Solving Optimal Stopping Problem by Using Computer Algebra Systems

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Abstract. The article deals with optimal stopping problem for arbitrary sequence of bounded random values (finite horizon). We obtained recurrent relations in the form of a Bellman equation and established a criteria of optimality for a stopping moment. Using this results we study how computer algebra systems can be applied to solve optimal stopping problems. We used Maple 14 to solve optimal stopping problem for geometric random walk and managed to find explicit formulas for this problem. Also for geometric random walk explicit solution for any continuous piecewise linear convex downward reward function has been found.

Keywords: optimal stopping problem, computer algebra systems, random walk, explicit solution

1 Introdution

Optimal stopping problem arises in areas of statistics (sequential hypothesis testing), mathematical finance (American options, the change-point problem) and others fields (see [1–4,6–13]). In [2,4,6,7,12,13] the main results for problems with infinite horizon were obtained. In this paper we consider optimal stopping problem with finite horizon.

In [1] they suppose, that stopping region is separated from continuation region with one point only. Under the assumption optimal stopping problem is solved with use of Wiener-Hopf factorization.

In [6] many examples of explicit solution for optimal stopping problems with finite horizon can be found (see also [3, 4, 12]).

In [8, 9] they suppose, that: i) a Markov process is observed; ii) reward functions are monotonic and convex downward. In this case sufficient conditions, that stopping region is separated from continuation region with one point only, are established.

In this work we obtain recurrent relations in the form of a Bellman equation (Theorem 3) and establish a criteria of optimality for a stopping moment (Theorems 4 and 5). Note that in literature we found only sufficient conditions, under which value function satisfies recurrent relations in the form of a Bellman equation.

It's well known that optimal stopping problem can be reduced to a free boundary problem for a Bellman type equation [13]. But to solve a free boundary problem is a challenging task itself. One of our purposes was to study how computer algebra systems can be applied to solve optimal stopping problems. We used Maple 14 to solve optimal stopping problem for geometric random walk and managed to find explicit formulas for this problem. Also for geometric random walk explicit solution for any continuous piecewise linear convex downward reward function has been found.

2 The Optimal Stopping Problem

Suppose we have: i) $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathsf{P})$ – a stochastic basis [12]; ii) $N \in \mathbb{N}^+$ – horizon; iii) $\mathcal{T}_n^N$ – set of stopping moments τ regarding the filtration $(\mathcal{F}_n)_{0\leq n\leq N}$ [12], with values in $\{n, \ldots, N\}$; iv) $(f_n, \mathcal{F}_n)_{0\leq n\leq N}$ – sequence of bounded random variables (in mathematical finance – utility of an observer); v) $\mathcal{L}^0(\Omega, \mathcal{F}_0)$ – set of all P-a. s. bounded $\mathcal{F}_0$-measurable random variables [7, §A.7].

We will study the following problem

$$\mathsf{E}[f_{\tau\wedge N}|\mathcal{F}_0] \to \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^N}, \tag{1}$$

where $\mathsf{E}[\,\cdot\,|\mathcal{F}_0]$ is a conditional expectation taken regarding to σ-algebra $\mathcal{F}_0$ (see definition in [7], definition of essential supremum can be found in [5, 7]).

The problem (1) is called optimal stopping problem (see [12]) where utility of an observer is maximized.

We denote $v_0^N \triangleq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^N} \mathsf{E}[f_{\tau\wedge N}|\mathcal{F}_0]$.

Definition 1. *A pair* $\left(\tau^*, v_0^N\right) \in (\mathcal{T}_0^N, \mathcal{L}^0(\Omega, \mathcal{F}_0))$ *is said to be a solution of the problem* (1) *if* $v_0^N = \mathsf{E}[f_{\tau^*\wedge N}|\mathcal{F}_0]$ P*-a. s. Stopping moment* $\tau^* \in \mathcal{T}_0^N$ *is called an optimal stopping moment,* $\mathcal{F}_0$*-measurable random variable* v_0^N *is called value function at moment 0.*

3 Recurrent Relations for Value Function. Criteria of Optimality of a Stopping Moment

We use a stochastic dynamic programming method to obtain a value function. For any $n \in \{1, \ldots, N\}$ we denote

$$v_n^N \triangleq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_n^N} \mathsf{E}[f_{\tau\wedge N}|\mathcal{F}_n]. \tag{2}$$

Definition 2 ([12]). v_n^N *is called value function at a moment* n*.*

We study value function properties below.

