The structure of the category of parabolic equations. II

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Abstract

This is the second part of the series consisting of two papers. Here we investigate the category $\mathcal{PE}$ of parabolic equations introduced in the first paper. The objects of this category are second order parabolic equations posed on arbitrary manifolds, and the morphisms generalize the notion of the quotient map by a symmetry group. We introduce a certain structure in $\mathcal{PE}$ formed by the lattice of subcategories. These subcategories are obtained by the restricting to equations of specific kind or to morphisms of specific kind or both. We investigate this structure using a language developed in the first paper. An example that deals with nonlinear reaction-diffusion equation is discussed in more detail.

Introduction

This paper is the second part of the series of two papers. In the first part [2] the author defined the category $\mathcal{PDE}$ of partial differential equations and its full subcategory $\mathcal{PE}$ that arises from second order parabolic equations on arbitrary manifolds. This paper is devoted to the investigation of the internal structure of the category $\mathcal{PE}$ by means of the special-purpose language developed in [2, section 4].

Recall the definition of the category of parabolic equations from [2, section 5]. Let us consider the class $P\left(X, T, \Omega\right)$ of differential operators on a connected smooth manifold X, which depend additionally on a parameter t ("time"), locally having the form

$$Lu = \sum_{i,j} b^{ij}(t,x,u)u_{ij} + \sum_{i,j} c^{ij}(t,x,u)u_i u_j + \sum_i b^i(t,x,u)u_i + q(t,x,u),$$
$$x \in X,\ t \in T,\ u \in \Omega$$

in some neighborhood of each point, in some (and then arbitrary) local coordinates $\left(x^i\right)$ on X. Here subscript i denotes partial derivative with respect to x^i, quadratic form $b^{ij} = b^{ji}$ is positive definite, and $c^{ij} = c^{ji}$. Both T and Ω may be bounded, semi-bounded or unbounded open intervals of $\mathbb{R}$. *The category $\mathcal{PE}$ of parabolic equations* is a subcategory of $\mathcal{PDE}$, whose objects are pairs $\mathbf{A} = (N, E)$, $N = T \times X \times \Omega$, where X is a connected smooth manifold, T and Ω are open intervals, E is an equation of the form $u_t = Lu$, $L \in P\left(X, T, \Omega\right)$. Theorem 1 of [2] asserts that every morphism in $\mathcal{PE}$ has the form

$$(t, x, u) \mapsto (t'(t), x'(t, x), u'(t, x, u)), \tag{1}$$

In: A.A. Makhnev, S.F. Pravdin (eds.): Proceedings of the 47th International Youth School-conference "Modern Problems in Mathematics and its Applications", Yekaterinburg, Russia, 02-Feb-2016, published at http://ceur-ws.org

with submersive $t'(t)$, $x'(t,x)$, and $u'(t,x,u)$. Isomorphisms in $\mathcal{PE}$ are exactly diffeomorphisms of the form (1).

Section 1 of this paper is devoted to the classification of parabolic equations in this framework and to the description of the internal structure of $\mathcal{PE}$. The proofs of Theorems 1-7 given in the section are postponed to Section 3.

Section 2 illustrates the using of this structure of $\mathcal{PE}$ on the example of the reaction-diffusion equation

$$u_t = a(u)\,(\Delta u + \eta\nabla u) + q(x,u), \quad x \in X,\ t \in \mathbb{R}, \tag{2}$$

posed on a Riemannian manifold X equipped with a vector field η. There are two exceptional cases: $a(u) = e^{\lambda u}H(u)$ and $a(u) = (u-u_0)^{\lambda}H(\ln(u-u_0))$, where $H(\cdot)$ is a periodic function; in these cases there are more morphisms then in a regular case. If only function $a(u)$ does not belong to one of these two exceptional classes, then Theorems 9-10 assert that every morphism from equation (2) may be transformed by an isomorphism (i.e. by a bijective global change of variables) of the quotient equation to the "canonical" morphism of very simple kind so that the "canonical" quotient equation has the same form as (2) with the same function $a(u)$ but is posed on another Riemannian manifold X', $\dim X' \le \dim X$.

1 The structure of $\mathcal{PE}$ and classification of parabolic equations

We formulate here the number of theorems describing the internal structure of $\mathcal{PE}$; the proofs of these theorems are given in Section 3 below. Certain parts of the structure of $\mathcal{PE}$ are depicted schematically on Fig. 1 (the full picture is not given here in view of its awkwardness).

Let us consider five full subcategories $\mathcal{PE}_k$ of $\mathcal{PE}$, $1 \le k \le 5$, whose objects are equations that can be written locally in the following form:

$$u_t = \sum_{i,j} b^{ij}(t,x,u)\,(u_{ij} + \lambda(t,x,u)u_iu_j) + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{PE}_1$}$$

$$u_t = a(t,x,u)\sum_{i,j}\bar{b}^{ij}(t,x)u_{ij} + \sum_{i,j} c^{ij}(t,x,u)u_iu_j + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{PE}_2$}$$

$$u_t = a(t,x,u)\sum_{i,j}\bar{b}^{ij}(t,x)\,(u_{ij} + \lambda(t,x,u)u_iu_j) + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{PE}_3$}$$

$$u_t = \sum_{i,j} b^{ij}(t,x)u_{ij} + \sum_{i,j} c^{ij}(t,x,u)u_iu_j + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{PE}_4$}$$

$$u_t = \sum_{i,j} b^{ij}(t,x)\,(u_{ij} + \lambda(t,x,u)u_iu_j) + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{PE}_5$}$$

Remark 1. Everywhere in the paper we use notation of a category equipped with a subscript and/or primes for its full subcategory. For example, $\mathcal{QPE}_k$, $\mathcal{QPE}'$, and $\mathcal{QPE}'_k$ defined below are full subcategories of $\mathcal{QPE}$.

Remark 2. In equations of the categories $\mathcal{PE}_2$ and $\mathcal{PE}_3$, function $a(\cdot)$ is determined up to multiplication by arbitrary function from $T \times X$ to $\mathbb{R}^+$; moreover, it is determined only locally. Nevertheless we can lead these equations to the equations of the same form but with globally defined function $a\colon T \times X \times \Omega \to \mathbb{R}^+$. For example, we can require that $a(t,x,u_0) \equiv 1$, where u_0 is a fixed point of Ω. Everywhere below we will assume that function a is globally determined on $T \times X \times \Omega$.

Theorem 1.

1. $\mathcal{PE}_1$ and $\mathcal{PE}_2$ are closed in $\mathcal{PE}$.

2. $\mathcal{PE}_3 = \mathcal{PE}_1 \cap \mathcal{PE}_2$ is closed in $\mathcal{PE}_1$, in $\mathcal{PE}_2$, and in $\mathcal{PE}$.

3. $\mathcal{PE}_4$ is closed in $\mathcal{PE}_2$ and in $\mathcal{PE}$.

4. $\mathcal{PE}_5 = \mathcal{PE}_3 \cap \mathcal{PE}_4$ is closed in $\mathcal{PE}_3$, in $\mathcal{PE}_4$, and in $\mathcal{PE}$.

Definition 1. $\mathcal{TPE}$, $\overline{\mathcal{QPE}}$, $\overline{\mathcal{SQPE}}$, $\overline{\mathcal{AQPE}}$, and $\overline{\mathcal{EPE}}$ are wide subcategories of $\mathcal{PE}$, whose morphisms have the following form:

$$(t,x,u) \to \begin{cases} (t, y(t,x), v(t,x,u)) & \text{for } \mathcal{TPE} \\ (t, y(t,x), \varphi(t,x)u + \psi(t,x)) & \text{for } \overline{\mathcal{QPE}} \\ (t, y(x), \varphi(t,x)u + \psi(t,x)) & \text{for } \overline{\mathcal{SQPE}} \\ (t, y(x), \varphi(x)u + \psi(x)) & \text{for } \overline{\mathcal{AQPE}} \\ (t, y(x), u) & \text{for } \overline{\mathcal{EPE}} \end{cases}$$

