Application of Boolean-valued models and FCA for the development of ontological models

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Abstract. The paper is devoted to the development of methods of ontology supported knowledge discovery in the field of mobile network subscribers. We develop the ontological model of the domain of mobile networks. This ontological model is intended to describe the behavior of mobile network subscribers. The ontological model is based on the four-level model of knowledge representation. In the paper, special attention is given to the third level which represents the set of cases from the domain. For the analysis of domain cases and for the generation of fuzzy knowledge about the domain we use Boolean-valued models, fuzzy models and Formal Concept Analysis. To describe the set of cases from the domain we use formal contexts; objects of these formal contexts are models representing sets of subscribers. We study formal contexts generated by Boolean-valued models, and formal concepts of these formal contexts. We obtain a description of theories of classes of domain cases in the language of formal concepts.

Keywords: Boolean-valued model, fuzzy model theory, ontology, ontological model, mobile networks, mobile network subscribers, FCA, formal context, formal concept

1 Introduction

The article is devoted to the development of an ontological model of the domain of mobile networks. The ontological model is based on the four-level model of knowledge representation [1, 2]. We apply model-theoretical methods [3], theory of fuzzy models [4] and FCA [5, 6] to construct the ontological model.

The present investigations continue the studies begun in [7, 8]. In [8] the ontological model of the domain of mobile networks was used to determine which tariffs and services of the mobile operator are most interesting and useful for a given mobile network user. In this article we expand the borders of our consideration of the domain. We are interested in revealing high-level characteristics of the subscriber: income level, social status, mobility, interests, preferences, etc. The elucidation of high-level characteristics is necessary to predict the behavior of the subscriber.

Thus we need to consider not only properties of individual subscribers, but also the interaction of subscribers (for example, calls between subscribers and so on). Therefore, in contrast to [8], we need to consider not only unary predicates, but also binary predicates to describe the domain ontology. A case model of this domain is represented as a set of countable algebraic systems.

Next, in the paper we describe a method for constructing a formal context for the case model (using the notion of a Boolean-valued model) and a method of transitioning to an object-clarified context.

This work is mainly theoretical. The results of practical implementation will be described in the next paper.

2 Ontological model of the domain of mobile networks

The ontological model of the domain of mobile networks is constructed on the basis of the four-level model of knowledge representation [1, 2]. We call a tuple

$$\boldsymbol{\mathcal{OM}} = \langle K, \sigma, T^a, T^s, T^f \rangle$$

an ontological model of the domain. Here K is the set of cases of the domain, σ is the signature of the domain, T^a is the analytic theory of the domain, T^s is the theory of the domain and T^f is the fuzzy theory of the domain.

The first step of the construction of the ontological model is the description of the ontology of the domain. From the model-theoretical point of view, the construction of the domain ontology consists of describing a signature σ (i.e. the set of key concepts) and a set of axiom $\mathcal{A}x_a$ of the given domain [9, 10]. The pair $\langle \sigma, \mathcal{A}x_a \rangle$ generates the analytic theory T^a, i.e. description of sense of the key concepts of the domain, definitions of used notions. The theory of the domain T^s represents general (universal) knowledge of the domain.

We consider six classes of concepts to determine the signature $\sigma_{\mathbb{M}}$ of the object domain $\mathbb{M}$ = "Mobile networks":

1. σ_P represents individual indicators of subscribers, and contains two parts: σ_{P_1} are traffics and σ_{P_2} are accruals.
2. σ_Q represents different tariff plans and services.
3. σ_R represents properties and characteristics of various tariff plans, services and options; for example, the number of free minutes of talk, the volume of the SMS package, etc.
4. σ_I represents concepts which express different interests of subscribers.
5. σ_S represents concepts which express subscriber's social status.
6. σ_T represents concepts which express relationships and interactions between subscribers: calls / SMS / MMS of one subscriber to another and so on.

Note that the concepts from the classes $\sigma_P, \sigma_Q, \sigma_R, \sigma_I, \sigma_S$ represent different properties of subscribers; these concepts are formalized as unary predicates. The concepts from the class σ_T represent relationships and interactions between subscribers; these concepts are formalized as binary predicates.

