ON A MOMENT PROBLEM FOR SETS OF POINTS IN THE COMPLEX PLANE

V.P. Tsvetov

Samara National Research University, Samara, Russia

Abstract. This paper deals with a uniqueness problem of determination a set of points in the complex plane by the degree-like moments. We discuss applications of these results to determine the similarity of flat polygonal lines in contour analysis method.

Keywords: moment problem, contour analysis, image processing.

Citation: Tsvetov VP. On a moment problem for sets of points in the complex plane. CEUR Workshop Proceedings, 2016; 1638: 411-418. DOI: 10.18287/1613-0073-2016-1638-411-418

1 Introduction

A moment problem is closely associated with many questions of functional analysis [1], integral geometry [2, 3], problems of interpolations, and function classifications [4]. Several problems concerning the uniqueness of solutions of operator equations can be reformulated in terms of determining continuous linear functional $f : X \to \mathrm{C}$ by the known values on a system of basic elements $\{z_m\}_{m \in I \subseteq \mathrm{N}_0}$ of a normed linear space X :

$$f(z_m) = s_m. \qquad (1)$$

In applications X is a space of complex-valued functions that are continuous on compact set K . Such space with the uniform norm is denoted as $C(K)$. By $C(K)^*$ we denote the linear space that topologically conjugates to $C(K)$. So $C(K)^*$ contains all continuous linear functionals $f : C(K) \to \mathrm{C}$. Based on the Riesz-Radon theorem we know that $C(K)^*$ is isometric to the space of Radon measures with compact support on K [1, 5]. Hence any linear functional can be uniquely represented in the form

$$f(z) = \int_K z(t)\, d\mu_f(t), \qquad (2)$$

here μ_f is a measure with compact support $K_f \subseteq K$ that is uniquely defined by the functional f .

In this paper we will be interested in the case of (1) or (2), where $K \subset \mathrm{R}^2$, $z_m(t) = z_m(x,y) = z^m = (x+iy)^m$, $i^2 = -1$, and $\mu_f(t)$ is a function of bounded variation with compact support $K_f \subseteq K \subset \mathrm{R}^2$. The integral in (2) is understood as an integral of Lebesgue-Stieltjes.

If $K_f = \left\{ t_j \mid t_j = (x_j, y_j),\ j \in 1..k \right\}$ is a finite set of points, then the integral in (2) is reduced to the finite sum

$$f(z) = \sum_{j=1}^{k} z(t_j) \cdot \mu_f(t_j) = \sum_{j=1}^{k} z_j \cdot \mu_j^f , \tag{3}$$

where $z_j = x_j + iy_j$, and $\mu_j^f \in \mathrm{C}$. So (1) takes one of the forms

$$f(z^m) = \int_{K \subset \mathrm{R}^2} z^m(t)\, d\mu_f(t) = \int_{K_f \subset \mathrm{R}^2} z^m(t)\, d\mu_f(t) = S_m, \tag{4}$$

or

$$f(z^m) = \sum_{j=1}^{k} z_j^m \cdot \mu_j^f = s_m. \tag{5}$$

We will show the uniqueness of a linear functional f (and so K_f) that is determined from (5) by a known finite number of values s_m . Then we extend this result to the special case of (4), when the compact support K_f is a polyline with a finite number of segments, and the integral is understood as a line integral along a plane curve.

Note that the moment problem (5) arises in a contour analysis based on the integral representations for Gaussian beams [6].

It is easy to give an example of different compact subsets $K_f \subseteq K \subset \mathrm{R}^2$, which produce an equal moments S_m to a corresponding functionals f , even in the case $m \in I = \mathrm{N}_0$.

As a compact $K \subset \mathrm{R}^2$ we consider a circle $K = \left\{ (x,y) \mid x^2 + y^2 \le r_0^2 \right\}$ and define a family of subcompacts:

$$K_1^{r_1} = \left\{ (x,y) \mid x^2 + y^2 = r_1^2 < r_0^2 \right\},$$

$$K_2^{r_2, r_3} = \left\{ (x,y) \mid r_2^2 \le x^2 + y^2 \le r_3^2 < r_0^2 \right\},$$

$$K_3^{r_4} = \left\{ (x,y) \mid x^2 + y^2 \le r_4^2 < r_0^2 \right\}.$$

Then we define continuous linear functionals

$$f_1^{r_1}(z) = \int_{K_1^{r_1}} z(x,y)\, dL \in C(K)^*, \tag{6}$$

$$f_2^{r_2,r_3}(z)=\iint\limits_{K_2^{r_2,r_3}} z(x,y)\,dxdy \in C(K)^*, \tag{7}$$

$$f_3^{r_4}(z)=\iint\limits_{K_1^{r_4}} z(x,y)\,dxdy \in C(K)^*, \tag{8}$$

where the integral in (6) is understood as a line integral.