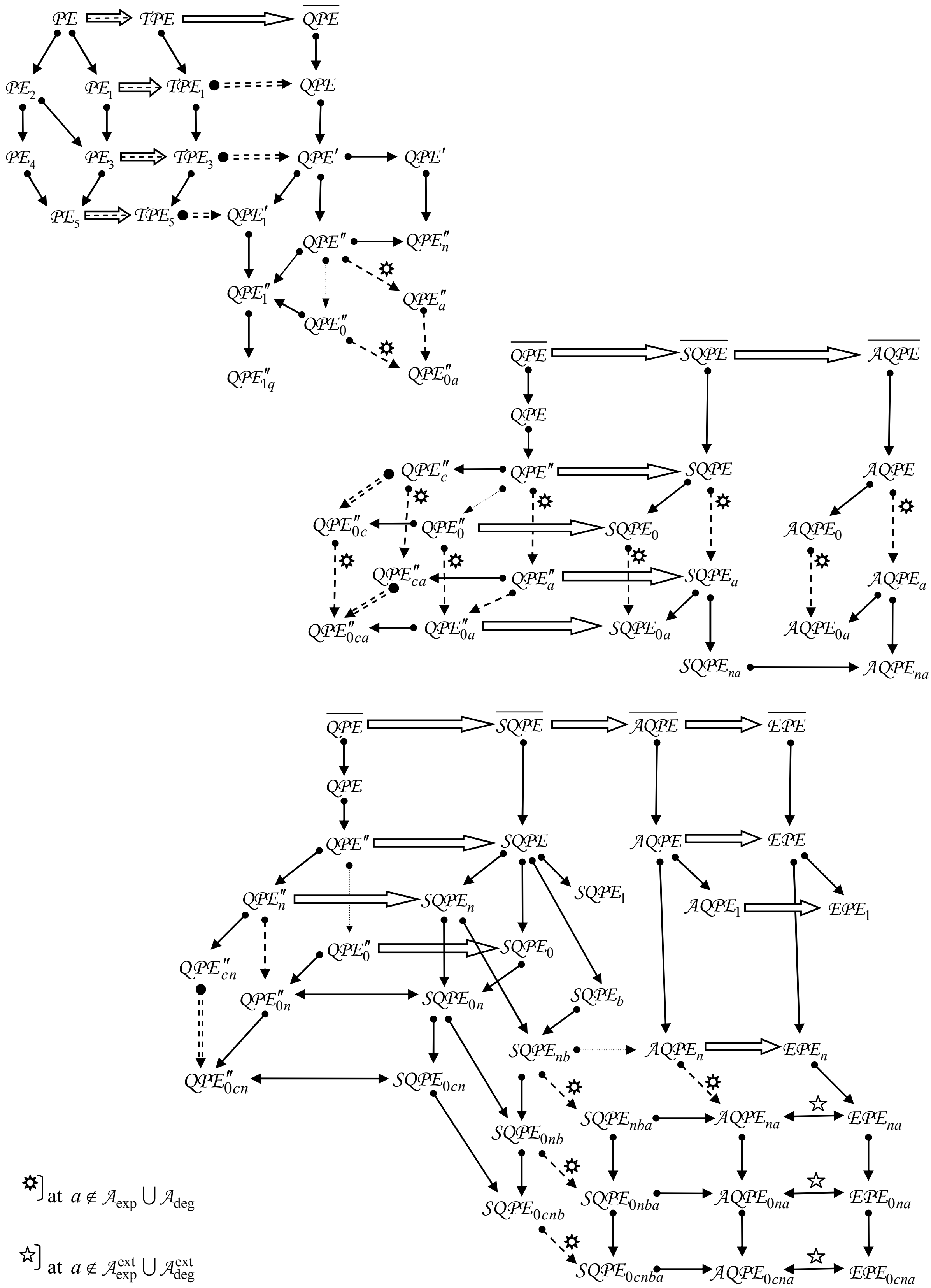

Figure 1: The part of the structure of the category of parabolic equations

Denote $\mathcal{TPE}_k = \mathcal{TPE} \cap \mathcal{PE}_k$.

Theorem 2.

1. *$\mathcal{TPE}$ is wide and plentiful in $\mathcal{PE}$.*
2. *$\mathcal{TPE}_k$ is closed in $\mathcal{TPE}$; it is wide and plentiful in $\mathcal{PE}_k$, $k = 1..5$.*

Definition 2. The *category $\mathcal{QPE}$ of quasilinear parabolic equations* is the full subcategory of $\overline{\mathcal{QPE}}$, whose objects are equations of the form

$$u_t = \sum_{i,j} b^{ij}(t,x,u)u_{ij} + \sum_i b^i(t,x,u)\, u_i + q(t,x,u), \tag{$\mathcal{QPE}$}$$

(in a local coordinates). In particular, morphisms of $\mathcal{QPE}$ are maps of the form

$$(t,x,u) \to (t, y(t,x), \varphi(t,x)u + \psi(t,x))\,.$$

Denote by $\mathcal{A}_{nc}(M,\Omega)$ the set of continuous positive functions $a\colon M \times \Omega \to \mathbb{R}$ that satisfy the condition

$$\forall m \in M \quad \exists u_1, u_2 \quad a(m,u_1) \neq a(m,u_2)\,. \tag{$\mathcal{A}_{nc}$}$$

Define full subcategories of $\mathcal{QPE}$, whose objects are equations of the following form:

$$u_t = a(t,x,u) \sum_{i,j} \bar{b}^{ij}(t,x)u_{ij} + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{QPE}'$}$$

$$u_t = a(t,x,u) \sum_{i,j} \bar{b}^{ij}(t,x)u_{ij} + \sum_i b^i(t,x,u)u_i + q(t,x,u),\ a \in \mathcal{A}_{nc}(T \times X) \tag{$\mathcal{QPE}'_n$}$$

$$u_t = \sum_{i,j} b^{ij}(t,x)u_{ij} + \sum_i b^i(t,x,u)u_i + q(t,x,u) \tag{$\mathcal{QPE}'_1$}$$

$$u_t = a(t,x,u) \left(\sum_{i,j} \bar{b}^{ij}(t,x)u_{ij} + \sum_i \bar{b}^i(t,x)u_i \right) + \sum_i \xi^i(t,x)u_i + q(t,x,u) \tag{$\mathcal{QPE}''$}$$

$$u_t = a(t,x,u) \left(\sum_{i,j} \bar{b}^{ij}(t,x)u_{ij} + \sum_i \bar{b}^i(t,x)u_i \right) + q(t,x,u) \tag{$\mathcal{QPE}''_0$}$$

$$u_t = a(u) \left(\sum_{i,j} \bar{b}^{ij}(t,x)u_{ij} + \sum_i \bar{b}^i(t,x)u_i \right) + \sum_i \xi^i(t,x)u_i + q(t,x,u) \tag{$\mathcal{QPE}''_a(a)$}$$

$$u_t = \sum_{i,j} b^{ij}(t,x)u_{ij} + \sum_i b^i(t,x)u_i + q(t,x,u), \tag{$\mathcal{QPE}''_1$}$$

$$u_t = \sum_{i,j} b^{ij}(t,x)u_{ij} + \sum_i b^i(t,x)u_i + q_1(t,x)u + q_0(t,x), \tag{$\mathcal{QPE}''_{1q}$}$$

where $a(\cdot)$ is a positive function. The family of categories $\mathcal{QPE}''_a(a)$ is parameterized by functions $a(\cdot)$, that is one assigns the category $\mathcal{QPE}''_a(a)$ to each continuous positive function $a\colon \Omega \to \mathbb{R}$.

We define additionally the full subcategory $\mathcal{QPE}_c$ of $\mathcal{QPE}$, whose objects are equations from $\mathcal{QPE}$ posed on a *compact* manifolds X.

Let us introduce the following notation for the intersections of enumerated "basic" subcategories: for a string σ we set $\mathcal{QPE}_\sigma = \cap \{\mathcal{QPE}_\alpha \colon \alpha \in \sigma\}$, $\mathcal{QPE}^\beta_\sigma = \mathcal{QPE}_\sigma \cap \mathcal{QPE}^\beta$. Particularly, $\mathcal{QPE}''_{0n}$ denotes the intersection $\mathcal{QPE}'_n \cap \mathcal{QPE}''_0$.

In the same manner as in Remark 2, we can obtain a global function $a(u)$ for any equation from $\mathcal{QPE}''_a(a)$, for example, by imposing the condition $a(u_0) = 1$. Such function $a(u)$ is independent of the choice of neighborhood in $T \times X \times \Omega$ and of local coordinates.

Theorem 3.

1. $\mathcal{QPE}$ *is closed in* $\overline{\mathcal{QPE}}$ *and is fully dense in* $\mathcal{TPE}_1$.

2. $\mathcal{QPE}_c$ *is closed in* $\mathcal{QPE}$.

3. $\mathcal{QPE}' = \mathcal{QPE} \cap \mathcal{PE}_2 = \mathcal{QPE} \cap \mathcal{PE}_3$ *is fully dense in* $\mathcal{TPE}_3$ *and is closed in* $\mathcal{QPE}$.

4. $\mathcal{QPE}'_1 = \overline{\mathcal{QPE}} \cap \mathcal{PE}_5 = \mathcal{QPE}' \cap \mathcal{PE}_5$ *is fully dense in* $\mathcal{TPE}_5$ *and is closed in* $\mathcal{QPE}'$.

5. $\mathcal{QPE}''$ *is closed in* $\mathcal{QPE}'$.

6. $\mathcal{QPE}''_1 = \mathcal{QPE}'' \cap \mathcal{PE}_5 = \mathcal{QPE}'' \cap \mathcal{QPE}'_1 = \mathcal{QPE}''_a(1)$ *is closed in* $\mathcal{QPE}'_1$, *in* $\mathcal{QPE}''$, *and in* $\mathcal{QPE}''_0$.

7. $\mathcal{QPE}''_{1q}$ *is closed in* $\mathcal{QPE}''_1$.

8. $\mathcal{QPE}'_n$ *is closed in* $\mathcal{QPE}'$.

9. $\mathcal{QPE}''_{0n}$ *is fully plentiful in* $\mathcal{QPE}''_n$.

10. $\mathcal{QPE}''_{0c}$ *is fully dense in* $\mathcal{QPE}''_c$.

Denote by $\mathcal{A}_{\exp}$ the set of functions of the form $a(u) = e^{\lambda u} H(u)$ and by $\mathcal{A}_{\deg}$ the set of functions of the form $a(u) = (u - u_0)^{\lambda} H(\ln(u - u_0))$, where λ, u_0 are arbitrary constants and $H(\cdot)$ is arbitrary non-constant periodic function.