The presented classes of concepts have a hierarchical structure which is described by a set of axioms "*hyponym-hypernym*". For these classes of concepts we present *axioms of completeness* and *axioms of including*. A detailed description of these axioms is given in [8]. We present a set of *axioms of irreflexivity* for the class of concepts σ_T. For example, for any concept $Q(x, y) \in \sigma_T$ we formulate an axiom

$\neg Q(x, x)$ which means that no subscriber can call to himself. This concludes the first step of the ontological model's constructing.

At the second step of the construction of the ontological model, we present the set $\mathcal{A}x_s$ of general statements about the domain. The set $\mathcal{A}x_s$ is considered to be true at the moment but may be changed at the future time. This knowledge is synthetic, in contrast to the analytical knowledge presented in the ontology. The synthetic knowledge does not follow from the meaning of the terms used in the description of the domain. The truth value of this knowledge depends on the present state of the real world. An example of such knowledge is the statement that a given tariff and a specific service of the mobile operator are not compatible.

The set of all axioms $\mathcal{A}x_a \cup \mathcal{A}x_s$ generates the theory of the domain T^s, i.e., a set of statements which are true in the given domain. This concludes the second step of the ontological model's constructing.

The third step of the building the ontological model is a formalization of empirical knowledge about the domain, i.e., knowledge about the concrete precedents (cases) of the given object area. In the domain $\mathbb{M}$ the set A of subjects is the set of various individuals and organizations using mobile network services, etc. Note that the set of subjects A is finite at each point of time, but it is constantly changing in dynamics. Therefore we may consider the set A as potentially infinite. Thus in this paper we consider a countable set A of subjects.

Let us consider a set $A = \{a_1, a_2, \dots\}$ of the subjects of the domain. Each ***case*** (instance) of the domain $\mathbb{M}$ determines an algebraic system $\mathfrak{A} = \langle A, \sigma_{\mathbb{M}} \rangle$, i.e., defines the signature $\sigma_{\mathbb{M}}$ on the set A.

Further, to simplify the formalization, we will consider models $\mathfrak{A} = \langle A, \sigma_{\mathbb{M}} \rangle$ in the signature $(\sigma_{\mathbb{M}})_A$, enriched by constants for all elements of the model:

$$(\sigma_{\mathbb{M}})_A = \sigma_{\mathbb{M}} \cup \{c_a \mid a \in A\}.$$

So, for simplicity, we denote $\sigma = \sigma_{\mathbb{M}}$, $\sigma_A = (\sigma_{\mathbb{M}})_A$ and $\mathfrak{A}_A = \langle A, \sigma_A \rangle$.

Note that not every algebraic system $\mathfrak{A} = \langle A, \sigma_{\mathbb{M}} \rangle$ is the ***case*** of the domain $\mathbb{M}$. We say that a model $\mathfrak{A}_A$ is a domain's case if $\mathfrak{A}_A \vDash \mathcal{A}x_a \cup \mathcal{A}x_s$.

To solve different problems we may consider different sets of cases $K \subseteq \mathbb{K}(\sigma_A)$. For example, the set of cases K may describe temporal "slices" of the domain, or geolocation "slices" of the domain. Also, the set K may be used to describe the behavior of different groups of subscribers.

Empirical knowledge completely depends on the selection of the set of cases $K \subseteq \mathbb{K}(\sigma_A)$ of the domain and serves as a source of generating a new knowledge about the given domain. Below we will be construct the case model of the domain based on the selected set of cases K.

Using the presented formalization of the case model, we can identify high-level characteristics of mobile network subscribers and make portraits of mobile users' segments. To do this, we generate different Boolean-valued and fuzzy models (see Section 3) and construct corresponding formal contexts for given case models (see Section 4).

3 Preliminaries: Boolean-valued models and fuzzy models

In the present paper a model (an algebraic system) is a tuple $\mathfrak{A} = \langle A;\ P_1, \dots, P_n, c_1, \dots, c_k \rangle$. The set $|\mathfrak{A}| = A$ is called the universe of the model, $P_1, \dots, P_n$ are predicates defined on the set A and $c_1, \dots, c_k$ are constants. The tuple $\sigma = \langle P_1, \dots, P_n, c_1, \dots, c_k \rangle$ is called the signature of the algebraic system $\mathfrak{A}$.

