)\right), \quad k = 0, 1, 2, \ldots,
$$

where $P_{U_{ad}} : H \to U_{ad}$ is the projection operator.

The method of choosing the step size λ_k is adjusted as required for decreasing the cost functional. Unlike methods based on the Armijo rule, this method does not need an inner loop for adjustment of λ_k that requires computing the value of J and, therefore, solving the problem (18).

Define $\widehat{J}(u) = J_\mu(\theta(u), \varphi(u), u)$, where $\{\theta(u), \varphi(u)\}$ is a solution of system (4). The method is as follows. If $\widehat{J}(u_{k+1}) \geq \widehat{J}(u_k)$, then return back to the control u_k and reduce λ_k by a factor of 2. Additionally, if $\widehat{J}(u_{k+1}) < \widehat{J}(u_k)$, $k = s, s+1, \ldots, s+m_0-1$, then λ_k is increased by a factor of 2. Here, $m_0 \geq 1$ is a prescribed integer parameter of a quantity of decreases of the cost functional, which is enough for increasing the step size λ_k.

The pseudocode of the algorithm is presented below.

Algorithm 1: Gradient descent method with a variable step size

```
Choose the parameters λ_0 and m_0.
Choose the initial guess u_0.
cost_func_decreases ← 0;
for k ← 0, 1, 2, ... do
    For the given u_k, find {θ_k, φ_k} from (18).
    Compute Ĵ(u_k).
    if k ≥ 1 and Ĵ(u_k) ≥ Ĵ(u_{k−1}) then
        u_k ← u_{k−1};
        λ_k ← λ_{k−1}/2;
        p^(k) ← p^(k−1);
        cost_func_decreases ← 0;
    else
        if k ≥ 1 then
            cost_func_decreases ← cost_func_decreases + 1;
        Find p^(k) from (19).
        if cost_func_decreases = m_0 then
            λ_k ← 2λ_{k−1};
            cost_func_decreases ← 0;
        else
            if k ≥ 1 then
                λ_k ← λ_{k−1};
    u_{k+1} ← P_{U_ad}(u_k − λ_k(μu_k − p_1^(k)));
```

8 Numerical example

Consider an example for the two-dimensional domain $\Omega = \{(x, y) \colon 0 \leq x, y \leq L\}$ which can be interpreted as a long rectangular channel in the three-dimensional space. The parameters values are taken as follow: $L = 10$ [cm], $\alpha = 3.3\ldots$ [cm], $\kappa_a = 0.01$ [cm^{-1}], $\beta = 1.5$ [cm/s], and $\gamma = \varepsilon/2(2 - \varepsilon)$, where $\varepsilon = 0.7$ is the emissivity coefficient of the boundary. The thermodynamical characteristics of the medium correspond to air at the normal atmospheric pressure and the temperature of 400 °C. The maximum temperature is chosen as $T_{\max} = 773$ K. This yields $a = 0.92$ [cm^2/s] and $b = 18.7$ [cm/s]. Notice that the absolute temperature is $T = T_{\max}\theta$. The boundary temperature $\theta_b = 0.5$.

The cost functional is defined by (3) and (17), where $G_1 = \Omega \setminus S$, $G_2 = \emptyset$, $\theta_d = 0.7$, and $\mu = 0.01$. Let the set of admissible controls be $U_{ad} = \{u \in L^2(\Omega) \colon u_1 \leq u \leq u_2\}$, where $u_1 = 0$, $u_2 = 1$ in $S = [x_1, x_2] \times [y_1, y_2]$, and $u_1 = u_2 = 0$ in $\Omega \setminus S$. Assume $x_1 = 0.65L$, $x_2 = 0.85L$, $y_1 = L/12$, $y_2 = 5L/12$.

For the numerical solution we use the software FreeFem++ [18]. The boundary-value problem (18) is solved by Newton's method. The initial guess for the optimal control is chosen as $u_0 = 0$, and parameters of the optimization algorithm are $\lambda_0 = 5$, $m_0 = 3$.

The computed optimal control is presented in Fig. 1. The graph of the optimal temperature is depicted in Fig. 2. The values of $\widehat{J}(u_k)$ and λ_k for different k are indicated in Figs. 3, 4. As it is seen in Fig. 4, the most frequent value of λ_k is 10, and the step size is adjusted as needed.