Theorem 4.

1. *If* $a \notin \mathcal{A}_{\exp} \cup \mathcal{A}_{\deg}$, *then* $\mathcal{QPE}''_a(a)$ *is fully plentiful in* $\mathcal{QPE}''$.

2. $\mathcal{QPE}''_{0a}(a)$ *is fully plentiful in* $\mathcal{QPE}''_a(a)$; *if* $a \notin \mathcal{A}_{\exp} \cup \mathcal{A}_{\deg}$, *then* $\mathcal{QPE}''_{0a}(a)$ *is fully plentiful in* $\mathcal{QPE}''_0$.

3. $\mathcal{QPE}''_{0ca}(a)$ *is fully dense in* $\mathcal{QPE}''_{ca}(a)$.

4. *Suppose* $\mathbf{A}$ *is an object of* $\mathcal{QPE}''_a(a)$, $F\colon \mathbf{A} \to \mathbf{B}$ *is a morphism in* $\mathcal{PE}$ *such that there is no object of* $\mathcal{QPE}''_a(a)$ *isomorphic to* $\mathbf{B}$ *in* $\mathcal{PE}$ *(that is* $a(\cdot) \in \mathcal{A}_{\exp} \cup \mathcal{A}_{\deg}$*). Then there exists an object of* $\mathcal{QPE}''$ *isomorphic to* $\mathbf{B}$ *such that the composition of* $F\colon \mathbf{A} \to \mathbf{B}$ *with this isomorphism is of the form*

$$(t, x, u) \to \begin{cases} (t, y(t,x), u + \psi(t,x)), & a \in \mathcal{A}_{\exp} \\ (t, y(t,x), v_0 + (u - u_0)\exp(\psi(t,x))), & a \in \mathcal{A}_{\deg} \end{cases}$$

In addition, for each $t \in T$ *and* $x_1, x_2 \in X$ *such that* $y(t, x_1) = y(t, x_2)$, *the difference* $\psi(t, x_2) - \psi(t, x_1)$ *is an integral multiple of* $\hat{H}$, *where* $\hat{H}$ *is the period of periodic function* H. *The same assertion holds if we replace* $\mathcal{QPE}''_a(a)$ *by* $\mathcal{QPE}''_{0a}(a)$ *and* $\mathcal{QPE}''$ *by* $\mathcal{QPE}''_0$.

Example. The equation

$$E\colon u_t = (2 + \sin u)\, u_{xx}$$

is an object of $\mathcal{QPE}''_{0a}(f)$, with $X = T = \Omega = \mathbb{R}$, $f(u) = 2 + \sin u$, and $f \in \mathcal{A}_{\exp}$. It admits both maps $(t, x, u) \mapsto (t, x \bmod 2\pi, u)$ and $(t, x, u) \mapsto (t, x \bmod 2\pi, u + x)$. In both cases $Y = S^1$. In the first case the quotient equation has the form $v_t = (2 + \sin v)\, v_{yy}$, so it is an object of $\mathcal{QPE}''_{0a}(f)$. In the second case the quotient equation has the form $v_t = (2 + \sin(v + y))\, v_{yy}$; it is an object of $\mathcal{QPE}''_0$, but is not isomorphic to any object of $\mathcal{QPE}''_{0a}(f)$.

Definition 3. The *category of semi-autonomous quasilinear parabolic equations* $\mathcal{SQPE}$ is the intersection $\overline{\mathcal{SQPE}} \cap \mathcal{QPE}''$. In other words, $\mathcal{SQPE}$ is the full subcategory of $\overline{\mathcal{SQPE}}$ and the wide subcategory of $\mathcal{QPE}''$, whose objects are equations of the form

$$u_t = a(t,x,u) \left(\sum_{i,j} \bar{b}^{ij}(t,x) u_{ij} + \sum_i \bar{b}^i(t,x) u_i \right) + \sum_i \xi^i(t,x) u_i + q(t,x,u), \qquad (\mathcal{SQPE})$$

and morphisms are maps of the form $(t, x, u) \mapsto (t, y(x), \varphi(t,x) u + \psi(t,x))$. Define additionally the following full subcategories of $\mathcal{SQPE}$:

$\mathcal{SQPE}_\sigma = \mathcal{SQPE} \cap \mathcal{QPE}''_\sigma$, where σ is one of possible subscripts of $\mathcal{QPE}''$;
$\mathcal{SQPE}_b$ is the category, whose objects are equations of the form

$$u_t = a(t,x,u)\left(\sum_{i,j} \bar{b}^{ij}(x)u_{ij} + \sum_i \bar{b}^i(t,x)u_i\right) + \sum_i \xi^i(t,x)u_i + q(t,x,u). \qquad (\mathcal{SQPE}_b)$$

Theorem 5.

1. *$\mathcal{SQPE}$ is closed in $\overline{\mathcal{SQPE}}$.*
2. *$\mathcal{SQPE}_0 = \overline{\mathcal{SQPE}} \cap \mathcal{QPE}''_0$, $\mathcal{SQPE}_n = \overline{\mathcal{SQPE}} \cap \mathcal{QPE}''_n$, and $\mathcal{SQPE}_b$ are closed in $\mathcal{SQPE}$.*
3. *$\mathcal{SQPE}_{0n}$ coincides with $\mathcal{QPE}''_{0n}$; it is closed in $\mathcal{SQPE}_0$ and in $\mathcal{SQPE}_n$.*
4. *$\mathcal{SQPE}_1 = \overline{\mathcal{SQPE}} \cap \mathcal{QPE}''_1 = \mathcal{SQPE}_a(1)$ is closed in $\mathcal{SQPE}_0$.*
5. *If $a \notin \mathcal{A}_{\exp} \cup \mathcal{A}_{\deg}$, then $\mathcal{SQPE}_a(a)$ is fully plentiful in $\mathcal{SQPE}$.*

Definition 4. The *category of autonomous quasilinear parabolic equations* $\mathcal{AQPE}$ is the full subcategory of $\overline{\mathcal{AQPE}}$, whose objects are equations of the form

$$u_t = a(x,u)\,(\Delta u + \eta\nabla u) + \xi\nabla u + q(x,u) \qquad (\mathcal{AQPE})$$

posed on a Riemann manifold X equipped with vector fields ξ, η.

Define full subcategories $\mathcal{AQPE}_\sigma = \mathcal{AQPE} \cap \mathcal{QPE}''_\sigma$ of $\mathcal{AQPE}$, where σ is one of possible subscripts of $\mathcal{QPE}''$. The objects of these categories are equations of the form

$$\begin{aligned}
u_t &= a(x,u)\,(\Delta u + \eta\nabla u) + \xi\nabla u + q(x,u), \quad a \in \mathcal{A}_{nc}(X), && (\mathcal{AQPE}_n)\\
u_t &= a(x,u)(\Delta u + \eta\nabla u) + q(x,u), && (\mathcal{AQPE}_0)\\
u_t &= a(u)(\Delta u + \eta\nabla u) + \xi\nabla u + q(x,u), && (\mathcal{AQPE}_a(a))\\
u_t &= \Delta u + \xi\nabla u + q(x,u). && (\mathcal{AQPE}_1)
\end{aligned}$$

Theorem 6.

1. *$\mathcal{AQPE}$ is closed in $\overline{\mathcal{AQPE}}$.*
2. *$\mathcal{AQPE}_n$ is closed in $\mathcal{AQPE}$ and full in $\mathcal{SQPE}_{bn}$.*
3. *$\mathcal{AQPE}_0$ and $\mathcal{AQPE}_1$ are closed in $\mathcal{AQPE}$.*
4. *If $a(\cdot) \notin \mathcal{A}_{\exp} \cup \mathcal{A}_{\deg}$, then $\mathcal{AQPE}_a(a)$ is fully plentiful in $\mathcal{AQPE}$.*
5. *$\mathcal{AQPE}_{na}(a)$ is closed in $\mathcal{SQPE}_{na}(a)$.*

Definition 5. Define the following full subcategories of $\overline{\mathcal{EPE}}$ (its morphisms are maps of the form $(t,x,u) \mapsto (t,y(x),u)$):

$$\begin{aligned}
\mathcal{EPE} &= \overline{\mathcal{EPE}} \cap \mathcal{AQPE},\\
\mathcal{EPE}_\sigma &= \overline{\mathcal{EPE}} \cap \mathcal{AQPE}_\sigma,\\
\mathcal{EPE}_a(a) &= \overline{\mathcal{EPE}} \cap \mathcal{AQPE}_a(a).
\end{aligned}$$

Denote by $\mathcal{A}^{\mathrm{ext}}_{\exp}$ the set of functions $a(u)$ of the form $a(u) = e^{\lambda u}H(u)$ and by $\mathcal{A}^{\mathrm{ext}}_{\deg}$ the set of functions of the form $a(u) = (u-u_0)^\lambda\, H(\ln(u-u_0))$, where λ, u_0 are arbitrary constants, $H(\cdot)$ is arbitrary periodic function (that is $\mathcal{A}_{\exp} \subset \mathcal{A}^{\mathrm{ext}}_{\exp}$, $\mathcal{A}_{\deg} \subset \mathcal{A}^{\mathrm{ext}}_{\deg}$).

Theorem 7.

