Crank-Nicolson Scheme for Space Fractional Heat Conduction Equation with Mixed Boundary Condition

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***Abstract*—This paper presents Crank-Nicolson scheme for space fractional heat conduction equation, formulated with Riemann-Liouville fractional derivative. Dirichlet and Robin boundary condition will be considered. To illustrate the accuracy of described method some computational examples will be presented as well.**

I. Introduction

In recent years the applications of mathematical models using the fractional order derivatives are very popular in technical science. Different types of phenomena in physics, biology, viscoelasticity, heat transfer, electrical engineering, control theory, fluid and continuum mechanics can be modeled by using the fractional order derivatives [6], [23], [10], [19], [33]. For example in paper [20] mathematical models of supercapacitors are considered. Authors investigated models based on electrical circuits in the form of RC ladder networks. These models are described using fractional order differential equations. Often we are not able to solve these models in an analytical way, so it is important to develop approximate methods of solving differential equations of fractional order.

Murio in paper [21] presents the implicit finite difference approximation for the time fractional diffusion equations with homogeneous Dirichlet boundary conditions, formulated by Caputo derivative. In paper [1], implicit finite difference method was used to solve time fractional heat conduction equation with Neumann and Robin boundary conditions. In these case also, the Caputo derivative was used. In paper [30] authors describes approximated solution of the fractional equation with Dirichlet and Neumann boundary conditions.

Paper [7] describes numerical method for diffusion equation with Dirichlet boundary conditions. To discretization fractional derivative authors used finite difference method and Kansa method. Paper [11] presents numerical solution of model describe by fractional differential equation with space fractional derivative and Dirichlet boundary conditions. Fractional derivative used in this model was Riemann-Liouville derivative. Authors used finite volume method.

Also Meerschaert and Tadjeran dealt with fractional differential equations [13], [12], [14], [35]. Paper [35] describes numerical solution of space fractional diffusion equation with boundary condition of the first kind, and in paper [12] authors presents finite difference method for two-dimensional fractional dispersion equation. In both papers, as the fractional derivative, the Riemann-Liouville derivative was used.

In this paper we present numerical solution of space fractional heat conduction equation. To the equation the Dirichlet and Robin boundary condition was added. In order to solve the equation, the Crank-Nicolson scheme was used. To illustrate the accuracy of described method some computational examples will be presented as well. The aim to achieve numerical solution of considered model is application it to solve inverse problems. Numerical solution of described model is called solution of direct problem. In the process of solving inverse problem, it is required to solve the direct problem many times. More about inverse problems of fractional order can be found in papers [2], [3], [4], [5]. In these papers, swarm optimization algorithms are introduced. More about intelligent algorithms and its application can be found in [24], [25], [26], [27], [28].

II. Formulation of the Problem

We discuss the following space fractional heat conduction equation

$$c\,\varrho\,\frac{\partial u(x,t)}{\partial t} = \lambda(x)\,\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + g(x,t), \tag{1}$$

defined in area $D = \{(x,t) : x \in [a,b], t \in [0,T)\}$, where $\lambda(x)$ is thermal conductivity coefficient, c and ϱ denote the specific heat and density. We assume that $\lambda(x) > 0$ and $\alpha \in (1,2]$. The initial condition is also posed

$$u(x,0) = f(x), \qquad x \in [a,b], \tag{2}$$

as well as the homogeneous Dirichlet and Robin boundary conditions

$$u(a,t) = q(t), \qquad t \in [0,T), \tag{3}$$

$$-\lambda(x)\frac{\partial u(b,t)}{\partial x} = h(t)(u(b,t) - u^{\infty}), \qquad t \in [0,T), \tag{4}$$

where h describes the heat transfer coefficient and u^{∞} is the ambient temperature.

Fractional derivative with respect to space, which occurs in equation (1), will be the Riemann-Liouville fractional derivative [23], [8] determined as follows

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^{n}}{\partial x^{n}} \int\limits_{a}^{x} u(s,t)(x-s)^{n-1-\alpha}\, ds, \quad (5)$$

where $\alpha \in (n-1, n]$ and $\Gamma(\cdot)$ is the Gamma function [34]. In case of $\alpha \in (1,2)$, the equation (1) describes super-diffusive process [15], [16], and for $\alpha = 2$, we get the classical heat conduction equation.