Theorem 1. *Assume that* $\sup_{n\in N} ||f_n||_{\mathcal{L}^0(\Omega,\mathcal{F}_n)} \leq C$*. Then* $|v_n^N| \leq C$ P*-a. s.*

Proof. Obviously,

$$|v_n^N| \leq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_n^N} \mathsf{E}(|f_\tau|\,|\mathcal{F}_n) \quad \mathsf{P}\text{-a.s.} \tag{3}$$

As $f_\tau = \sum_{i=n}^{N} f_i 1_{\{\tau=i\}}$, we have $|f_\tau| \leq \sum_{i=n}^{N} |f_i| 1_{\{\tau=i\}} \leq C$ P-a.s.

Using previous preposition and (3) we obtain that $|v_n^N| \leq C$ P-a.s.

Theorem 2. *Under assumptions of Theorem 1 sequence* $(v_n^N, \mathcal{F}_n)_{0\leq n\leq N}$ *is a supermartingale.*

Proof. By definition of v_n^N we observe P-a.s.

$$v_n^N \triangleq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_n^N} \mathsf{E}[f_\tau|\mathcal{F}_n] \geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{n+1}^N} \mathsf{E}[f_\tau|\mathcal{F}_n] = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{n+1}^N} \mathsf{E}(\mathsf{E}[f_\tau|\mathcal{F}_{n+1}]|\mathcal{F}_n). \tag{4}$$

Using properties of essentual supremum for any n we obtain P-a.s.

$$v_n^N \geq \mathsf{E}(\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{n+1}^N} \mathsf{E}[f_\tau|\mathcal{F}_{n+1}]|\mathcal{F}_n) = \mathsf{E}(v_{n+1}^N|\mathcal{F}_n). \tag{5}$$

The proof is complete.

Let us study a recurrent relations describing evolution of value function in backward time.

Theorem 3. $(v_n^N, \mathcal{F}_n)_{0\leq n\leq N}$ *is a sequence of value functions if and only if it satisfies recurrent relations* P*-a. s.*

$$v_n^N = \max\left\{f_n; \mathsf{E}[v_{n+1}^N|\mathcal{F}_n]\right\}, \quad v_n^N|_{n=N} = f_N. \tag{6}$$

Proof. Necessity. Let us derive that if $(v_n^N, \mathcal{F}_n)_{0\leq n\leq N}$ is a sequence of value functions, then recurrent relations (6) are satisfied.

First we will prove that for any $n \in \{0, \ldots, N\}$ the following inequality is satisfied P-a.s.

$$v_n^N \geq \max\left\{f_n; \mathsf{E}[v_{n+1}^N|\mathcal{F}_n]\right\}. \tag{7}$$

Note, that:

- random variable $1_{\{\tau=n\}} f_n$ is $\mathcal{F}_n$-measurable;
- conditional expectation has a tower property. Due to the definition and properties of essentual supremum [5] for any $n \in \{0, \ldots, N\}$ we can conclude that P-a.s.

$$
\begin{aligned}
v_n^N = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_n^N} \mathsf{E}\left[\sum_{i=n}^{\tau\wedge N} 1_{\{\tau=i\}} f_i \middle| \mathcal{F}_n\right] \geq \\
\geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{n+1}^N} \left\{ 1_{\{\tau=n\}} f_n + 1_{\{\tau>n\}} \mathsf{E}\left[\mathsf{E}\left[\sum_{i=n+1}^{\tau\wedge N} 1_{\{\tau=i\}} f_i \middle| \mathcal{F}_{n+1}\right] \middle| \mathcal{F}_n \right] \right\} = \\
= 1_{\{\tau=n\}} f_n + 1_{\{\tau>n\}} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{n+1}^N} \mathsf{E}\left[\mathsf{E}\left[f_{\tau\wedge N} | \mathcal{F}_{n+1}\right] | \mathcal{F}_n \right] = \\
= 1_{\{\tau=n\}} f_n + 1_{\{\tau>n\}} \mathsf{E}\left[\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{n+1}^N} \mathsf{E}\left[f_{\tau\wedge N} | \mathcal{F}_{n+1}\right] \middle| \mathcal{F}_n \right] = \\
= 1_{\{\tau=n\}} f_n + 1_{\{\tau>n\}} \mathsf{E}\left[v_{n+1}^N | \mathcal{F}_n \right]. \quad (8)
\end{aligned}
$$

Left part does not depend on a stopping moment $\tau \in \mathcal{T}_n^N$. If we apply ess sup by $\tau \in \mathcal{T}_n^N$ to a right part of (8), we will obtain (7).