1. *$\mathcal{EPE}$ is closed in $\overline{\mathcal{EPE}}$ and wide in $\mathcal{AQPE}$.*
2. *$\mathcal{EPE}_n$, $\mathcal{EPE}_0$, $\mathcal{EPE}_1$, and $\mathcal{EPE}_a(a)$ are closed in $\mathcal{EPE}$.*

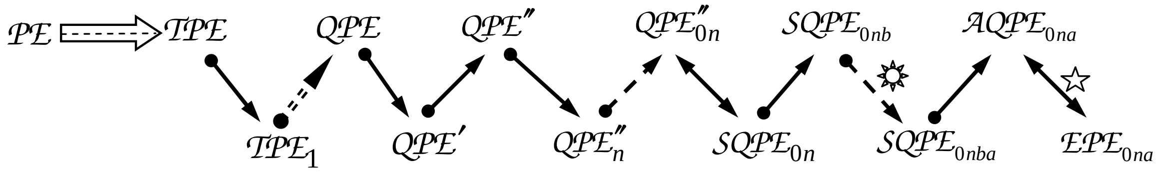

Figure 2: The sequence of arrows from $\mathcal{PE}$ to $\mathcal{EPE}_{0na}(a)$

3. *If $a \notin \mathcal{A}^{\rm ext}_{\rm exp} \cup \mathcal{A}^{\rm ext}_{\rm deg}$, then $\mathcal{EPE}_a(a)$ coincides with $\mathcal{AQPE}_a(a)$.*

Let us consider the sequence depicted on Fig. 2. Selecting the "weakest" arrow in this sequence, we obtain the following result.

Theorem 8.

1. *If $a \notin \mathcal{A}_{\rm exp} \cup \mathcal{A}_{\rm deg}$, $a \neq \mathrm{const}$ then $\mathcal{AQPE}_{0a}(a)$ is fully plentiful in $\mathcal{TPE}$ and plentiful in $\mathcal{PE}$.*

2. *If $a \notin \mathcal{A}^{\rm ext}_{\rm exp} \cup \mathcal{A}^{\rm ext}_{\rm deg}$, $a \neq \mathrm{const}$ then $\mathcal{EPE}_{0a}(a)$ is fully plentiful in $\mathcal{TPE}$ and plentiful in $\mathcal{PE}$.*

2 Factorization of the reaction-diffusion equation

Let us consider a nonlinear reaction-diffusion equation

$$u_t = a(u)\,(\Delta u + \eta \nabla u) + q(x,u)$$

for an unknown function $u(t,x)$, $u\colon T \times X \to \Omega$, where T and Ω are open intervals of $\mathbb{R}$ and X is a connected Riemann manifold equipped with a vector field η and a function $q\colon X \times T \to \Omega$. This equation defines the object $\mathbf{A}$ of $\mathcal{PE}$.

The following two theorems are the immediate corollaries of Theorem 8.

Theorem 9. *Let $F\colon \mathbf{A} \to \mathbf{B}$ be a morphism of $\mathcal{PDE}$ and $\mathbf{B}$ be an object of $\mathcal{PE}$. Suppose that $a(u)$ can be written neither in a form $e^{\lambda u} H(u)$ nor in a form $(u-u_0)^\lambda H(\ln(u-u_0))$ with $\lambda \neq 0$, u_0 being real constants, $H(\cdot)$ being a periodic function. Then there exists an isomorphism $I\colon \mathbf{B} \to \mathbf{B}'$ of $\mathcal{PE}$ (in other words, a bijective global change of variables of the form* (1) *in the quotient equation) transforming F to the morphism $I \circ F$ of the form*

$$(t,x,u) \mapsto (t, x'(x), u)$$

such that the quotient equation $\mathbf{B}'$ is the reaction-diffusion equation

$$v_t = a(v)\,(\Delta v + \eta' \nabla v) + q'(x',v) \tag{3}$$

for an unknown function $v\colon T \times X' \to \Omega$, posed on some Riemannian manifold X' equipped with a vector field ξ' and a function $q'\colon X' \times T \to \Omega$.

Theorem 10. *Let $F\colon \mathbf{A} \to \mathbf{B}$ be a morphism of $\mathcal{PDE}$ and $\mathbf{B}$ be an object of $\mathcal{PE}$. Suppose that either $a(u) = a_0 e^{\lambda u}$ or $a(u) = a_0 (u-u_0)^\lambda$ for some real constants $\lambda \neq 0$, u_0, a_0. Then there exists an isomorphism $I\colon \mathbf{B} \to \mathbf{B}'$ of $\mathcal{PE}$ (in other words, a bijective global change of variables of the form* (1) *in the quotient equation) transforming F to the morphism $I \circ F$ of the form*

$$(t,x,u) \mapsto (t, x'(x), \varphi(x)u + \psi(x))$$

for some smooth functions $\varphi\colon X \to \mathbb{R}\backslash\{0\}$, $\psi\colon X \to \mathbb{R}$, such that the quotient equation B' is the reaction-diffusion equation (3) *for an unknown function $v\colon T \times X' \to \Omega'$, posed on some Riemannian manifold X' equipped with a vector field η' and a function $q'\colon X' \times T \to \Omega'$.*

3 Proofs of Theorems 1-8

Proof of Theorem 1

The map $(t,x,u) \mapsto (\tau(t), y(t,x), v(t,x,u))$ is a morphism in $\mathcal{PE}$ if and only if

$$\begin{cases} \tau_t B^{kl} = \sum\limits_{i,j} b^{ij} y_i^k y_j^l \\ \tau_t C^{kl} = (\ln U_v)_v B^{kl} + U_v \sum\limits_{i,j} c^{ij} y_i^k y_j^l \\ \tau_t B^k = \sum\limits_{i,j} b^{ij} y_{ij}^k + 2\sum\limits_{i,j} b^{ij} (\ln U_v)_j y_i^k + 2\sum\limits_{i,j} c^{ij} U_j y_i^k + \sum\limits_i b^i y_i^k - y_t^k \\ \tau_t Q = U_v^{-1} \left(\sum\limits_{i,j} b^{ij} U_{ij} + \sum\limits_{i,j} c^{ij} U_i U_j + \sum\limits_i b^i U_i + q(t,x,U) - U_t \right) \end{cases} \tag{4}$$

where function $u = U(t,x,v)$ is the inverse of the $v(t,x,u)$. The quotient equation is written as $v_\tau = \sum_{k,l} B^{kl} v_{kl} + \sum_{k,l} C^{kl} v_k v_l + \sum_k B^k v_k + Q$. Here and below indexes i, j relate to x, indexes k, l relate to y.

By definition, all $\mathcal{PE}_k$ are full subcategories of $\mathcal{PE}$.

1. Let us prove that $\mathcal{PE}_1$ is closed in $\mathcal{PE}$. Suppose $\mathbf{A} \in \mathrm{Ob}_{\mathcal{PE}_1}$, $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{PE}$. Then $c^{ij} = \lambda(t,x,u) b^{ij}$. From the second equation of system (4) we get

$$C^{kl}(\tau,y,v) = B^{kl}(\tau,y,v)\left[\tau_t^{-1} (\ln U_v)_v + \lambda(t,x,u) U_v\right].$$

The quadratic form B^{kl} is non-degenerated at any point (τ,y,v), so the expression in square brackets is a function of (τ,y,v): $\tau_t^{-1}(\ln U_v)_v + \lambda(t,x,u)U_v = \Lambda(\tau,y,v)$, and $C^{kl}(\tau,y,v) = \Lambda(\tau,y,v) B^{kl}(\tau,y,v)$. Thus $\mathbf{B} \in \mathrm{Ob}_{\mathcal{PE}_1}$.

Let us show that $\mathcal{PE}_2$ is closed in $\mathcal{PE}$. Suppose $\mathbf{A} \in \mathrm{Ob}_{\mathcal{PE}_2}$, $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{PE}$. Then $b^{ij} = a(t,x,u)\bar{b}^{ij}(t,x)$. Using the first equation of system (4), we obtain

$$\tau_t B^{kl} = a(t,x,u) \left(\sum_{i,j} \bar{b}^{ij} y_i^k y_j^l \right)_{(t,x)}.$$

Taking into account that the quadratic form B^{kl} is non-degenerated, we obtain that $B^{11} \neq 0$ everywhere. From the equality

$$\frac{B^{kl}}{B^{11}}(\tau,y,v) = \frac{\sum_{i,j} \bar{b}^{ij} y_i^k y_j^l}{\sum_{i,j} \bar{b}^{ij} y_i^1 y_j^1}(t,x)$$

we obtain that this fraction is function of (t,y). Thus

$$B^{kl}(\tau,y,v) = A(\tau,y,v)\bar{B}^{kl}(\tau,y)$$

for $A(\tau,y,v) = B^{11}(\tau,y,v)$ and some functions $\bar{B}^{kl}(t,y)$. Therefore, $\mathbf{B} \in \mathrm{Ob}_{\mathcal{PE}_2}$. $\Box$

2. $\mathcal{PE}_3 = \mathcal{PE}_1 \cap \mathcal{PE}_2$ is closed in $\mathcal{PE}$, in $\mathcal{PE}_1$, and in $\mathcal{PE}_2$, because $\mathcal{PE}_1$ and $\mathcal{PE}_2$ are closed in $\mathcal{PE}$. $\Box$