For all $m \in \mathrm{N}_0$ functions $z_m(x,y)=z^m=(x+iy)^m$ form an orthogonal system over the scalar products

$$\langle z_1,z_2\rangle_1=\int\limits_{K_1^{r_1}} z_1(x,y)\overline{z_2(x,y)}\,dL,$$

$$\langle z_1,z_2\rangle_2=\iint\limits_{K_2^{r_2,r_3}} z_1(x,y)\overline{z_2(x,y)}\,dxdy,$$

$$\langle z_1,z_2\rangle_3=\iint\limits_{K_1^{r_4}} z_1(x,y)\overline{z_2(x,y)}\,dxdy,$$

So for all $m>0$ we have the zero moments

$$f_1^{r_1}(z^m)=S_m^1=\int\limits_{K_1^{r_1}} z^m\,dL=0,$$

$$f_2^{r_2,r_3}(z^m)=S_m^2=\iint\limits_{K_2^{r_2,r_3}} z^m\,dxdy=0,$$

$$f_3^{r_4}(z^m)=S_m^3=\iint\limits_{K_1^{r_4}} z^m\,dxdy=0,$$

and non zero moments $S_0^1=2\pi r_1$, $S_0^2=\pi\left(r_3^2-r_2^2\right)$, $S_0^3=\pi r_4^2$, for $m=0$.

Then we fix $0<r_1<\frac{r_0^2}{2}$ and set $r_4=\sqrt{2r_1}$, $r_3^2=r_4^2+r_2^2$. It's clear that for all $0<r_2^2<r_0^2-2r_1$ we obtain an infinite number of different continuous linear functionals $f_1^{r_1}$, $f_2^{r_2,r_3}$, $f_3^{r_1}$ with compact supports $K_1^{r_1}$, $K_2^{r_2,r_3}$, $K_3^{r_4}$, and equal moments $f_1^{r_1}(z^m)$, $f_2^{r_2,r_3}(z^m)$, $f_3^{r_1}(z^m)$.

This is explained by the fact that the set of functions $z_m(x,y)=z^m=(x+iy)^m$ is not a dense set in $C(K)$. According to the Weierstrass theorem, a dense set is formed by the system of polynomials $p_{m,n}(x,y)=x^m y^n$, $m,n\in \mathrm{N}_0$. Based on the representations $x=\mathrm{Re}(z)=\frac{z+\bar{z}}{2}$, $y=\mathrm{Im}(z)=\frac{z-\bar{z}}{2}$, we can say that the system of functions $P_{m,n}(x,y)=z^m\,\bar{z}^n$ has the same quality.

Thus any continuous linear functionals $f\in C(K)^*$, including those that are defined by (6)-(8) will be uniquely determined by the system of values

$$f(z^m \,\overline{z}^n) = S_{m,n},\ m,n \in \mathrm{N}_0. \qquad (9)$$

If we consider the linear functionals defined by integrals over plain domains or rectifiable curves, then the system of moments (9) will uniquely determine bounded curve or plain domain.

2 Moment problem for a finite set of points

Let's consider a moment problem in the following form. From a given system of equations

$$\sum_{j=1}^{k} z_j^m \cdot \mu_j = s_m,\ m \in 0..M,\ M \in \mathrm{N}, \qquad (10)$$

we want to determine a set of pairs of complex numbers $S = \left\{\left(z_j, \mu_j\right)\right\}_{j=1}^{k}$ under the assumption that all the numbers z_j are pairwise distinct, and $\mu_j \neq 0$. In (10) we put $z^0 = 1$ for all $z \in \mathrm{C}$.

Two sets $S = \left\{\left(z_j, \mu_j\right)\right\}_{j=1}^{k}$ and $\tilde{S} = \left\{\left(\tilde{z}_j, \tilde{\mu}_j\right)\right\}_{j=1}^{k}$ will be considered as equivalent $S \approx \tilde{S}$, if there exists a total bijection $h: 1..k \to 1..k$ that is $z_j = \tilde{z}_{h(j)}$ and $\mu_j = \tilde{\mu}_{h(j)}$ (in other words, if they match up to a permutation).

We are interested in the uniqueness of the solution for the problem above, which is understood as equivalent in the above sense.