Let us derive that the following inequality is satisfied P-a. s.

$$
v_n^N \leq \max\left\{ f_n; \mathsf{E}[v_{n+1}^N | \mathcal{F}_n] \right\}. \quad (9)
$$

Due to the fact that random variable $1_{\{\tau=n\}} f_n$ is $\mathcal{F}_n$-measurable, definitions of ess sup and v_{n+1}^N, tower property of conditional expectation we can conclude that P-a. s.

$$
\begin{aligned}
v_n^N = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_n^N} \left\{ 1_{\{\tau=n\}} f_n + 1_{\{\tau>n\}} \mathsf{E}\left[\mathsf{E}\left[f_{\tau\wedge N} | \mathcal{F}_{n+1}\right] | \mathcal{F}_n \right] \right\} \leq \\
\leq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_n^N} \left\{ 1_{\{\tau=n\}} f_n + 1_{\{\tau>n\}} \mathsf{E}\left[\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{n+1}^N} \mathsf{E}\left[f_{\tau\wedge N} | \mathcal{F}_{n+1}\right] \middle| \mathcal{F}_n \right] \right\} = \\
= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_n^N} \left\{ 1_{\{\tau=n\}} f_n + 1_{\{\tau>n\}} \mathsf{E}\left[v_{n+1}^N | \mathcal{F}_n \right] \right\} = \max\left\{ f_n; \mathsf{E}[v_{n+1}^N | \mathcal{F}_n] \right\}. \quad (10)
\end{aligned}
$$

Obviously, $v_n^N|_{n=N} = \mathsf{E}[f_N^N | \mathcal{F}_N] = f_N$ P-a. s. Hence with (7) and (9) we obtain (6).

Sufficiency. Proof of sufficiency is well known and can be found in [2,7,13].

Remark 1. As we stated above, proof of sufficiency in Theorem 3 is well known. Note, that necessity is new.

Using Theorem 3 we obtain necessary and sufficient conditions, under which a pair $\left(\tau^*, v_0^N\right) \in (\mathcal{T}_0^N, \mathcal{L}^0(\Omega, \mathcal{F}_0))$ is a solution of the problem (1).

Theorem 4. *Stopping moment τ^* is an optimal stopping moment if and only if $\tau^* = \min\left\{0 \leq n \leq N : f_n = v_n^N\right\}$.*

Proof. Let us define $\tau^* = \min\left\{0 \leq n \leq N : v_n^N = f_n\right\}$. Obliviously, $v_{\tau^*}^N = f_{\tau^*}$ P-a. s. Suppose the stopping moment τ^* is not an optimal stopping moment, i. e. there exists a stopping moment τ such that $\tau \geq \tau^*$.

Note that sequence $\{v_n^N\}$ is a supermartingale with respect to the measure P (Theorem 2). Due to Doob-Meyer decomposition theorem for any $n \in N_0$ it follows that

$$v_n^N = v_0 + M_n - A_n, \tag{11}$$

where M_n is a martingale with respect to the probability measure P, A_n is an increasing sequence. Obviously, $v_{\tau^*}^N = \sum_{i=0}^N v_i^N 1_{\{\tau^*=i\}}$ and $v_\tau^N = \sum_{i=0}^N v_i^N 1_{\{\tau=i\}}$. Hence with (11) we can conclude $v_{\tau^*}^N = v_0 + M_{\tau^*} - A_{\tau^*}$ and $v_\tau^N = v_0 + M_\tau - A_\tau$ P-a. s. From this two formulas we obtain

$$v_\tau^N = v_{\tau^*}^N + (M_\tau - M_{\tau^*}) - (A_\tau - A_{\tau^*}) \quad \mathsf{P}\text{-a. s.} \tag{12}$$

Due to the fact that $A_\tau - A_{\tau^*} \geq 0$ and from (12) we can conclude immediately that

$$v_\tau^N \leq v_{\tau^*}^N + (M_\tau - M_{\tau^*}) \quad \mathsf{P}\text{-a. s.} \tag{13}$$

Let us apply conditional expectation $\mathsf{E}\left[\,\cdot\,|\mathcal{F}_n\right]$ to (13). Thus we have P-a. s.