A formula having no free variables is called a sentence. For a signature σ we denote:

$$\boldsymbol{F(\sigma) \leftrightharpoons \{\varphi \mid \varphi\ \text{is a formula of the signature}\ \sigma\}},$$
$$\boldsymbol{S(\sigma) \leftrightharpoons \{\varphi \mid \varphi\ \text{is a sentence of the signature}\ \sigma\}}\ \text{and}$$
$$\boldsymbol{K(\sigma) \leftrightharpoons \{\mathfrak{A} \mid \mathfrak{A}\ \text{is a model of the signature}\ \sigma\}}.$$

For a sentence $\varphi \in S(\sigma)$ and a model $\mathfrak{A} \in K(\sigma)$ we denote $\mathfrak{A} \vDash \varphi$ if the sentence φ is true in the model $\mathfrak{A}$.

Definition 1 [11]. Let $\mathbb{B}$ be a complete Boolean algebra and $\tau: \mathrm{S}(\sigma_\mathrm{A}) \to \mathbb{B}$. A triple $\mathfrak{A}_\mathbb{B} = \langle A, \sigma_A, \tau \rangle$ is called a ***Boolean-valued model*** if the following conditions hold:

$$\tau(\neg\varphi) = \overline{\tau(\varphi)}; \quad \tau(\varphi \vee \psi) = \tau(\varphi) \cup \tau(\psi);$$
$$\tau(\varphi \,\&\, \psi) = \tau(\varphi) \cap \tau(\psi); \quad \tau(\varphi \to \psi) = \overline{\tau(\varphi)} \cup \tau(\psi);$$
$$\tau(\forall x \varphi(x)) = \bigcap_{a \in A} \tau(\varphi(c_a)); \ \tau(\exists x \varphi(x)) = \bigcup_{a \in A} \tau(\varphi(c_a)).$$

We use the notion of **a fuzzy model** to formalize the estimated knowledge of the object domain (the fourth level of the construction of the ontological model). Each fuzzy model is a special case of the **fuzzification** of some Boolean-valued model [12, 13].

Definition 2 [4]. Let $\mathfrak{A}_\mathbb{B} = \langle A, \sigma_\mathrm{A}, \tau \rangle$ be a Boolean-valued model, where $\tau: \mathrm{S}(\sigma_\mathrm{A}) \to \mathbb{B}$, and $h: \mathbb{B} \to [0,1]$ be a mapping for which $h(0) = 0$ and $h(1) = 1$. We define a valuation $\mu: \mathrm{S}(\sigma_\mathrm{A}) \to [0,1]$ as a composition $\mu(\varphi) = h(\tau(\varphi))$. A triple $\mathfrak{A}_\mu = \langle A, \sigma_A, \mu \rangle$ is called a ***fuzzification*** of the Boolean valued-model $\mathfrak{A}_\mathbb{B}$ via the mapping h.

Definition 3 [4]. The mapping $h: \mathbb{B} \to [0,1]$ is called an ***additive homomorphism*** if

(1) h preserves the order, i.e., h is a homomorphism $h: \mathbb{B} \to [0,1]$ of posets with constants 0 and 1;

(2) h is additive, i.e.,
$$a \cap b = 0 \ \Rightarrow \ h(a \cup b) = h(a) + h(b) \ \text{for any}\ a,\ b \in \mathbb{B}.$$

Definition 4 [4]. A ***fuzzy model*** $\mathfrak{A}_\mu = \langle A, \sigma_A, \mu \rangle$ is a fuzzification of a Boolean-valued model $\mathfrak{A}_\mathbb{B} = \langle A, \sigma_\mathrm{A}, \tau \rangle$ via an additive homomorphism.

4 Main results: formal contexts for Boolean-valued models

Remark 5. Let $\mathfrak{A}_{\mathbb{B}} = \langle A, \sigma_A, \tau\rangle$ be a Boolean-valued model and $\tau\colon S(\sigma_A) \to \mathbb{B}$. Consider the formal context $(\mathbb{B}, S(\sigma_A), I)$, where $b\, I\, \varphi \iff \tau(\varphi) = b$. Note that the non-empty extents of formal concepts are one-element sets.

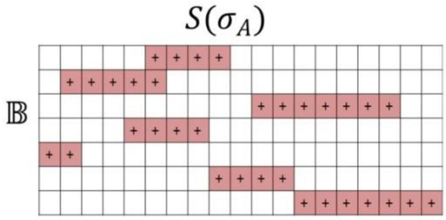

Fig. 1. The formal context $(\mathbb{B}, S(\sigma_A), I)$.