Suppose that there are two sets S and $\tilde{S}$ that satisfies (10). Therefore we have the following system of equations

$$\sum_{j=1}^{k}\left(z_j^m \cdot \mu_j - \tilde{z}_j^m \cdot \tilde{\mu}_j\right) = 0,\ m \in 0..M, \qquad (11)$$

Denote by $\left\{z_j\right\} \cap \left\{\tilde{z}_j\right\}$ the intersection of sets $\left\{z_j \mid j \in 1..k\right\}$ and $\left\{\tilde{z}_j \mid j \in 1..k\right\}$. Now we define the sets of indices $J_{z\tilde{z}} = \left\{j \mid z_j \in \left\{z_j\right\} \cap \left\{\tilde{z}_j\right\}\right\}$, $J_{\tilde{z}z} = \left\{j \mid \tilde{z}_j \in \left\{z_j\right\} \cap \left\{\tilde{z}_j\right\}\right\}$, $J_z = 1..k \setminus J_{z\tilde{z}}$, $J_{\tilde{z}} = 1..k \setminus J_{\tilde{z}z}$.

Let $s = \left|J_{z\tilde{z}}\right| = \left|J_{\tilde{z}z}\right| \leq k$. Without loss of generality, we assume that $J_{z\tilde{z}} = J_{\tilde{z}z} = 1..s$ ($s > 0$), and $J_{z\tilde{z}} = J_{\tilde{z}z} = \varnothing$ ($s = 0$). It is clear that in this case $J_z = J_{\tilde{z}} = s+1..k$ ($s < k$) and $J_z = J_{\tilde{z}} = \varnothing$ ($s = k$).

Denote by $h: J_{z\tilde{z}} \to J_{\tilde{z}z}$ the total bijection that takes each j_1 to j_2 in accordance with the rule $z_{j_1} = \tilde{z}_{j_2}$. Then the system of equalities (11) takes the form

$$\sum_{j\in J_z} z_j^m \cdot \mu_j - \sum_{j\in J_{\tilde z}} \tilde z_j^m \cdot \tilde\mu_j - \\ - \sum_{j\in J_{z\cap\tilde z}} \left(\mu_j - \tilde\mu_{h(j)}\right)\cdot z_j^m = 0,\ m\in 0..M. \tag{12}$$

Further denote

$$\xi_j = \begin{cases} \mu_j - \tilde\mu_{h(j)}, 1\le j\le s \\ \mu_j, \qquad s+1\le j\le k \\ -\tilde\mu_{h(j-k+s),}\ k+1\le j\le 2k-s \end{cases},$$

$$\alpha_{ij} = \begin{cases} z_j^{i-1}, 1\le j\le k, 1\le i\le M+1 \\ \tilde z_{h(j-k+s)}^{i-1}, k+1\le j\le 2k-s, 1\le i\le M+1 \end{cases},$$

and rewrite (12) in the form

$$\sum_{j=1}^{2k-s} \alpha_{ij}\cdot\xi_j = 0,\ i\in 1..M+1. \tag{13}$$

We note that the leading principal minors of the matrix $\left(\alpha_{ij}\right)$ are Vandermonde determinants composed by powers of distinct numbers. Thus the rank of this matrix is equal to $\min\left(M+1, 2k-s\right)$. Let us consider (13) as a system of linear equations with variables ξ_j. If $M+1\ge 2k-s$, the system (13) can have only the trivial solution. In this case $J_z = J_{\tilde z} = \varnothing$, $J_{z\tilde z} = J_{\tilde z z} = 1..k$, and for all $j\in 1..k$ we obtain

$z_j = \tilde z_{h(j)}, \mu_j = \tilde\mu_{h(j)}.$

It's mean that $S\approx\tilde S$.

Obviously that $M+1\ge 2k-s$ for all $s\ge 0$, if $M+1\ge 2k$. Hence, if $M\ge 2k-1$, then the moment problem (10) can have only one solution. The converse may be proved in such a way as is done in [7].

3 Moment problem for polyline with a finite number of segments

Let's consider the unclosed polyline $L^k\subset \mathrm{R}^2$ with k nodes $\left\{t_j = \left(x_j, y_j\right)\right\}_{j=1}^k$, and without self-intersections. We assume that associated nodes t_{j-1}, t_j, t_{j+1} does not lie on the same straight line, and each node can belong to no more than two segments of a polyline. So polyline has $k-1$ segments.