3. Suppose $\mathbf{A} \in \mathrm{Ob}_{\mathcal{PE}_4}$ and $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism of $\mathcal{PE}$. From the first equation of (4) we obtain that $B^{kl}(\tau,y,v)$ is independent of v. Hence $B^{kl} = B^{kl}(\tau,y)$, $\mathcal{PE}_4$ is closed in $\mathcal{PE}$, so it is closed in $\mathcal{PE}_2$ too. $\Box$

4. Since $\mathcal{PE}_3$ and $\mathcal{PE}_4$ are closed in $\mathcal{PE}$, we obtain that $\mathcal{PE}_5 = \mathcal{PE}_3 \cap \mathcal{PE}_4$ is closed in $\mathcal{PE}$, $\mathcal{PE}_3$ and $\mathcal{PE}_4$. $\Box$

Proof of Theorem 2

1. By definition, $\mathcal{TPE}$ is wide in $\mathcal{PE}$.

Suppose $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{PE}$. By Theorem 1 from [2], the function $\tau(t)$ is non-degenerated, so we can consider the inverse function $t(\tau)$. The map $(\tau,y,v) \to (t(\tau),y,v)$ is an isomorphism in $\mathcal{PE}$. Note that the superposition of F with this isomorphism is a morphism in $\mathcal{TPE}$. Therefore $\mathcal{TPE}$ is plentiful in $\mathcal{PE}$. $\Box$

2. $\mathcal{TPE}_k$ is closed in $\mathcal{PE}$, while $\mathcal{TPE}$ is wide and plentiful in $\mathcal{PE}$. Thus $\mathcal{TPE}_k = \mathcal{PE}_k \cap \mathcal{TPE}$ is closed in $\mathcal{TPE}$ and also it is wide and plentiful in $\mathcal{PE}_k$. $\Box$

Proof of Theorem 3

Using system (4), we see that the map $(t, x, u) \to (t, y, \varphi u + \psi)$ is a morphism in $\mathcal{QPE}$ if and only if

$$\begin{cases} B^{kl} = \sum\limits_{i,j} b^{ij} y_i^k y_j^l \\ B^k = \sum\limits_{i,j} b^{ij} y_{ij}^k + 2 \sum\limits_{i,j} b^{ij} (\ln \bar{\varphi})_j y_i^k + \sum\limits_i b^i y_i^k - y_t^k \\ Q\bar{\varphi} = \left(\sum\limits_{i,j} b^{ij} \bar{\varphi}_{ij} + \sum\limits_i b^i \bar{\varphi}_i - \bar{\varphi}_t \right) v + \left(\sum\limits_{i,j} b^{ij} \bar{\psi}_{ij} + \sum\limits_i b^i \bar{\psi}_i - \bar{\psi}_t \right) + q\left(t, x, \bar{\varphi} v + \bar{\psi}\right) \end{cases}, \tag{5}$$

where $\bar{\varphi} = \varphi^{-1}$, $\bar{\psi} = -\varphi^{-1}\psi$, so $U = \bar{\varphi} v + \bar{\psi}$. By definition, all subcategories of $\overline{\mathcal{QPE}}$ considered in the Theorem are full subcategories of $\overline{\mathcal{QPE}}$.

1a. If $c^{ij} = 0$ and v is linear in u, then $C^{kl} = 0$. It follows from the second equation of system (4) that $\mathcal{QPE}$ is closed in $\overline{\mathcal{QPE}}$.

1b. Let $F\colon \mathbf{A} \to \mathbf{B}$, $(t, x, u) \mapsto (t, y(t, x), v(t, x, u))$ be a morphism in $\mathcal{TPE}_1$, and $\mathbf{A}, \mathbf{B} \in \mathrm{Ob}_{\mathcal{QPE}}$. Using the second equation of system (4), we get $(\ln U_v)_v B^{kl} = C^{kl} = 0$. It follows that U is linear in v, v is linear in u, F is a morphism in $\mathcal{QPE}$, and $\mathcal{QPE}$ is full in $\mathcal{TPE}$.

1c. Suppose $\mathbf{A} \in \mathrm{Ob}_{\mathcal{TPE}_1}$. Fix $u_0 \in \Omega_{\mathbf{A}}$ and consider the map $F\colon (t, x, u) \mapsto (t, x, v(t, x, u))$, where

$$v(t, x, u) = \int\limits_{u_0}^{u} \exp\left(\int\limits_{u_0}^{\xi} \lambda(t, x, \varsigma)\, d\varsigma \right) d\xi.$$

F defines an isomorphism in $\mathcal{TPE}_1$ from $\mathbf{A}$ to $\mathbf{B}$ with

$$C^{ij} = (\ln U_v)_v b^{ij} + U_v \lambda b^{ij} = v_u^{-1} \left(\lambda - (\ln v_u)_u\right) = 0.$$

Therefore every object of $\mathcal{TPE}_1$ is isomorphic in $\mathcal{TPE}_1$ to some object of $\mathcal{QPE}$, and $\mathcal{QPE}$ is full in $\mathcal{TPE}_1$. □

2. The image of a compact under a continuous map is compact. The surjectivity of the map completes the proof. □

3. $\mathcal{TPE}_3$ is closed in $\mathcal{PE}_1$, $\mathcal{QPE}$ is fully dense in $\mathcal{PE}_1$. □

4. $\mathcal{TPE}_5$ is closed in $\mathcal{TPE}_3$, and $\mathcal{QPE}'$ is fully dense in $\mathcal{TPE}_3$. Equality $\mathcal{QPE}'_1 = \mathcal{TPE}_5 \cap \mathcal{QPE}'$ completes the proof.

5. Let $\mathbf{A} \in \mathrm{Ob}_{\mathcal{QPE}''}$, and suppose $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{QPE}'$. From the first equation of system (5) we obtain

$$a(t, x, u) = A(t, y, v)\bar{a}(t, x), \tag{6}$$

where $\bar{a}(t, x) = B^{11}(t, y(t, x)) \Big/ \left(\sum_{i,j} b^{ij}(t, x) y_i^1 y_j^1(t, x) \right)$.

From the second equation of (5) we obtain

$$B^k(t, y, v) = A(t, y, v)\omega^k(t, x) + \mu^k(t, x), \tag{7}$$

where

$$\omega^k(t, x) = \bar{a} \left(\sum_{i,j} \bar{b}^{ij} y_{ij}^k + 2 \sum_{i,j} \bar{b}^{ij} (\ln \bar{\varphi})_j y_i^k + \sum_i \bar{b}^i y_i^k \right), \quad \mu^k(t, x) = \sum_i \xi^i y_i^k - y_t^k.$$

Further we will need the following statement:

Lemma 1 ((about the extension of a function))**.** *Suppose M, N are C^r-manifolds, $1 \le r \le \infty$, $F\colon M \to N$ is a surjective C^r-submersion, $\mu\colon M \to \mathbb{R}$ is a C^s-function, $0 \le s \le r$ (if $s = 0$, then μ is continuous). Take*

$$N_0 = \left\{ n \in N\colon \quad \mu|_{F^{-1}(n)} = \mathrm{const} \right\},$$

$$M_0 = F^{-1}\left(N_0\right) = \left\{ m \in M\colon\ \forall m' \in M\ \left[F\left(m'\right) = F\left(m\right)\right] \Rightarrow \left[\mu\left(m'\right) = \mu\left(m\right)\right] \right\},$$

$F_0 = F|_{M_0}$, $\mu_0 = \mu|_{M_0}$, and define a function $\nu_0\colon N_0 \to \mathbb{R}$ by the formula $\nu_0 F_0 = \mu_0$ (see Fig. 3(a)). Then ν_0 can be extended from N_0 to the entire manifold N so that the extended function $\nu\colon N \to \mathbb{R}$ has class C^s of smoothness (see Fig. 3(b); both diagrams Fig. 3(a, b) are commutative).

$$\begin{tikzcd}
M_0 \arrow[rr, "F_0"] \arrow[dd, tail] \arrow[dr, "\mu_0"'] & & N_0 \arrow[dd, tail] \arrow[dl, "\nu_0"] \\
& \mathbb{R} & \\
M \arrow[ur, "\mu"] \arrow[rr, two heads, "F"] & & N
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
M_0 \arrow[rr, "F_0"] \arrow[dd, tail] \arrow[dr, "\mu_0"'] & & N_0 \arrow[dd, tail] \arrow[dl, "\nu_0"] \\
& \mathbb{R} & \\
M \arrow[ur, "\mu"] & & N \arrow[ul, "\nu"']
\end{tikzcd}$$

a   *b*

Figure 3: The extension of a function

Proof of Lemma 1

Take an open covering $\{V_i\colon i \in I\}$ of N such that for every V_i there is a C^r-smooth section $p_i\colon V_i \to M$ over V_i, $F \circ p_i = \mathrm{id}|_{V_i}$ (such a covering exists, because F is submersive and surjective). Let $\{\lambda_i\}$ be a C^r-partition of unity subordinated to $\{V_i\}$ [1, section 2.2]. Let

$$\nu_i\left(n\right) = \left\{ \begin{array}{ll} \lambda_i\left(n\right)\mu\left(p_i\left(n\right)\right), & n \in V_i \\ 0, & n \notin V_i \end{array} \right. .$$

Then $\nu\left(n\right) = \sum\limits_{i \in I} \nu_i\left(n\right)$ is a desired function. $\Box$

Proof of Theorem 3 (continuation)

Fix k. In the notations and assumption of Lemma 1, replace F by the map $(t, x) \mapsto (t, y(t, x))$ and the continuous function μ by $\mu^k(t, x)$. We obtain that there exists a continuous function $\nu^k(t, y)$ satisfying the following property for each (t_0, y_0): if $\mu^k(t, x)$ is constant on the inverse image of (t_0, y_0) with respect to the map $(t, x) \mapsto (t, y(t, x))$, then $\nu^k\left(t_0, y_0\right)$ coincides with this constant. Let now

$$\bar{B}^k(t, y, v) = \left(B^k(t, y, v) - \nu^k(t, y)\right) / A(t, y, v)\,. \tag{8}$$

Consider the following two cases for every point (t_0, y_0):

Case 1: The function $A(t_0, y_0, v)$ is independent of v. Then (7) implies that $B^k(t_0, y_0, v)$ is independent of v; (8) implies that $\bar{B}^k$ is independent of v.