III. Numerical Solution

In this section we describe numerical solution of equation (1) using the finite difference method. Let $N, M \in \mathbb{N}$ be the size of grid in space and time, respectively. We denote grid steps $\Delta x = \frac{(b-a)}{N}$ and $\Delta t = T/M$. Therefore, we get following grid

$$S = \big\{(x_i, t_k),\ x_i = i\,\Delta x,\ t_k = k\,\Delta t, i = 0,1,\ldots,N, \\ k = 0,1,2,\ldots,M\big\}. \quad (6)$$

We assume the following notation $\lambda_i = \lambda(x_i)$, $g_i^k = g(x_i, t_k)$, $f_i = f(x_i)$, $h_k = h(t_k)$. Values of approximate function in points (x_i, t_k), we denoted by U_i^k.

In order to approximate fractional derivative (5), we used right-shifted Grünwald formula [17]

$$\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} = \frac{1}{\Gamma(-\alpha)} \lim_{N\to\infty} \frac{1}{r^{\alpha}} \sum_{j=0}^{N} \frac{\Gamma(j-\alpha)}{\Gamma(j+1)} u(x-(j-1)r, t), \quad (7)$$

where $N \in \mathbb{N}$ and $r = \frac{x-a}{N}$. We denote

$$\omega_{\alpha,j} = \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}. \quad (8)$$

Discretize the equation (1) and using approximation of fractional derivative, we get

$$\frac{U_i^{k+1} - U_i^k}{\Delta t} = \frac{\lambda_i}{2c\varrho(\Delta x)^{\alpha}} \Big(\sum_{j=0}^{i+1} \omega_{\alpha,j} U_{i-j+1}^{k+1} + \sum_{j=0}^{i+1} \omega_{\alpha,j} U_{i-j+1}^{k}\Big) + \frac{g_i^{k+\frac{1}{2}}}{c\varrho}. \quad (9)$$

Now, we approximate boundary condition of the third kind, so we obtain

$$U_N^k = \frac{\lambda_N U_{N-1}^k + \Delta x h^k u^{\infty}}{\lambda_N + \Delta x h^k}. \quad (10)$$

Having regard to both boundary condition, the equation (9) may be written in matrix form

$$(I_{N-1} - A_1)\underline{U}^{k+1} = (I_{N-1} + A_2)\underline{U}^k + \underline{G}^{k+\frac{1}{2}}\Delta t, \quad k = 0,1,2,\ldots, \quad (11)$$

where

$$\underline{U}^k = [U_1^k, U_2^k, \ldots, U_{N-1}^k]^T,$$

$$\underline{G}^{k+\frac{1}{2}}\Delta t = \Big[\frac{g_1^{k+\frac{1}{2}}\Delta t}{c\varrho}, \ldots, \frac{g_{N-2}^{k+\frac{1}{2}}\Delta t}{c\varrho},$$

$$\frac{g_{N-1}^{k+\frac{1}{2}}\Delta t}{c\varrho} + \frac{\lambda_{N-1}\Delta t u^{\infty}}{2(\Delta x)^{\alpha}}\Big(\frac{\Delta x h^k}{\lambda_N + \Delta x h^k} + \frac{\Delta x h^{k+1}}{\lambda_N + \Delta x h^{k+1}}\Big)\Big],$$

I_{N-1} is the identity matrix of size $N-1 \times N-1$, and matrices A_1 and A_2 are defined as follows

$$i = 1,2,\ldots,N-1, \quad j = 1,2,\ldots,N-1,$$

$$a_{ij}^1 = \begin{cases} \frac{\lambda_i \Delta t}{2c\varrho(\Delta x)^{\alpha}}\omega_{\alpha,i-j+1} & j \le i-1, \\ \frac{\lambda_i \Delta t}{2c\varrho(\Delta x)^{\alpha}}\omega_{\alpha,1} & j = i \ \wedge\ i \ne N-1, \\ \frac{\lambda_i \Delta t}{2c\varrho(\Delta x)^{\alpha}}(\omega_{\alpha,1} + \frac{\lambda_N}{\lambda_N + \Delta x h^{k+1}}) & j = i = N-1, \\ \frac{\lambda_i \Delta t}{2c\varrho(\Delta x)^{\alpha}}\omega_{\alpha,0} & j = i+1, \\ 0 & j > i+1, \end{cases}$$