For a formal context (G, M, I) by $\underline{\mathfrak{B}}(G, M, I)$ we denote the lattice of formal concepts of the formal context (G, M, I).

Definition 6 [14]. Let $\mathfrak{A}_{\mathbb{B}} = \langle A, \sigma_A, \tau\rangle$ be a Boolean-valued model, where $\tau\colon S(\sigma_A) \to \mathbb{B}$. Consider an atom $b \in \mathrm{At}(\mathbb{B})$. Define a model $\mathfrak{A}_b \in \mathbb{K}(\sigma_A)$ by setting

$$\mathfrak{A}_b \vDash P(c_1, \ldots, c_n) \iff b \leq \tau\big(P(c_1, \ldots, c_n)\big)$$

for any $P, c_1, \ldots, c_n \in \sigma_A$.

Proposition 7 [14]. For the model $\mathfrak{A}_b$ and for an arbitrary sentence $\varphi \in S(\sigma_A)$, we have $\mathfrak{A}_b \vDash \varphi \iff b \leq \tau(\varphi)$.

A Boolean-valued model $\mathfrak{A}_{\mathbb{B}} = \langle A, \sigma_A, \tau\rangle$ is called ***atomic*** if the Boolean algebra $\mathbb{B}$ is atomic.

Definition 8. Let $\mathfrak{A}_{\mathbb{B}}$ be an atomic Boolean-valued model. Denote $At(\mathbb{B}) = \{a \in \mathbb{B} \mid a \text{ is an atom}\}$. Consider the formal context $(At(\mathbb{B}), S(\sigma_A), I_\tau)$, where

$$a\, I_\tau\, \varphi \iff a \leq \tau(\varphi).$$

Proposition 9. Let $(C, \Gamma) \in \underline{\mathfrak{B}}(At(\mathbb{B}), S(\sigma_A), I_\tau)$. Then Γ is a theory of the signature σ_A, i.e., for any $\varphi \in S(\sigma_A)$ if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$.

Proof. Let $(C, \Gamma) \in \underline{\mathfrak{B}}(At(\mathbb{B}), S(\sigma_A), I_\tau)$. Let us show that Γ is a theory. Suppose that $\varphi \in S(\sigma_A)$ and $\Gamma \vdash \varphi$. Prove that $\varphi \in \Gamma$. We have $C' = \Gamma$; consequently, we must only show that $\varphi \in C'$.

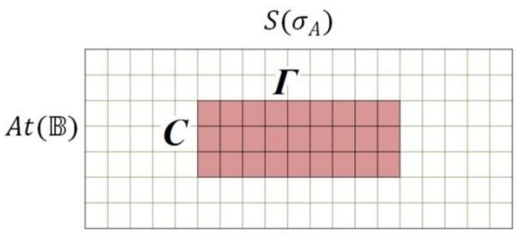

Fig. 2. The formal context $(At(\mathbb{B}), S(\sigma_A), I_\tau)$.

Consider $b \in C$, then $b \in At(\mathbb{B})$. It is true that $C' = \Gamma$, so for any $\psi \in \Gamma$, we have $b\ I_\tau\ \psi$, therefore $b \le \tau(\psi)$. Hence, by Proposition 7, we have $\mathfrak{A}_b \vDash \psi$.

Consequently, for any $\psi \in \Gamma$, it is true that $\mathfrak{A}_b \vDash \psi$, which means that $\mathfrak{A}_b \vDash \Gamma$. Involving the fact that $\Gamma \vdash \varphi$, we conclude that $\mathfrak{A}_b \vDash \varphi$. Therefore, by Proposition 7, we have $b \le \tau(\varphi)$, so $b\ I_\tau\ \varphi$.