$$\mathsf{E}\left[v_\tau^N|\mathcal{F}_n\right] \leq \mathsf{E}\left[{}_{\tau^*}^N|\mathcal{F}_n\right] = \mathsf{E}\left[f_{\tau^*}|\mathcal{F}_n\right]. \tag{14}$$

As $\mathsf{E}\left[v_\tau^N|\mathcal{F}_n\right] \geq \mathsf{E}\left[f_\tau^N|\mathcal{F}_n\right]$ from (14) for any τ we obtain

$$\mathsf{E}\left[f_\tau^N|\mathcal{F}_n\right] \leq \mathsf{E}\left[f_{\tau^*}^N|\mathcal{F}_n\right]. \tag{15}$$

Note, that (15) is a necessary and sufficient condition of optimality for the stopping moment τ^*. The proof is complete.

Using theorems 3 and 4 we can derive another criterion of optimality for stopping moment $\tau^* \in \mathcal{T}_0^N$.

Theorem 5. *A pair* $\left(\tau^*, v_0^N\right) \in (\mathcal{T}_0^N, \mathcal{L}^0(\Omega, \mathcal{F}_0))$ *is a solution of the problem* (1) *if and only if:*

1. *sequence* $(v_n^N, \mathcal{F}_n)_{0 \leq n \leq \tau^* \wedge N}$ *is a martingale in respect to measure* P*;*
2. $v_n^N|_{n=\tau^*\wedge N} = f_{\tau^*\wedge N}$ P*-a. s.*

Proof. Sufficiency. Let $(v_n^N, \mathcal{F}_n)_{0 \leq n \leq \tau^* \wedge N}$ be a martingale with respect to measure P and $v_n^N|_{n=\tau^*\wedge N} = f_{\tau^*\wedge N}$ P-a. s. Hence with solution definition we can conclude required equalities

$$v_0^N = \mathsf{E}\left[v_{\tau^*\wedge N}^N|\mathcal{F}_0\right] = \mathsf{E}\left[f_{\tau^*\wedge N}|\mathcal{F}_0\right]. \tag{16}$$

Necessity. Let $\left(\tau^*, v_0^N\right)$ be a solution of the problem (1). From the proof of Theorem 3 we conclude that P-a. s.

$$\begin{aligned} v_n^N &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_n^N} (1_{\{\tau=n\}} f_n + 1_{\{\tau>n\}} \mathsf{E}\left[v_{n+1}^N|\mathcal{F}_n\right]) = \\ &= 1_{\{\tau^*=n\}} f_n + 1_{\{\tau^*>n\}} \mathsf{E}\left[v_{n+1}^N|\mathcal{F}_n\right]. \end{aligned} \tag{17}$$

As $v_{\tau^*}^N = f_{\tau^*}$, we have $1_{\{\tau^*=n\}} v_n^N = 1_{\{\tau^*=n\}} f_n$. By (6) and (17) we may find

$$v_n^N = \max\left\{f_n; \mathsf{E}[v_{n+1}^N|\mathcal{F}_n]\right\} \geq \mathsf{E}[v_{n+1}^N|\mathcal{F}_n] = \mathsf{E}[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{n+1}^N} \mathsf{E}(f_\tau|\mathcal{F}_{n+1})|\mathcal{F}_n] \geq$$
$$\geq \mathsf{E}[\mathsf{E}(f_{\tau^*}|\mathcal{F}_{n+1})|\mathcal{F}_n] = \mathsf{E}[f_{\tau^*}|\mathcal{F}_n]. \quad (18)$$

Let us apply conditional expectation $\mathsf{E}[\,\cdot\,|\mathcal{F}_0]$ to (18). Thus we have

$$\mathsf{E}[v_n^N|\mathcal{F}_0] \geq \mathsf{E}[f_{\tau^*}|\mathcal{F}_0] = v_0^N. \quad (19)$$

Moreover, $v_0^N = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^N} \mathsf{E}(f_\tau|\mathcal{F}_0) \geq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_n^N} \mathsf{E}(f_\tau|\mathcal{F}_n) = v_n^N$. So, we have

$$v_0^N \geq \mathsf{E}(v_n^N|\mathcal{F}_0). \quad (20)$$

So, from (19) and (20) it follows that $(v_n^N, \mathcal{F}_n)_{0\leq n\leq \tau^*\wedge N}$ is a martingale with respect to the measure P.