The optimal controls for $\mu = 0.1$ and $\mu = 0.001$ are presented in Figs. 5, 6 for comparison. It can be easily proved from (16) that in the case of $\mu = 0$ the optimal control satisfies an analog of the bang-bang principle, that is $\widehat{u}(x) = u_1(x)$ or $u_2(x)$ for a.e. $x \in \Omega$ where $p_1(x) \neq 0$. Notice that the optimal control comes near to a bang-bang control as $\mu \to 0$. The optimal control for $\mu = 0$ is depicted in Fig. 7. This bang-bang control was computed by an optimization algorithm of the gradient descent type, see [9, 10].

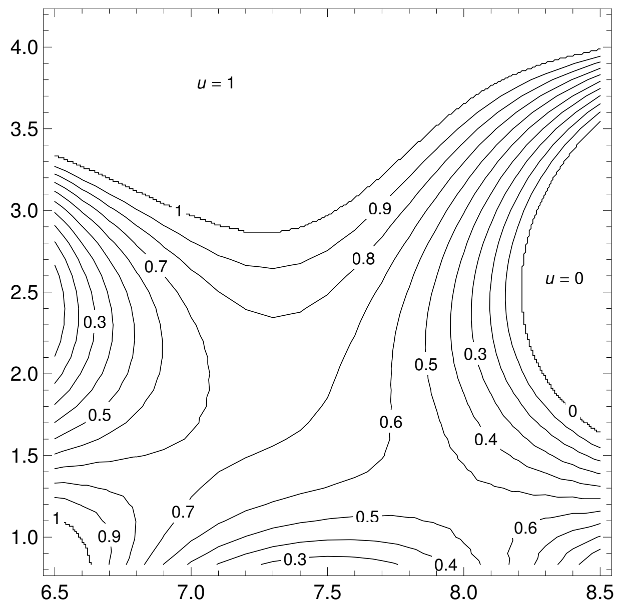

Fig. 1. Optimal control ($\mu = 0.01$)

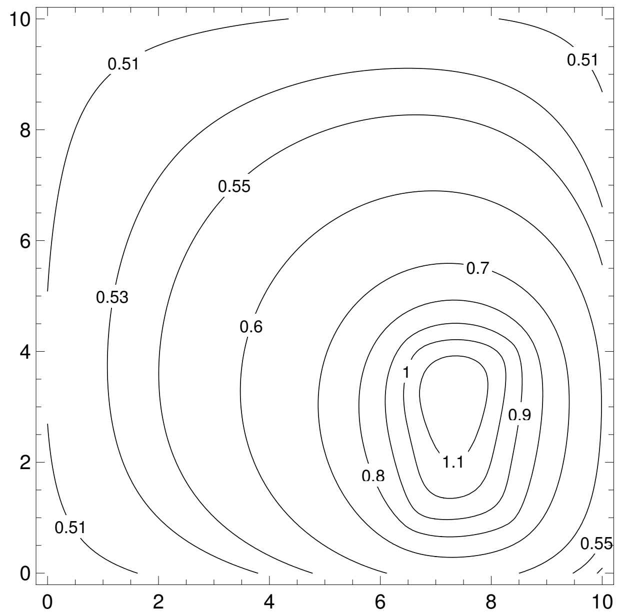

Fig. 2. Optimal temperature ($\mu = 0.01$)

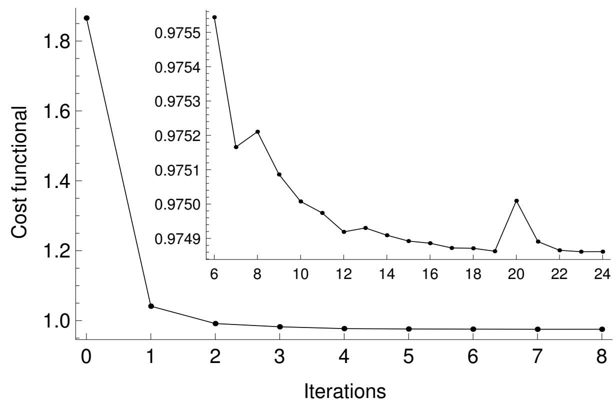

Fig. 3. Cost functional $\widehat{J}(u_k)$ at different iterations ($\mu = 0.01$)