Thus, we have proved that $b\ I_\tau\ \varphi$ for any $b \in C$. Then $\varphi \in C' = \Gamma$. Therefore, we have shown that for any $\varphi \in S(\sigma_A)$, if $\mathrm{T} \vdash \varphi$ then $\varphi \in \mathrm{T}$. Hence, Γ is a theory. ■

Theorem 10. Let $\mathfrak{A}_{\mathbb{B}} = \langle A, \sigma_A, \tau \rangle$ be an atomic Boolean-valued model. There exists an atomic Boolean-valued model $\mathfrak{A}_{\mathbb{B}_1} = \langle A, \sigma_A, \tau_1 \rangle$, where $\tau_1: S(\sigma_A) \to \mathbb{B}_1$, and an epimorphism g: $\mathbb{B} \to \mathbb{B}_1$ such that

1) the formal context $(At(\mathbb{B}_1), S(\sigma_A), I_{\tau_1})$ is object-clarified;

2) for any $\varphi \in S(\sigma_A)$ we have $g(\tau(\varphi)) = \tau_1(\varphi)$;

3) for any $b \in At(\mathbb{B})$, if $g(b) \neq 0$ then $g(b) \in At(\mathbb{B}_1)$;

4) for any $b \in At(\mathbb{B})$ and $\varphi \in S(\sigma_A)$, if $g(b) \neq 0$ then $b\ I_\tau\ \varphi \Leftrightarrow g(b)\ I_{\tau_1} \varphi$.

Proof. Let $\mathfrak{A}_{\mathbb{B}} = \langle A, \sigma_A, \tau \rangle$ be a Boolean-valued model and $\mathbb{B}$ be atomic.

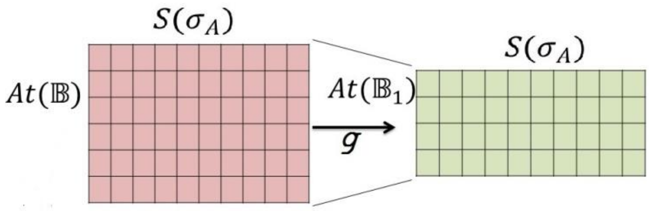

Fig. 3. Epimorphism $g: \mathbb{B} \to \mathbb{B}_1$

Suppose that the formal context $(At(\mathbb{B}), S(\sigma_A), I_\tau)$ is not object-clarified. Consider an equivalence relation $\sim$ on the set $At(\mathbb{B})$ of the atoms of the Boolean algebra $\mathbb{B}$ defined as follows: for $a, b \in At(\mathbb{B})$ we have $a \sim b$ iff for any $\varphi \in S(\sigma_A)$ it is true that $a\ I_\tau\ \varphi \Leftrightarrow b\ I_\tau\ \varphi$.

Consider the quotient set $\mathbb{B}/_\sim$, and denote $H = At(\mathbb{B})/_\sim$.

We chose exactly one element $c_{[a]} \in [a]$ in each equivalence class $[a] \in At(\mathbb{B})/_\sim$. Denote

$$C = \{c_{[a]} \mid [a] \in H\} \subseteq At(B) \quad \text{and} \quad d = \bigcup_{(b \in (At(B) \setminus C))} b.$$

Consider a principal ideal $I = \hat{d} = \{b \in \mathbb{B} \mid b \le d\}$ of the Boolean algebra $\mathbb{B}$: $I \lhd \mathbb{B}$.

Consider a Boolean algebra $\mathbb{B}_1 = {}^{\mathbb{B}}/_I$ which is the quotient algebra of the Boolean algebra $\mathbb{B}$ by the ideal I.

Consider an epimorphism $g: \mathbb{B} \to \mathbb{B}_1$ defined as follows: $g(b) = {}^{b}/_I$ for any $b \in \mathbb{B}$; here ${}^{b}/_I$ is the quotient class of the element b.

Define a mapping $\tau_1: S(\sigma_A) \to \mathbb{B}_1$ as follows:

$$\tau_1(\varphi) = {}^{\tau(\varphi)}/_{I} \quad \text{for} \quad \varphi \in S(\sigma_A).$$

Lemma 11. $\mathfrak{A}_{\mathbb{B}_1} = \langle A, \sigma_A, \tau_1 \rangle$ is a Boolean-valued model.