Case 2: For given (t_0, y_0) the set $\{A(t_0, y_0, v)\colon v \in \Omega\}$ contains more than one element. Then (7) implies that the restriction of $\mu^k(t_0, x)$ to the inverse image of a point (t_0, y_0) is constant. Thus $\mu^k(t_0, x) = \nu^k(t_0, y_0)$ on this inverse image, and $\bar{B}^k = \omega^k(t, x)$ is independent of v in this case too.

In both cases $B^k(t, y, v) = A(t, y, v)\bar{B}^k(t, y) + \nu^k(t, y)$. So, the equation **B** has the form

$$v_t = A(t, y, v)\left(\sum_{k,l} \bar{B}^{kl}(t, y)v_{kl} + \sum_k \bar{B}^k(t, y)v_k\right) + \sum_k \nu^k(t, y)v_k + Q(t, y, v),$$

and $\mathbf{B}$ is an object of $\mathcal{QPE}''$.

F is a morphism in $\mathcal{QPE}''$ if and only if the following system holds; we will use this system in the proof of the rest of the theorem.

$$\begin{cases} a(t,x,u) = A(t,y,v)\bar{a}(t,x) \\ \bar{B}^{kl}(t,y) = \bar{a}\sum_{i,j}\bar{b}^{ij}y_i^k y_j^l(t,x) \\ y_t^k + \Xi^k - \sum_i \xi^i y_i^k = a(t,x,u)\left(\sum_{i,j}\bar{b}^{ij}y_{ij}^k + 2\sum_{i,j}\bar{b}^{ij}\left(\ln\bar{\varphi}\right)_j y_i^k + \sum_i \bar{b}^i y_i^k - B^k/\bar{a}\right) \\ Q\bar{\varphi} = \left(\sum_{i,j} a\bar{b}^{ij}\bar{\varphi}_{ij} + \sum_i \left(a\bar{b}^i + \xi_i\right)\bar{\varphi}_i - \bar{\varphi}_t\right)v + \\ \quad + \left(\sum_{i,j} a\bar{b}^{ij}\bar{\psi}_{ij} + \sum_i \left(a\bar{b}^i + \xi_i\right)\bar{\psi}_i - \bar{\psi}_t\right) + q\left(t,x,\bar{\varphi}v + \bar{\psi}\right) \end{cases} \tag{9}$$

6. $\mathcal{QPE}_1''$ is closed in $\mathcal{QPE}''$ and in $\mathcal{QPE}_1'$, because $\mathcal{QPE}''$ and $\mathcal{QPE}_1'$ are closed in $\mathcal{QPE}'$. $\mathcal{QPE}_1''$ is closed in $\mathcal{QPE}_0''$, because $\mathcal{QPE}_0''$ is the subcategory of $\mathcal{QPE}''$. □

7. Suppose $\mathbf{A} \in \mathrm{Ob}_{\mathcal{QPE}_{1q}''}$, and $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{QPE}_1''$. From the third equation of (5) we get

$$Q(t,y,v) =$$
$$\left(\sum_{i,j} b^{ij}\bar{\varphi}_{ij} + \sum_i b^i\bar{\varphi}_i + q_1(t,x) - \bar{\varphi}_t\right)\bar{\varphi}^{-1}v + \left(\sum_{i,j} b^{ij}\bar{\psi}_{ij} + \sum_i b^i\bar{\psi}_i + q_0(t,x) - \bar{\psi}_t\right)\bar{\varphi}^{-1} =$$
$$Q_1(t,x)v + Q_0(t,x),$$

so Q_1, Q_0 are functions of (t,y), and $\mathbf{B} \in \mathrm{Ob}_{\mathcal{QPE}_{1q}''}$. Thus $\mathcal{QPE}_{1q}''$ is closed in $\mathcal{QPE}_1''$. □

8. Suppose $\mathbf{A} \in \mathrm{Ob}_{\mathcal{QPE}_n'}$, $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{QPE}'$. For given (t_0, y_0) let us fix arbitrary x_0 such that $y(t_0, x_0) = y_0$. Since $a \in \mathcal{A}_{nc}(T \times X)$, from (6) we get

$$A(t_0, y_0, v) = a\left(t_0, x_0, \bar{\varphi}(t_0, x_0)v + \bar{\psi}(t_0, x_0)\right)\bar{a}(t_0, x_0) \neq \mathrm{const}.$$

Finally, we obtain $A \in \mathcal{A}_{nc}(T \times Y)$, and $\mathbf{B} \in \mathrm{Ob}_{\mathcal{QPE}_n'}$, so $\mathcal{QPE}_n'$ is closed in $\mathcal{QPE}'$. □

9. Suppose $\mathbf{A} \in \mathrm{Ob}_{\mathcal{QPE}_{0n}''}$, $\mathbf{B} \in \mathrm{Ob}_{\mathcal{QPE}_n''}$. Substituting $\xi_i = 0$ in the third equation of (9), we get

$$y_t^k + \Xi^k(t,y) = a(t,x,u)\left(\sum_{i,j}\bar{b}^{ij}y_{ij}^k + 2\sum_{i,j}\bar{b}^{ij}\left(\ln\bar{\varphi}\right)_j y_i^k + \sum_i \bar{b}^i y_i^k - B^k/\bar{a}\right)(t,x).$$

Since left hand side is independent of u and $a \in \mathcal{A}_{nc}(T \times X)$, both sides of this equality vanish, and we get

$$y_t^k = -\Xi^k(t,y) \tag{10}$$

The function $y(t,x)$ satisfies the ordinary differential equation (10) with smooth right hand side, so for any t, t' the equality $y(t,x_1) = y(t,x_2)$ implies that $y(t',x_1) = y(t',x_2)$. Let 1-parameter transformation group $g_s\colon T \times Y \to T \times Y$ be given by $(t, y(t,x)) \mapsto (t+s, y(t+s,x))$. This group is correctly defined when $T = \mathbb{R}$; otherwise transformations g_s are partially defined, nevertheless reasoning below remains correct after small refinement.

The composition $g_s g_{-s}$ is identity for every s , so g_s is bijective. $\{g_s\}$ is the flow map of the smooth vector field $\partial_t - \sum_k \Xi^k(t,y)\partial_{y^k}$, so transformations $\{g_s\}$ are smooth by both t and y.

Define the map $z(t,y)$ by the equality $g_{-t}(t,y) = (0, z(t,y))$. Then the map $G\colon T \times Y \to T \times Y$, $(t,y) \mapsto (t, z(t,y))$ is an isomorphism in $\mathcal{QPE}''$ such that $z(t, y(t,x)) = z(0, y(0,x))$ for every x, t. Therefore $G \circ F \in \mathrm{Hom}_{\mathcal{QPE}_0''}$. □

10. Suppose $\mathbf{A}$ is an object of $\mathcal{QPE}''_c$. Since X is compact, there exists a solution $y\colon T \times X \to X$ of the linear PDE $\partial y^k / \partial t = \sum_i \xi^i(t,x) \partial y^k / \partial x^i$. Then the isomorphism $(t,x,u) \mapsto (t, y(t,x), u)$ maps $\mathbf{A}$ to some object of $\mathcal{QPE}''_0$. Thus $\mathcal{QPE}''_{0c}$ is closed in $\mathcal{QPE}''_c$. $\square$

Proof of Theorem 4

If $a \neq \text{const}$, then $\mathcal{QPE}''_{0a}(a)$ is fully plentiful in $\mathcal{QPE}''_a(a)$ thanks to the part 9 of Theorem 3.

If $a = \text{const}$, then $\mathcal{QPE}''_a(a)$ coincides with $\mathcal{QPE}''_1$, which is closed in $\mathcal{QPE}''$ by Theorem 3. So $\mathcal{QPE}''_a(a)$ is fully plentiful in $\mathcal{QPE}''$.

Suppose now that $a \neq \text{const}$, $\mathbf{A} \in \mathrm{Ob}_{\mathcal{QPE}''_a(a)}$, and $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{QPE}''$. Let us see on equation (6) as a functional one:

$$a\left(\bar{\varphi}(t,x)v + \bar{\psi}(t,x)\right) = A(t,y,v)\bar{a}(t,x). \tag{11}$$

We have three cases:

Case 1. $a(u) = He^{\lambda u}$, $\lambda, H = \text{const}$, and $\lambda \neq 0$. Substituting $a(u)$ to (11), we get $\lambda\bar{\varphi}(t,x)v - \ln A(t,y,v) = \left(\ln \bar{a} - \lambda\bar{\psi} - \ln H\right)$. The right hand side of this equality is a function of (t,x), so $\bar{\varphi} = \bar{\varphi}(t,y)$, and the isomorphism $(t,y,v) \mapsto (t,y,\bar{\varphi}(t,y)v)$ maps $\mathbf{B}$ to some object of $\mathcal{QPE}''_a(a)$.