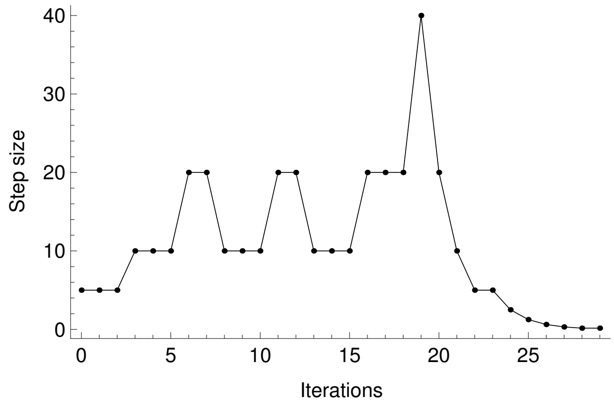

Fig. 4. Step size λ_k at different iterations ($\mu = 0.01$)

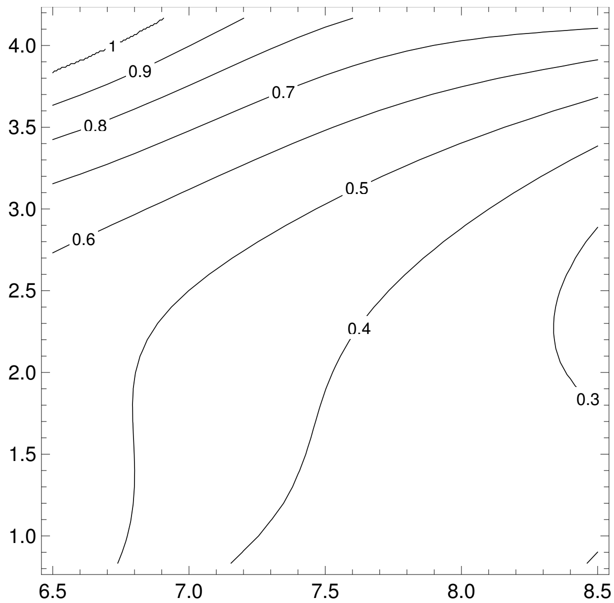

Fig. 5. Optimal control ($\mu = 0.1$)

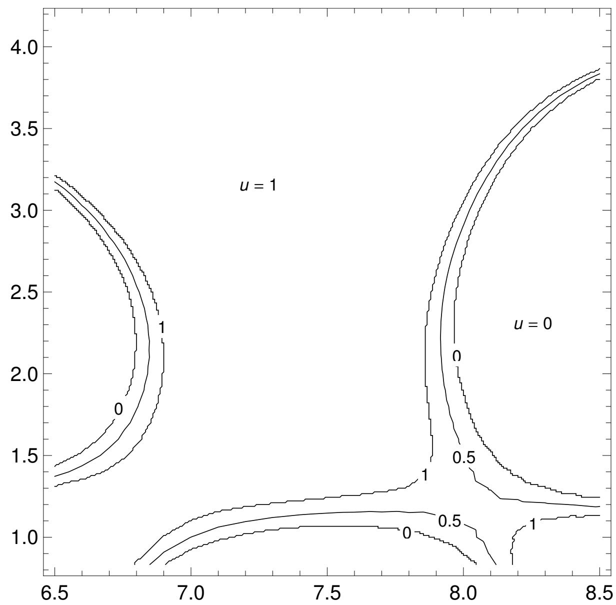

Fig. 6. Optimal control ($\mu = 0.001$)

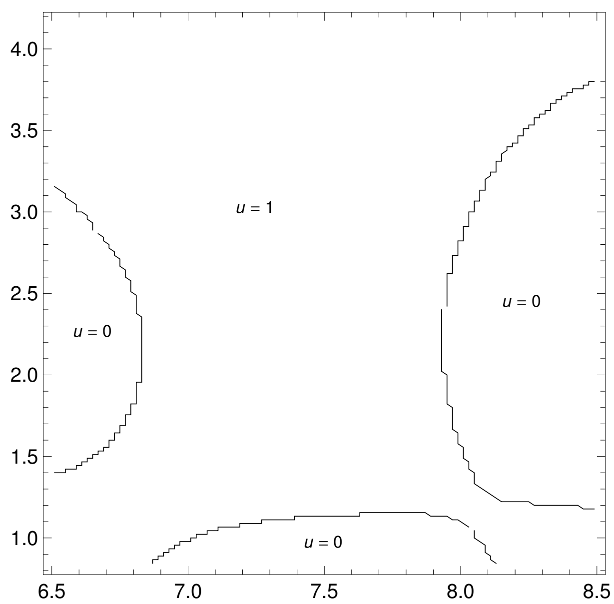

Fig. 7. Bang-bang optimal control ($\mu = 0$)

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