Proof. Consider $\varphi, \psi \in S(\sigma_A)$. We have

$$\tau_1(\neg\varphi) = {}^{\tau(\neg\varphi)}/_{I} = {}^{\overline{\tau(\varphi)}}/_{I} = \overline{{}^{\tau(\varphi)}/_{I}} = \overline{\tau_1(\varphi)};$$

$$\tau_1(\varphi \vee \psi) = {}^{\tau(\varphi \vee \psi)}/_{I} = {}^{\tau(\varphi) \cup \tau(\psi)}/_{I} = {}^{\tau(\varphi)}/_{I} \cup {}^{\tau(\psi)}/_{I} = \tau_1(\varphi) \cup \tau_1(\psi);$$

$$\tau_1(\varphi \,\&\psi) = {}^{\tau(\varphi\&\psi)}/_{I} = {}^{\tau(\varphi) \cap \tau(\psi)}/_{I} = {}^{\tau(\varphi)}/_{I} \cap {}^{\tau(\psi)}/_{I} = \tau_1(\varphi) \cap \tau_1(\psi);$$

$$\tau_1(\varphi \to \psi) = {}^{\tau(\varphi \to \psi)}/_{I} = {}^{\overline{\tau(\varphi)} \cup \tau(\psi)}/_{I} = \overline{{}^{\tau(\varphi)}/_{I}} \cup {}^{\tau(\psi)}/_{I} = \overline{\tau_1(\varphi)} \cup \tau_1(\psi).$$

Consider $\varphi(x) \in F(\sigma_A)$. We have

$$\tau_1\big(\forall x\varphi(x)\big) = {}^{\tau(\forall x\varphi(x))}/_{I} = {}^{\cap_{a\in A}\tau(\varphi(c_a))}/_{I} =$$

$$= \bigcap_{a\in A} {}^{\tau(\varphi(c_a))}/_{I} = \bigcap_{a\in A} \tau_1\big(\varphi(c_a)\big);$$

$$\tau_1\big(\exists x\varphi(x)\big) = {}^{\tau(\exists x\varphi(x))}/_{I} = {}^{\cup_{a\in A}\tau(\varphi(c_a))}/_{I} =$$

$$= \bigcup_{a\in A} {}^{\tau(\varphi(c_a))}/_{I} = \bigcup_{a\in A} \tau_1\big(\varphi(c_a)\big).$$

Note that the equalities

$${}^{\cap_{a\in A}\tau(\varphi(c_a))}/_{I} = \bigcap_{a\in A} {}^{\tau(\varphi(c_a))}/_{I}$$

and

$${}^{\cup_{a\in A}\tau(\varphi(c_a))}/_{I} = \bigcup_{a\in A} {}^{\tau(\varphi(c_a))}/_{I}$$

are true in virtue of the fact that the Boolean algebra $\mathbb{B}$ is complete.

Thus, we have shown that $\mathfrak{A}_{\mathbb{B}_1} = \langle A, \sigma_{\mathrm{A}}, \tau_1 \rangle$ is Boolean-valued model.

The lemma is proved. ■

Continue the proof of the theorem. First we prove that the set of atoms $At(\mathbb{B}_1) = \left\{ {}^{b}/_{I} \mid b \in C \right\}$.

Let $b \in C$. Then $b \notin I$ and so, ${}^{b}/_{I} \neq 0$. If ${}^{c}/_{I} \neq 0$ and ${}^{c}/_{I} \leq {}^{b}/_{I}$ then

$${}^{c}/_{I} = {}^{c}/_{I} \cap {}^{b}/_{I} = {}^{c \cap b}/_{I}.$$

So, $c \cap b \neq 0$ and $c \cap b \leq b$, therefore, $c \cap b = b$. Hence, ${}^{c \cap b}/_{I} = {}^{b}/_{I}$ and so, ${}^{c}/_{I} = {}^{b}/_{I}$. Thus, ${}^{b}/_{I}$ is an atom of the Boolean algebra $\mathbb{B}_1$.

On the other hand, let ${}^{c}/_{I}$ be an atom of the Boolean algebra $\mathbb{B}_1$. Then ${}^{c}/_{I} \neq 0$ and so, $c \neq 0$. The Boolean algebra $\mathbb{B}$ is atomic, hence the set of atoms $At(c) = \{b \in At(\mathbb{B}) \mid b \leq c\} \neq \emptyset$ and $c = \bigcup_{b \in At(\mathbb{B})} b$. Involving the fact that ${}^{c}/_{I} \neq 0$ we conclude that $c \notin I = \hat{d}$, consequently, $c \nleq d$.