Remark 2. Unlike [7] and others, Theorem 4 is a criterion for optimal stopping moment.

4 Examples based on using Maple 14

Theorems 1–5 allow us to use computer algebra systems to solve the optimal stopping problem. In this section we present some new examples of solutions for optimal stopping problems based on using computer algebra system Maple 14.

Suppose that random sequence $(S_n, \mathcal{F}_n)_{0\leq n\leq N}$ satisfies the recursive relation $S_{n+1} = S_n(1+\rho_{n+1})$, $S_n|_{n=0} = S_0$ P-a. s., where $\{\rho_n\}_{0\leq n\leq N}$ is a sequence of jointly independent identically distributed random values taking values in the compact set of $\{a, b\}$, $a, b \in \mathbb{R}^1$, where $-1 < a < 0 < b < \infty$.

Let $p^* = \frac{|a|}{|a|+b}$, $q^* = 1 - p^*$. This means that random sequence $\{S_n\}_{0\leq n\leq N}$ is a martingale with respect to the measure P and fintration $(\mathcal{F}_n)_{0\leq n\leq N}$, where $\mathcal{F}_n = \sigma\{S_0, \dots, S_n\}$.

We can prove easily that v_n^N is a Markov random function. There exists Borel function $v_n^N(x)$ such as $v_n^N|_{x=S_n} = v_n^N$ and it satisfies the following formula

$$v_n^N(x) = \max(f_n(x),\, v_{n+1}^N(x(1+a))p^* + v_{n+1}^N(x(1+b))q^*). \quad (21)$$

Definition 3 ([12]). *For any* $n \in \{0, \dots, N\}$ *set* $\mathcal{D}_n \triangleq \left\{x \in R^+ : f_n(x) = v_n^N(x)\right\}$ *is called stopping region at a moment* n.

Example 1. Suppose that:

- for any $n \in \{0, \dots, N\}$ there is function $f_n(x) = \beta^n (x-K)^+$, where $K > 0$, $0 < \beta \leq 1$;
- $-1 < a < 0 < b < \infty$;

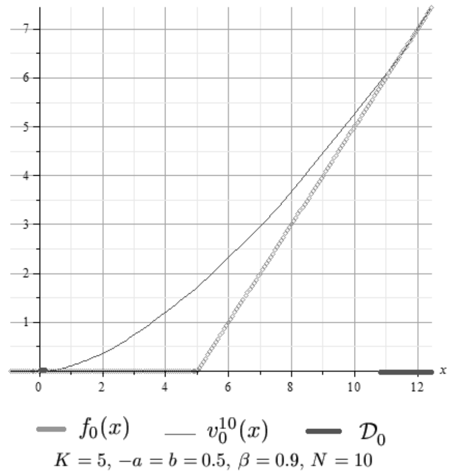

Fig. 1. Functions $f_0(x)$, $v_0^{10}(x)$ and stopping region $\mathcal{D}_0$

– for any $n \in \{0, \dots, N\}$ probabilities p^* and q^* are known (i. e. $\{S_n\}_{0 \le n \le N}$ is a martingale with respect to P and filtration $(F_n)_{0 \le n \le N}$).

This problem arises in trading of American options (see [1, 10, 13]).

Let $K = 5$, $-a = b = 0.5$, $\beta = 0.9$, $N = 10$. At Fig. 1 see plots of functions $f_0(x)$, $v_0^{10}(x)$ and stopping region $\mathcal{D}_0$.

We can conclude that in this example for any $n \in \{0, \dots, N\}$ stopping region has the following form: $\mathcal{D}_n = (0, x^*_{1,n}] \cup [x^*_{2,n}, \infty)$, where $x^*_{1,n}$, $x^*_{2,n} \in \mathbb{R}^+$ are stopping region's boundaries (see Fig. 2).

Example 2. Suppose for any $n \in \{0, \dots, N\}$:

$$f_n(x) = \begin{cases} 0, & x \le 1, \\ 1, & 1 < x \le 2, \\ 2, & 2 < x \le 3, \\ 3, & x > 3. \end{cases}$$

Let $a = -0.05$, $b = 0.05$, $p = q = 0.5$, $N = 10$. At Fig. 3 there are plots of functions $f_0(x)$, $v_0^{10}(x)$ and of stopping region $\mathcal{D}_0$.