$$a_{ij}^2 = \begin{cases} \frac{\lambda_i \Delta t}{2c\varrho(\Delta x)^{\alpha}}\omega_{\alpha,i-j+1} & j \le i-1, \\ \frac{\lambda_i \Delta t}{2c\varrho(\Delta x)^{\alpha}}\omega_{\alpha,1} & j = i \ \wedge\ i \ne N-1, \\ \frac{\lambda_i \Delta t}{2c\varrho(\Delta x)^{\alpha}}(\omega_{\alpha,1} + \frac{\lambda_N}{\lambda_N + \Delta x h^{k}}) & j = i = N-1, \\ \frac{\lambda_i \Delta t}{2c\varrho(\Delta x)^{\alpha}}\omega_{\alpha,0} & j = i+1, \\ 0 & j > i+1, \end{cases}$$

Solving systems of equations defined by (11), we obtain approximate values of temperatures in points of grid (6).

In paper [35] authors presents proof of unconditionally stability of presented method in case of homogeneous Dirichlet boundary condition. Doing similar proof it can be proven that described in this paper method is unconditionally stable.

IV. Experimental results

In this section we presents numerical examples of described method.

Example 1: Let consider equation (1), defined in area

$$D = \{(x,t) : x \in [0,1], t \in [0,1]\},$$

with following data

$$\alpha = 1.7, \quad \lambda(x) = \frac{1}{6}\Gamma(2.2)x^{2.8}, \quad c = \varrho = 1, \quad u^{\infty} = 50,$$

$$g(x,t) = -(1+x)x^3 e^{-t}, \quad h(t) = -\frac{0.550901 e^{-t}}{e^{-t} - 50}.$$

To equation we added an initial condition

$$u(x,0) = x^3, \quad x \in [0,1].$$

Exact solution of this problem is function

$$u(x,t) = x^3 e^{-t}.$$

Maximal and average absolute errors will be defined by formulas

$$\Delta_{\max} = \max_{\substack{0 \le i \le N \\ 1 \le k \le M}} |u_i^k - U_i^k|,$$

$$\Delta_{\text{avg}} = \frac{1}{(N+1)(M+1)} \sum_{i=0}^{N} \sum_{k=0}^{M} |u_i^k - U_i^k|.$$

Table I shows errors of approximate solution for different grids.

TABLE I. GRIDS, MAXIMAL ABSOLUTE ERRORS $\Delta_{\max}$ AND AVERAGE ABSOLUTE ERRORS Δ_{avg} (EXAMPLE 1)

Grid $N \times M$	$\Delta_{\max}$	Δ_{avg}
10×10	$6.24590 \cdot 10^{-2}$	$1.23295 \cdot 10^{-2}$
10×50	$6.20448 \cdot 10^{-2}$	$1.26218 \cdot 10^{-2}$
20×20	$3.09731 \cdot 10^{-2}$	$5.58958 \cdot 10^{-3}$
50×50	$1.23473 \cdot 10^{-2}$	$2.08623 \cdot 10^{-3}$
100×100	$6.16827 \cdot 10^{-3}$	$1.01756 \cdot 10^{-3}$
100×200	$6.16490 \cdot 10^{-3}$	$1.01786 \cdot 10^{-3}$
100×300	$6.16429 \cdot 10^{-3}$	$1.01808 \cdot 10^{-3}$
200×100	$3.08626 \cdot 10^{-3}$	$5.02667 \cdot 10^{-4}$
300×100	$2.05917 \cdot 10^{-3}$	$3.34012 \cdot 10^{-4}$

With increase in the first dimension N of the grid, a fixed second dimension M, errors decrease. For example, for $M = 100$ and $N = 100, 200, 300$ absolute errors not exceed $6.17 \cdot 10^{-3}$, $3.09 \cdot 10^{-3}$, $2.06 \cdot 10^{-3}$. Increasing the dimension of the grid with respect to time for a fixed dimension N insignificantly impact on errors of approximate solution.