We have $d = \bigcup_{b \in At(\mathbb{B}) \setminus C} b$ and $c = \bigcup_{b \in At(c)} b$, hence, if $At(c) \subseteq At(\mathbb{B}) \setminus C$ then $c \leq d$. Therefore, $At(c) \nsubseteq At(\mathbb{B}) \setminus C$, so $At(c) \cap C \neq \emptyset$.

Consequently, there exists an atom $b \in At(c) \cap C$. Hence, $b \in C$ and $b \leq c$, so ${}^{b}/_{I} \leq {}^{c}/_{I}$. As we proved above, $b \in C$ implies that ${}^{b}/_{I}$ is an atom of the Boolean algebra $\mathbb{B}_1$. Involving the fact that ${}^{c}/_{I}$ is an atom we conclude that ${}^{b}/_{I} = {}^{c}/_{I}$ and so, ${}^{c}/_{I} \in \{{}^{e}/_{I} \mid e \in C\}$. Therefore, $At(\mathbb{B}_1) = \{{}^{b}/_{I} \mid b \in C\}$.

Thus, that for any $b \in At(\mathbb{B})$ if $b \in At(\mathbb{B}) \setminus C$ then $b \leq d$, hence, ${}^{b}/_{I} = 0$. If $b \in C$ then ${}^{b}/_{I} \in At(\mathbb{B}_1)$. Consequently, the statement (3) holds.

Next, by the definition of the epimorphism $g\colon \mathbb{B} \to \mathbb{B}_1$ we have

$$g(\tau(\varphi)) = {}^{\tau(\varphi)}/_{I} = \tau_1(\varphi).$$

Therefore, the statement (2) holds.

Lemma 12. For every $b \in At(\mathbb{B})$, and every $\varphi \in S(\sigma_A)$, if $g(b) \neq 0$ then

$$b\, I_\tau\, \varphi \iff g(b)\, I_{\tau_1} \varphi.$$

Proof. Consider $b \in At(\mathbb{B})$ and $\varphi \in S(\sigma_A)$. Let $g(b) \neq 0$. Let us prove that $b\, I_\tau\, \varphi \iff g(b)\, I_{\tau_1} \varphi$.

($\Rightarrow$) Suppose that $b\, I_\tau\, \varphi$ holds. Then $b \leq \tau(\varphi)$, hence ${}^{b}/_{I} \leq {}^{\tau(\varphi)}/_{I} = \tau_1(\varphi)$. Therefore, we have $\left({}^{b}/_{I}\right) I_{\tau_1} \varphi$ and so, $g(b) I_{\tau_1}\, \varphi$ holds.

($\Leftarrow$) Suppose that $b\, I_\tau\, \varphi$ doesn't hold. Then $b \nleq \tau(\varphi)$, so, $b \neq b \cap \tau(\varphi)$. The element b is an atom and we have $b \neq b \cap \tau(\varphi) \leq b$. Then $b \cap \tau(\varphi) = 0$, hence ${}^{b}/_{I} \cap {}^{\tau(\varphi)}/_{I} = {}^{b \cap \tau(\varphi)}/_{I} = 0$.

Since ${}^{b}/_{I} = g(b) \neq 0$, we have ${}^{b}/_{I} \neq {}^{b}/_{I} \cap {}^{\tau(\varphi)}/_{I}$, hence, ${}^{b}/_{I} \nleq {}^{\tau(\varphi)}/_{I} = \tau_1(\varphi)$, i.e., $g(b) \nleq \tau_1(\varphi)$. Therefore, $g(b) I_{\tau_1}\, \varphi$ doesn't hold.

The lemma is proved. ■

Thus, we have proved the statement (4). Now let us prove that the formal context $\left(At(\mathbb{B}_1), S(\sigma_A), I_{\tau_1}\right)$ is object-clarified – the statement (1).