Case 2. $a(u) = H\left(u - u_0\right)^{\lambda}$, $\lambda, H, u_0 = \text{const}$, and $\lambda \neq 0$. Substituting $a(u)$ to (11), we get

$$\left(v + \bar{\varphi}^{-1}(t,x)\left(\bar{\psi}(t,x) - u_0\right)\right)^{\lambda} = A(t,y,v)H^{-1}\bar{\varphi}^{-\lambda}\bar{a}(t,x).$$

Thus $\bar{\varphi}^{-1}\left(\bar{\psi} - u_0\right) = q(t,y)$ for some function q, so the object $\mathbf{B}$ maps by the isomorphism $(t,y,v) \mapsto (t,y,v + q(t,y) + u_0)$ to some object of $\mathcal{QPE}''_a(a)$.

Case 3. Suppose now that $a(u)$ is neither $He^{\lambda u}$ nor $H\left(u - u_0\right)^{\lambda}$. Denote $\bar{x} = (t,x)$, $\bar{y} = (t,y)$, $\alpha = \ln a$. Fix arbitrary $\bar{y}_0 \in T \times Y$ and denote $Z = \{\bar{x}\colon \bar{y}\left(\bar{x}\right) = \bar{y}_0\} \subset T \times X$. Since (11), for any $\bar{x}_0, \bar{x}_1 \in Z$ and $\bar{\varphi}_i = \bar{\varphi}\left(\bar{x}_i\right)$, $\bar{\psi}_i = \bar{\psi}\left(\bar{x}_i\right))$ the value $\alpha\left(\bar{\varphi}_1 z + \bar{\psi}_1\right) - \alpha\left(\bar{\varphi}_0 z + \bar{\psi}_0\right)$ is independent of v. Let $G = G\left(\bar{y}_0\right)$ be the additive subgroup of $\mathbb{R}$ generated by the set $\{\ln\bar{\varphi}\left(\bar{x}\right) - \ln\bar{\varphi}\left(\bar{x}_0\right) : \bar{x} \in Z\}$.

We have the following two subcases.

Case 3.1: $G \neq \{0\}$. Put $\hat{H}_1 = \ln\bar{\varphi}_1 - \ln\bar{\varphi}_0 \in G - \{0\}$, $u_0 = \left(\bar{\psi}_0 - \bar{\psi}_1\right)/(\bar{\varphi}_1 - \bar{\varphi}_0)$. Substituting $v = \left(w + u_0 - \bar{\psi}_0\right)/\bar{\varphi}_0$, for any w we have $\alpha\left(e^{\hat{H}_1}w + u_0\right) - \alpha\left(w + u_0\right) = c = \text{const}$. Consider the function $\beta\left(x\right) = \alpha\left(e^x + u_0\right)$. Since $\beta\left(x + \hat{H}_1\right) = \beta(x) + c$, for $\lambda = c/\hat{H}_1$ the function $\beta\left(x\right) - \lambda x$ is $\hat{H}_1$-periodic. Therefore,

$$a(u) = \left(u - u_0\right)^{\lambda} H\left(\ln\left(u - u_0\right)\right),$$

where H is $\hat{H}_1$-periodic, $H \neq \text{const}$, since the case "$H = \text{const}$" have been considered above. Let $\hat{H} > 0$ be the smallest positive period of H. For all $\bar{x} \in Z$ the number $\ln\bar{\varphi}\left(\bar{x}\right) - \ln\bar{\varphi}_0$ is a multiple of $\hat{H}$, so $\bar{\varphi}\left(\bar{x}\right) \in \left\{\bar{\varphi}_0 \exp\left(k\hat{H}\right) : k \in \mathbb{Z}\right\}$ for any $\bar{y}_0$. Since $a(u)$ is independent of $\bar{y}_0$, $\hat{H}$ is independent of $\bar{y}_0$ too.

Case 3.2: $G = \{0\}$, that is $\bar{\varphi}|_Z \equiv \bar{\varphi}_0 = \text{const}$. Here we have two possible sub-subcases:

Case 3.2.a: $\bar{\psi}\big|_Z \neq \text{const}$, that is $\exists \bar{x}_0, \bar{x}_1 \in Z\colon \bar{\psi}\left(\bar{x}_1\right) - \bar{\psi}\left(\bar{x}_0\right) = \hat{H}_1 \neq 0$. Then $\alpha\left(u + \hat{H}_1\right) - \alpha(u) = \text{const}$. By the same token as in case 3.1 we get $a(u) = H(u)e^{\lambda u}$, where $\lambda = \text{const}$ and H is a periodic function with the smallest period $\hat{H} > 0$. Note that such a representation of $a(u)$ is unique. Substituting this to (11), we obtain that $\forall\bar{y}\ \forall\bar{x}_0, \bar{x}_1 \in Z_{\bar{y}}$ the number $\bar{\psi}\left(\bar{x}_1\right) - \bar{\psi}\left(\bar{x}_0\right)$ is a multiple of $\hat{H}$.

Case 3.2.b: $\bar{\psi}\big|_Z = \text{const}$ for given $\bar{y}_0$. We already considered the cases $a(u) = H(u)e^{\lambda u}$ and $a(u) = \left(u - u_0\right)^{\lambda} H\left(\ln\left(u - u_0\right)\right)$, so we can assume now without loss of generality that a is not of this form. Then for every $\bar{y}_0$ we have $\bar{\psi}\big|_Z = \text{const}$, $\bar{\varphi} = \bar{\varphi}\left(\bar{y}\right)$, and $\bar{\psi} = \bar{\psi}\left(\bar{y}\right)$. Thus the isomorphism $(t,y,v) \to \left(t, y, \bar{\varphi}(t,y)v + \bar{\psi}\left(t,y\right)\right)$ maps $\mathbf{B}$ to some object of $\mathcal{QPE}''_a\left(a\right)$.

The proof of the full density of $\mathcal{QPE}''_{0ca}(a)$ in $\mathcal{QPE}''_{ca}\left(a\right)$ is similar to the proof of part 10 in Theorem 3. $\square$

Proof of Theorem 5

1. $\mathcal{QPE}''$ is closed in $\overline{\mathcal{QPE}}$, and $\overline{\mathcal{SQPE}}$ is the subcategory of $\overline{\mathcal{QPE}}$. Therefore $\mathcal{SQPE}$ is closed in $\overline{\mathcal{SQPE}}$. □

2. $\mathcal{SQPE}_n$ is closed in $\overline{\mathcal{SQPE}}$ for the same reason as in Part 1 of this Theorem. This implies that $\mathcal{SQPE}_n$ is closed in $\mathcal{SQPE}$.

Suppose $\mathbf{A}$ is an object of $\mathcal{SQPE}_0$, $F\colon \mathbf{A}\to\mathbf{B}$ is a morphism in $\mathcal{SQPE}$. Then $B^k(t,y,v)=A(t,y,v)\omega^k(t,x)$, where ω^k is defined as in (7). Hence ω^k is a function of (t,y), and B is an object of $\mathcal{SQPE}_0$.

Suppose $\mathbf{A}$ is an object of $\mathcal{SQPE}_b$, $F\colon \mathbf{A}\to\mathbf{B}$ is a morphism in $\mathcal{SQPE}$. From the first equation of (5) we obtain

$$\frac{\bar{B}^{kl}}{\bar{B}^{11}}(t,y)=\frac{\sum_{i,j}\bar{b}^{ij}y_i^ky_j^l}{\sum_{i,j}\bar{b}^{ij}y_i^1y_j^1}(x).$$

The right hand side is independent of t, so it is a function of y; denote this function by $\bar{B}'^{kl}(y)$. Then $A\bar{B}^{kl}=A'(t,y,v)\bar{B}'^{kl}(y)$, where $A'=AB^{11}$. It follows that $\mathbf{B}$ is an object of $\mathcal{SQPE}_b$, and $\mathcal{SQPE}_b$ is closed in $\mathcal{SQPE}$. □

3. Let us recall that $\mathcal{SQPE}_{0n}$ is closed in $\mathcal{QPE}''_{0n}$. So it is sufficient to prove that any morphism in $\mathcal{QPE}''_{0n}$ is also a morphism in $\mathcal{SQPE}_{0n}$. Suppose that $F\colon \mathbf{A}\to\mathbf{B}$ is a morphism in $\mathcal{QPE}''_{0n}$. Then $y_t^k(t,x)=A(t,y,v)\omega^k(t,x)$, where

$$\omega^k=-\bar{B}^k+\bar{a}\left(\sum_{i,j}\bar{b}^{ij}y_{ij}^k+2\sum_{i,j}\bar{b}^{ij}\left(\ln\bar{\varphi}\right)_j y_i^k+\sum_{i,j}\bar{b}^iy_i^k\right).$$

Since the left hand side of this equality is independent of v and $A\in\mathcal{A}_{nc}(Y)$, we conclude that $\omega^k=0$. Thus F is a morphism in $\mathcal{SQPE}_{0n}$. Finally, $\mathcal{SQPE}_{0n}=\mathcal{QPE}''_{0n}$, is closed in $\mathcal{QPE}''_0$ and is fully dense in $\mathcal{QPE}''_n$. □