Distribution of errors in points of the grid is presented on figure 1.

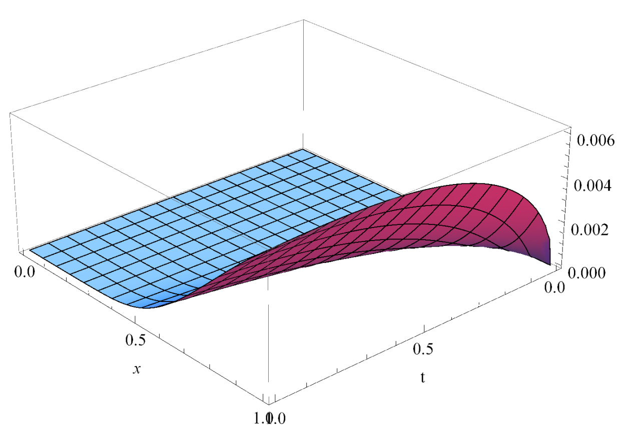

Fig. 1. Distribution of errors ($N = M = 100$) (example 1)

Figure 2 presents exact, approximate solutions and errors for time $t = 1$.

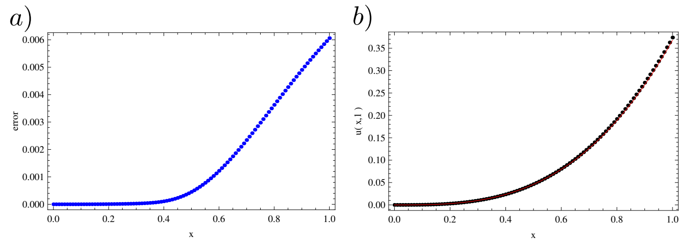

Fig. 2. Distribution of errors for approximate solution (a) and approximate solution (points), exact solution (solid line) (b) in time $t = 1$ ($N = M = 100$) (example 1)

Example 2: Let consider again equation (1) with following data

$$\alpha = 1.9, \quad \lambda(x) = \frac{1}{2}, \quad c = \varrho = 1, \quad u^{\infty} = 100, \quad a = 1, \quad b = 2,$$

$$g(x,t) = -2t(x-1) + 0.0525569(t^2-1)\Big(\frac{32.7273 - 32.7273x}{(x-1)^{1.9}} +$$

$$+ \frac{18.1818}{(x-1)^{0.9}} + \frac{15.5455 - 31.0909x + 15.5455x^2}{(x-1)^{2.9}}\Big),$$

$$h(t) = \frac{\frac{1}{2}(1-t^2)}{99+t^2}.$$

and with initial condition

$$u(x,0) = x - 1, \quad t \in [0,1].$$

Exact solution of this problem is given by function

$$u(x,t) = (x-1)(1-t^2).$$

Table II presents errors of approximate solution for different grids.

TABLE II. GRIDS, MAXIMAL ABSOLUTE ERRORS $\Delta_{\max}$ AND AVERAGE ABSOLUTE ERRORS Δ_{avg} (EXAMPLE 2)

Grid $N \times M$	$\Delta_{\max}$	Δ_{avg}
10×10	$7.14165 \cdot 10^{-4}$	$4.40379 \cdot 10^{-4}$
10×50	$6.25414 \cdot 10^{-4}$	$3.82786 \cdot 10^{-4}$
20×20	$4.16933 \cdot 10^{-4}$	$2.54149 \cdot 10^{-4}$
50×50	$2.11043 \cdot 10^{-4}$	$1.22171 \cdot 10^{-4}$
100×100	$1.2352 \cdot 10^{-4}$	$6.88743 \cdot 10^{-5}$
100×200	$1.23129 \cdot 10^{-4}$	$6.84358 \cdot 10^{-5}$
100×300	$1.23059 \cdot 10^{-4}$	$6.83813 \cdot 10^{-5}$
200×100	$7.64731 \cdot 10^{-5}$	$3.8691 \cdot 10^{-5}$
300×100	$5.90598 \cdot 10^{-5}$	$2.74809 \cdot 10^{-5}$

Similarly as in example 1, reducing the grid step Δx results in a significant reduction in average errors δ_{avg} and maximum errors Δ_{max} of approximate solution. Also, the density of the grid in the time domain results in a slight decrease in errors. Distribution of errors in the area of D, for example 2 is presented in the figure 3.