As before, we define the map $z:\mathrm{R}^2 \to \mathrm{C}$ in accordance with the rule $z = z(t) = z(x,y) = x + iy$. For each of $k-1$ segments we consider the parameterizations $L_j^k : [0,1] \to \mathrm{R}^2$ that given by $\tau \mapsto L_j^k(\tau) = t_j + (t_{j+1} - t_j)\cdot\tau.$

Let's denote

$z_j = z(t_j) = z(x_j, y_j) = x_j + iy_j,$

$z_j(\tau_j) = z(L_j^k(\tau)) = z_j + (z_{j+1} - z_j)\cdot\tau,$

$\varphi_j = \arg(z_{j+1} - z_j),$

and take a system of complex moments S_m with $m \in 0..M \subset \mathrm{N}_0$

$$S_m = \int_{L^k} z^m \, dL^k = \sum_{j=1}^{k-1} \int_0^1 z_j^m(\tau)\left|dz_j(\tau)\right|. \tag{14}$$

After simple calculations we obtain

$$S_m = \frac{1}{m+1}\cdot\Big(-e^{-i\varphi_1}\cdot z_1^{m+1} + \\ +\sum_{j=1}^{k-1}\left(e^{-i\varphi_{j-1}} - e^{-i\varphi_j}\right)\cdot z_j^{m+1} + e^{-i\varphi_{k-1}}\cdot z_k^{m+1}\Big). \tag{15}$$

Let $s_m = m\cdot S_{m-1}$, $\mu_1 = -e^{-i\varphi_1}$, $\mu_k = e^{-i\varphi_{k-1}}$, $\mu_j = e^{-i\varphi_{j-1}} - e^{-i\varphi_j}$ with $j \in 2..k-1$. It's easy that

$$\sum_{j=1}^{k}\mu_j = -e^{-i\varphi_1} + \sum_{j=2}^{k-1}\left(e^{-i\varphi_{j-1}} - e^{-i\varphi_j}\right) + e^{-i\varphi_{k-1}} = 0,$$

and therefore (15) can be written as (10).

By assumption of polyline we have

$$\mu_1 = -e^{-i\varphi_1} = -\frac{|z_2 - z_1|}{z_2 - z_1} \neq 0, \tag{16}$$

$$\mu_k = e^{-i\varphi_{k-1}} = \frac{|z_k - z_{k-1}|}{z_k - z_{k-1}} \neq 0. \tag{17}$$

$$\mu_j = e^{-i\varphi_{j-1}} - e^{-i\varphi_j} = \frac{|z_j - z_{j-1}|}{z_j - z_{j-1}} - \frac{|z_{j+1} - z_j|}{z_{j+1} - z_j} \neq 0, \tag{18}$$

with $j \in 2..k-1$.

From the results of the previous section it follows that no exists more than one sets of points $\{z_j\}_{j=1}^k$ and the weights $\{\mu_j\}_{j=1}^k$ that satisfy (14) with $m \in 0..2k$.

It is easy to prove that the pairs $\{(z_j, \mu_j)\}_{j=1}^k$ completely determine the nodes and segments of the polyline L^k. Indeed, μ_1 uniquely determines the value of $e^{i\varphi_1} = -\mu_1^{-1}$ and thus the direction $w = \left(-\mathrm{Re}\left(\mu_1^{-1}\right), -\mathrm{Im}\left(\mu_1^{-1}\right)\right)$ from node $t_1 = \left(\mathrm{Re}\left(z_1\right), \mathrm{Im}\left(z_1\right)\right)$ to the node t_2. Thus $t_2 = t_1 + r \cdot w$ for some real value $r > 0$. At least one such point must exist in the node set $\left\{t_j\right\}_{j=1}^{k}$ If there are several nodes that satisfy the relations $t_j = t_1 + r_j \cdot w$ with some $r_j > 0$, then we should choose the one which corresponds to the minimum value of r_i. This follows from our assumption that the points t_{j-1}, t_j, t_{j+1} should not lie on the same straight line and each node can belong to no more than two segments of a polyline. The same applies for μ_k, which together with z_k uniquely identifies the segment in the polyline $\left(t_{k-1}, t_k\right)$. As for the other pairs of values, we can use the obvious equality $e^{-i\varphi_j} = -\sum_{s=1}^{j} \mu_s = \sum_{s=0}^{n-j} \mu_{n-s}$ and consistently hold the previous arguments, starting with one of the node t_1 or t_k.