4. $\mathcal{QPE}''_1$ is closed in $\overline{\mathcal{QPE}}$, so $\mathcal{SQPE}_1$ is closed in $\overline{\mathcal{SQPE}}$ and, consequently, is closed in $\mathcal{SQPE}_0$. □

5. The proof is similar to the proof of part 1 of Theorem 4. □

Proof of Theorem 6

From (5)-(6) and the fact that $\mathcal{SQPE}_b$ is closed in $\overline{\mathcal{SQPE}}$ it follows that the map $(t,x,u)\mapsto(t,y,\varphi u+\psi)$ is a morphism in $\mathcal{SQPE}$ with the source from $\mathcal{AQPE}$ if and only if the following conditions are satisfied:

$$\begin{cases}
A(t,y,v) & =a(x,u)\bar{a}(t,x)\\
\bar{B}^{kl}(y) & =\bar{a}(t,x)\nabla y^k\nabla y^l\\
B^k(t,y,v) & =A(t,y,v)\bar{B}^k(t,y)+C^k(t,y)=\\
 & =a(x,u)\left(\Delta y^k+\left(\eta+2\nabla\left(\ln\bar{\varphi}\right)\right)\nabla y^k\right)+\xi\nabla y^k\\
Q\bar{\varphi} & =\left(a\left(\Delta\bar{\varphi}+\eta\nabla\bar{\varphi}\right)+\xi\nabla\bar{\varphi}-\bar{\varphi}_t\right)v+\\
 & \quad+\left(a\left(\Delta\bar{\psi}+\eta\nabla\bar{\psi}\right)+\xi\nabla\bar{\psi}-\bar{\psi}_t\right)+q\left(t,x,\bar{\varphi}v+\bar{\psi}\right)
\end{cases}\tag{12}$$

1. Suppose $F\colon \mathbf{A}\to\mathbf{B}$ is a morphism in $\overline{\mathcal{AQPE}}$, $\mathbf{A}$ is an object of $\mathcal{AQPE}$. From the second equation of system (12) it follows that $\bar{a}=\bar{a}(x)$. Using the first equation of (12) and taking into account that $\bar{\varphi}$, $\bar{\psi}$ are independent of t, we see that $A=A\left(y,v\right)$ is independent of t. It follows from the third equation of (12) that B^k is independent of t, $B^k(y,v)=A(y,v)\bar{B}^k(t,y)+C^k\left(t,y\right)$. From this formula, by the same token as in the proof of part 4 of Theorem 3, we obtain existing of functions $\mathrm{H}^k(y)$, $\Xi^k(y)$ such that $B^k=A(y,v)\mathrm{H}^k(y)+\Xi^k(y)$. Substituting $u=\bar{\varphi}(x)v+\bar{\psi}(x)$ in the last equation of (12), we obtain that Q is independent of t. This implies that the target $\mathbf{B}$ of the morphism F has the form

$$v_t=A(y,v)\left(\sum_{k,l}\bar{B}^{kl}(y)v_{kl}+\sum_k\mathrm{H}^k(y)v_k\right)+\sum_k\Xi^k(y)v_k+Q(y,v).$$

We prove so far that $\mathbf{B}$ has such a form only locally. Nevertheless, we can lead it to an equation of the same form but with globally defined function $A(y,v)$, for example by the way described in Remark 2. Then quadratic form $\bar{B}^{kl}$ is defined on the whole manifold Y, so we can equip Y with a Riemannian metric $\bar{B}^{kl}$ and finally get $\mathbf{B} \in \mathrm{Ob}_{\mathcal{AQPE}}$. $\square$

2. $\mathcal{AQPE}_n = \mathcal{AQPE} \cap \mathcal{SQPE}_n$ is closed in $\mathcal{AQPE}$, because $\mathcal{SQPE}_n$ is closed in $\mathcal{SQPE}$.

Let $F\colon \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathcal{SQPE}_{bn}$, and both source and target of F are objects of $\mathcal{AQPE}_n$. Then $\bar{a}$ is independent of t, and

$$a\left(x, \bar{\varphi}(t,x)v + \bar{\psi}\,(t,x)\right) = A\,(y(x),v)\,\bar{a}(x). \tag{13}$$

Let $x = x_0$. Suppose that the set $\left\{\left(\bar{\varphi}\,(t,x_0)\,, \bar{\psi}\,(t,x_0)\right)\right\}$ has more than one element, and consider the intervals

$$I\,(v) = \left\{\left(\bar{\varphi}\,(t,x_0)\,v + \bar{\psi}\,(t,x_0)\right) : t \in T_{\mathbf{A}}\right\} \subseteq \mathbb{R}.$$

Then $a\,(x_0,u)$ is constant on any interval $u \in I\,(v)$, because the right hand side of (13) is independent of t. Note that $I\,(v)$ is a continuous function of v in the Hausdorff metric, and $\forall t\ \bar{\varphi}\,(t,x_0) \neq 0$. If at any v the interval $I\,(v)$ does not collapses into a point, then $a\,(x_0,u)$ is constant on $\bigcup I\,(v)$. But this contradicts to the condition $a \in \mathcal{A}_{nc}\,(X)$. Therefore $I\,(v_0)$ degenerates into a point at some v_0, $\bar{\varphi}\,(t,x_0)\,v_0 + \bar{\psi}\,(t,x_0) \equiv u_0$, so $\bar{\varphi}v + \bar{\psi} = \bar{\varphi}\,(t,x_0)\,(v - v_0) + u_0$. By the assumption, $\mathrm{card}\left\{\left(\bar{\varphi}\,(t,x_0)\,, \bar{\psi}\,(t,x_0)\right)\right\} > 1$, so the set $\{\bar{\varphi}\,(t,x_0)\}$ is non-degenerated interval. Therefore, $a\,(x_0,u)$ is constant on the sets $\{u < u_0\}$ and $\{u > u_0\}$. But this contradicts to the condition $a \in \mathcal{A}_{nc}(X)$ and continuity of a. This contradiction shows that for each x_0 the functions $\bar{\varphi}$, $\bar{\psi}$ are independent of t. Consequently F is a morphism in $\mathcal{AQPE}$, and $\mathcal{AQPE}_n$ is the full subcategory of $\mathcal{SQPE}_{bn}$. $\square$

3. Since $\mathcal{SQPE}_0$ and $\mathcal{SQPE}_1$ are closed in $\mathcal{SQPE}$, the subcategories $\mathcal{AQPE}_0$ and $\mathcal{AQPE}_1$ are closed in $\mathcal{AQPE}$. $\square$

4. If $a \notin \mathcal{A}_{\mathrm{exp}} \cup \mathcal{A}_{\mathrm{deg}}$, then $\mathcal{AQPE}_a(a)$ is plentiful in $\mathcal{AQPE}$ by the same arguments as used in the proof of part 1 of Theorem 4, after replacement of $\bar{x}$, $\bar{y}$ to x, y respectively. $\square$

5. Let $F\colon \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathcal{SQPE}_{na}(a)$, $\mathbf{A}$ be an object of $\mathcal{AQPE}_{na}(a)$. Then

$$a\left(\bar{\varphi}(t,x)v + \bar{\psi}(t,x)\right) = A(v)\bar{a}(x).$$

As we proved in part 2, the functions $\bar{\varphi}$, $\bar{\psi}$ are independent of t, F is a morphism in $\mathcal{AQPE}$, and

$$\mathbf{B} \in \mathrm{Ob}_{\mathcal{AQPE}} \cap \mathrm{Ob}_{\mathcal{SQPE}_{na}(a)} = \mathrm{Ob}_{\mathcal{AQPE}_{na}(a)}\,.$$

Since $\mathcal{AQPE}_{na}(a)$ is full in $\mathcal{AQPE}_n$, we see that F is a morphism in $\mathcal{AQPE}_{na}(a)$. $\square$

Proof of Theorem 7

1. $\mathcal{EPE}$ is closed in $\overline{\mathcal{EPE}}$, because $\mathcal{AQPE}$ is closed in $\overline{\mathcal{AQPE}}$. $\square$

2. $\mathcal{EPE}_n$, $\mathcal{EPE}_0$, $\mathcal{EPE}_1$ are closed in $\mathcal{EPE}$, because $\mathcal{AQPE}_n$, $\mathcal{AQPE}_0$, $\mathcal{AQPE}_1$ are closed in $\mathcal{AQPE}$. $\square$

Suppose $F\colon \mathbf{A} \to \mathbf{B}$ is a morphism in $\mathcal{EPE}$ and $\mathbf{A} \in \mathrm{Ob}_{\mathcal{EPE}_a(a)}$. Then the first equation of (5) has the form $A(y,u)\bar{B}^{kl}(y) = a(u)\nabla y^k \nabla y^l$. Hence $\nabla y^k \nabla y^l = g^{kl}(y)$ for some functions g^{kl}. For $\bar{B}^{kl} = g^{kl}(y)$ we have $A(y,u) = a(u)$. So $\mathbf{A}$ is an object of $\mathcal{EPE}_a(a)$, and $\mathcal{EPE}_a(a)$ is closed in $\mathcal{EPE}$. $\square$

3. The proof is similar to the proof of Theorem 3. $\square$

Acknowledgements. This work was partially supported by the RFBR grants 15-01-02352 and 15-51-06001 (Russia).

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