We also present distribution of errors for approximate solution in time $t = 1$ (figure 4).

Example 3: Now, we take the following data

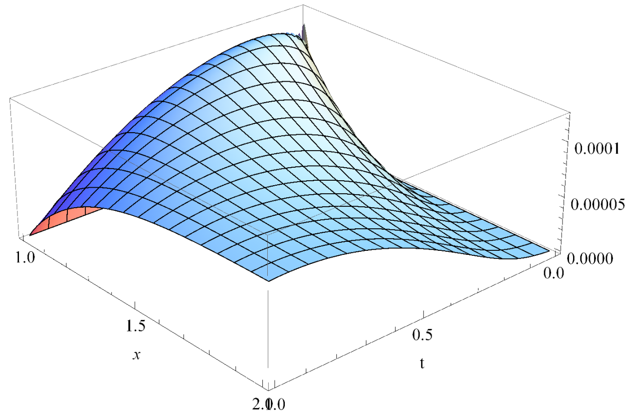

Fig. 3. Distribution of errors ($N = M = 100$) (example 2)

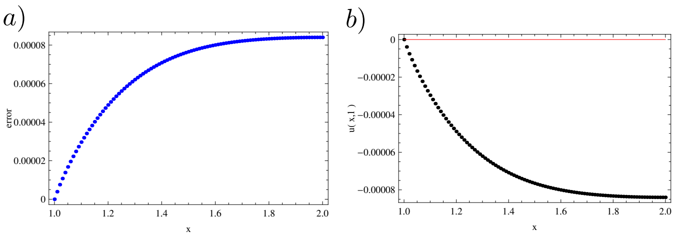

Fig. 4. Distribution of errors for approximate solution (a) and approximate solution (points), exact solution (solid line) (b) in time $t = 1$ ($N = M = 100$) (example 2)

$$\alpha = 1.6, \quad \lambda(x) = 1, \quad c = \varrho = 1, \ u^{\infty} = 120, \ a = 1, \ b = 2,$$

$$g(x,t) = -\frac{5.15228}{(x-1)^{0.6}} +$$

$$+\frac{t^3(0.0751374 + 1.87843(x-1) + 0.375687x)}{(x-1)^{0.6}} + 3xt^2(1-x),$$

$$h(t) = \frac{8 - 3t^3}{112 + 2t^3}, \quad f(x) = 8(x-1).$$

Exact solution of this problem is represented by function

$$u(x,t) = 2(x-1)(4 - \frac{1}{2}xt^3).$$

Just as it did in the previous examples, we will examine the errors depending on the density of the grid. In table III errors of approximate solutions are presented.

TABLE III. GRIDS, MAXIMAL ABSOLUTE ERRORS $\Delta_{\max}$ AND AVERAGE ABSOLUTE ERRORS Δ_{avg} (EXAMPLE 3)

Grid $N \times M$	$\Delta_{\max}$	Δ_{avg}
10×10	$1.2654 \cdot 10^{-1}$	$4.8953 \cdot 10^{-2}$
50×50	$2.48109 \cdot 10^{-2}$	$1.2146 \cdot 10^{-2}$
70×70	$1.77869 \cdot 10^{-2}$	$8.93166 \cdot 10^{-3}$
100×100	$1.25078 \cdot 10^{-2}$	$6.41806 \cdot 10^{-3}$
150×150	$8.38441 \cdot 10^{-3}$	$4.38665 \cdot 10^{-3}$

V. Conclusion

In this paper numerical solution of space heat conduction equation is presented. To the equation Robin and Neumann boundary conditions were added. Author used Crank-Nicolson scheme, and right-shifted Grünwald formula for approximation fractional Riemann-Liouville derivative. To illustrate the accuracy of presented method three examples are presented. In each example results are good. With an increase in the density of the grid, errors of approximate solution decreases.

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