A similar result holds for the closed polyline L^k defined by the nodes $\left\{t_j = \left(x_j, y_j\right)\right\}_{j=1}^{k}$ with $t_1 = t_k$. In this case, the analogue of (15) takes the form (10) if we substitute $k-1$ for k and put $\mu_1 = e^{-i\varphi_{k-1}} - e^{-i\varphi_1}$.

4 Applications to the recognition of similarity for planar contours

A moment problem has a relation to the image analysis [8], including analysis of similarity discrete planar contours [6, 9].

We say that two sets $M_1 \subseteq \mathrm{C}$ and $M_2 \subseteq \mathrm{C}$ are called similar, if there exist $\lambda_0, \kappa_0 \in \mathrm{C}$ such that $M_2 = \left\{z' \mid z' = \lambda_0 + \kappa_0 \cdot z, z \in M_1\right\}$. Here λ_0 describes parallel transport of points M_1, $\left|\kappa_0\right|$ is equal to the coefficient of similarity, and $\arg\left(\kappa_0\right)$ corresponds to the angle of rotation ($\kappa_0 \neq 0$).

If the similar sets M_1, M_2 are finite, then

$$z'_j - \frac{1}{k}\sum_{s=1}^{k} z'_s = \lambda_0 + \kappa_0 \cdot z_j - \frac{1}{k}\sum_{s=1}^{k}\left(\lambda_0 + \kappa_0 \cdot z_s\right) =$$

$$\kappa_0 \cdot \left(z_j - \frac{1}{k}\sum_{s=1}^{k} z_s\right),$$

and so without loss of generality, we assume that $\sum_{j=1}^{k} z_j = \sum_{j=1}^{k} z'_j = 0$ or the same $\lambda_0 = 0$.

Let's consider two unclosed polyline $L^k, L'^k \subset \mathrm{R}^2$ with the same number of segments. As follows from the previous section, each of them is uniquely determined by a system of moments

$$\sum_{j=1}^{k} z_j^m \cdot \mu_j = \sum_{j=1}^{k} z_j^m \cdot \mu\left(z_{j-1}, z_j, z_{j+1}\right) = s_m, \tag{19}$$

$$\sum_{j=1}^{k} z_j'^m \cdot \mu_j' = \sum_{j=1}^{k} z_j'^m \cdot \mu\left(z_{j-1}', z_j', z_{j+1}'\right) = s_m', \tag{20}$$

with $m \in 0..2k-1$ and $\mu\left(z_{j-1}, z_j, z_{j+1}\right)$ that satisfy (16)-(18)

Given that

$$\mu\left(\kappa_0 \cdot z_{j-1}, \kappa_0 \cdot z_j, \kappa_0 \cdot z_{j+1}\right) = \frac{|\kappa_0|}{\kappa_0} \mu\left(z_{j-1}, z_j, z_{j+1}\right),$$

we finally obtain the similarity criterion for polyline L^k, L'^k in the following form

$$s_m' = |\kappa_0| \cdot \kappa_0^{m-1} \cdot s_m,\ m \in 0..2k-1. \tag{21}$$

Criterion (21) also holds for closed polylines.

References

1. Kantorovich LV, Akilov GP. Functional analysis. Moscow: "Nauka" Publisher; 1984. [In Russian]
2. Lavrentiev MM, Saveliev LYa Operator theory and ill posed problems. Novosibirsk: "Institut Matematiki" Publisher; 2010. [In Russian]
3. Khakhlyutin VP. On a problem of integral geometry in the plane (English. Russian original) Sov. Math., Dokl., 1992; 44(2): 574-576; translation from Dokl. Akad. Nauk SSSR, 1991; 320(4): 832-834.
4. Akhiezer NI. The classical moment problem: And some related questions in analysis. Moscow: "Fizmatlit" Publisher, 1961. [In Russian]
5. Edwards RE. Functional analysis. N.Y.: Holt,-Rinehart & Winston, 1965.
6. Volostnikov VG, Kishkin SA, Kotova SP. Contour analysis and modern optics of Gaussian beams. Computer Optics, 2014; 38(3): 476-481. [In Russian]
7. Gantmacher FR. The Theory of Matrices. Moscow: "Nauka" Publisher, 1960. [In Russian]
8. Pratt WK. Digital image processing. N.Y.: John Wiley & Sons, Inc, 1978.
9. Furman YaA, Krevetsky AV, Peredreev AK, Rojentsov AA, Khafizov RG, Egoshina IL, Leukhin AN. Introduction into a contour analysis. Ed by Furman YaA. Moscow: "Fizmatlit" Publisher, 2003. [In Russian]