5 Optimal Stopping Problem Solution for Any Piecewise Linear Continues Convex Downward Reward Function

In this section we solve optimal stopping problem with geometric random walk in a case of piecewise linear continues convex downward reward functions.

Assumption 1. Let function $f(x) = A_i^0 x + B_i^0$ be continues, where $x \in \left(c_i^0, c_{i+1}^0\right]$, $i = 0, \dots, T_0$ and for every i: $A_i^0 \le A_{i+1}^0$. Let $c_0^0 = 0$ and $c_{T_0} = \infty$.

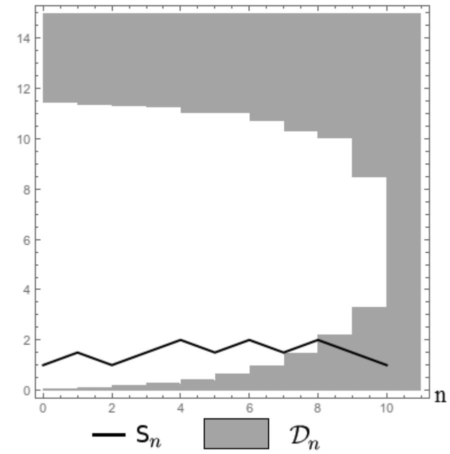

Fig. 2. Stopping regions in dependence of n

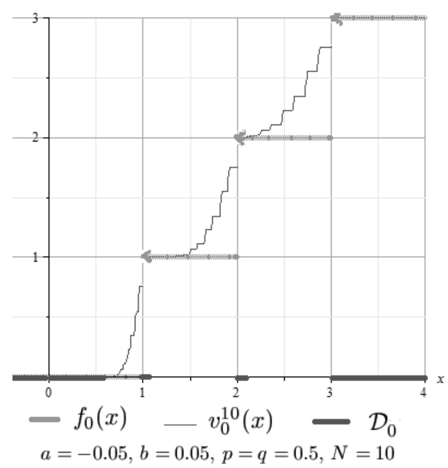

Fig. 3. Functions $f_0(x)$, $v_0^{10}(x)$ and stopping region $\mathcal{D}_0$

Theorem 6. *Under Assumption 1 the following is true.*

1. *For any $k \in \{0, \dots, N\}$:*

   (a) *functions $v_k^N(x)$ are piecewise linear, continues, convex downward and satisfy*

$$v_{k+1}^N(x) = A_j^{k+1} x + B_j^{k+1}, \quad x \in \left(c_j^{k+1}, c_{j+1}^{k+1}\right]$$

where

$$A_j^{k+1} = p^*(1+a)A_{i-t}^k + q^*(1+b)A_{i+m}^k, \quad B_j^{k+1} = p^*B_{i-t}^k + q^*B_{i+m}^k,$$
$$c_j^{k+1} = \max\left\{\frac{c_{i-t}^k}{1+a}; \frac{c_{i+m}^k}{1+b}\right\}, \quad c_{j+1}^{k+1} = \min\left\{\frac{c_{i-t+1}^k}{1+a}; \frac{c_{i+m+1}^k}{1+b}\right\},$$

if $c_j^{k+1} \le c_{j+1}^{k+1}$.

(b) Stopping region $\mathcal{D}^k$ *contains not more then* T_0 *intervals such that*

$$\left(0; \frac{c_1^0}{(1+b)^k}\right], \quad \left[\frac{c_{T_0}^0}{(1+a)^k}; \infty\right)$$

and

$$\left[\frac{c_{i-1}^0}{(1+a)^k}; \frac{c_i^0}{(1+b)^k}\right], \quad i = 2, \ldots, T_0$$

and there exists a number k^* *such that for any* $k \ge k^*$ *stopping region*

$$\mathcal{D}^k = \left(0; \frac{c_1^0}{(1+b)^k}\right] \cup \left[\frac{c_T^0}{(1+a)^k}; \infty\right).$$

2. (a) $v_\infty^N(x) = A^\infty x + B^\infty$, *where*

$$\min_{i=\overline{0,T_0}} A_i^0 \le A^\infty \le \max_{i=\overline{0,T_0}} A_i^0, \quad \min_{i=\overline{0,T_0}} B_i^0 \le B^\infty \le \max_{i=\overline{0,T_0}} B_i^0;$$

(b) $\mathcal{D}^\infty = \varnothing$.

The proof of the theorem is almost obvious, but large. That is why the paper does not contain it.

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