Consider $b_1, b_2 \in C$. Let ${}^{b_1}/_{I} \neq {}^{b_2}/_{I}$. Then there are atoms $a_1, a_2 \in At(\mathbb{B})$ such that $b_1 = c_{[a_1]}$ and $b_2 = c_{[a_2]}$. Hence, $b_1 \in [a_1]$ and $b_2 \in [a_2]$. Since $b_1 \neq b_2$, we have $[a_1] \neq [a_2]$, so, $b_1 \nsim b_2$. Therefore, there is a sentence $\varphi \in S(\sigma_A)$ such that only one of the statements $b_1\, I_\tau\, \varphi$ and $b_2\, I_\tau\, \varphi$ holds. Suppose that $b_1\, I_\tau\, \varphi$ holds and $b_2\, I_\tau\, \varphi$ doesn't hold. We conclude by Lemma 12 that $g(b_1)\, I_{\tau_1}\, \varphi$ holds and $g(b_2)\, I_{\tau_1}\, \varphi$ doesn't hold. Therefore, the rows corresponding to the objects ${}^{b_1}/_{I}$ and ${}^{b_2}/_{I}$ of the formal context $(At(\mathbb{B}_1), S(\sigma_A), I_{\tau_1})$ are different. Thus, the formal context $(At(\mathbb{B}_1), S(\sigma_A), I_{\tau_1})$ is object-clarified.

The theorem is proved. ■

Recall that $\mathbb{K}(\sigma_A) = \{\langle\{c_a^{\mathfrak{A}} \mid a \in A\}, \sigma_A\rangle \mid c_a^{\mathfrak{A}} \neq c_b^{\mathfrak{A}} \text{ for } a \neq b\}$ and $Th(K) = \{\varphi \in S(\sigma_A) \mid K \vDash \varphi\}$ is the theory of the class $K \subseteq \mathbb{K}(\sigma_A)$.

We represent the set of the cases of the domain (3rd level of the ontological model) as a class of models $K \subseteq \mathbb{K}(\sigma_A)$. So, it is interesting and important to solve the following

Problem 13. How to describe the theories of the classes $K \subseteq \mathbb{K}(\sigma_A)$.

Theorem 14. Let T be a theory of the signature σ_A. There exists a class $K \subseteq \mathbb{K}(\sigma_A)$ such that $T = Th(K)$ if and only if there exists a Bolean-valued model $\mathfrak{A}_{\mathbb{B}}$ such that

$$(T', T) \in \underline{\mathfrak{B}}(At(\mathbb{B}), S(\sigma_A), I_\tau).$$

Proof. Let T be a theory in the signature σ_A.

($\Rightarrow$) Consider a class $K \subseteq \mathbb{K}(\sigma_A)$. Let $T = Th(K)$. Denote $K_0 = \mathbb{K}(\sigma_A)$. Consider a Boolean algebra $\mathbb{B} = \langle\wp(K_0); \cup, \cap, -, \emptyset, K_0\rangle$ and a mapping $\tau: S(\sigma_A) \to \mathbb{B}$ defined as follows: $\tau(\varphi) = \{\mathfrak{B} \in K \mid \mathfrak{B} \vDash \varphi\}$ for a sentence $\varphi \in S(\sigma_A)$.

Lemma 15. $\mathfrak{A}_{\mathbb{B}} = \langle A, \sigma_a, \tau\rangle$ is a Boolean-valued model.

Proof. Let $\varphi, \psi \in S(\sigma_A)$. Then

$$\tau(\neg\varphi) = \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \neg\varphi\} = \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \nvDash \varphi\} =$$
$$K \backslash \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \varphi\} = K_0 \backslash \tau(\varphi) = \overline{\tau(\varphi)};$$
$$\tau(\varphi \vee \psi) = \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash (\varphi \vee \psi)\} =$$
$$= \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \varphi\} \cup \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \psi\} = \tau(\varphi) \cup \tau(\psi);$$
$$\tau(\varphi \& \psi) = \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash (\varphi \& \psi)\} =$$
$$= \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \varphi\} \cap \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \psi\} = \tau(\varphi) \cap \tau(\psi);$$
$$\tau(\varphi \to \psi) = \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash (\overline{\varphi} \vee \psi)\} = \overline{\tau(\varphi)} \cup \tau(\psi).$$

Let $\varphi(x) \in F(\sigma_A)$. Then

$$\tau(\forall x\varphi(x)) = \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \forall x\varphi(x)\} =$$
$$= \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \varphi(a) \text{ for any } a \in A\} = \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \varphi(c_a) \text{ for any } a \in A\} =$$
$$= \bigcap_{a \in A} \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \varphi(c_a)\} = \bigcap_{a \in A} \tau(\varphi(c_a));$$
$$\tau(\exists x\varphi(x)) = \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \exists x\varphi(x)\} =$$
$$\{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \varphi(a) \text{ for some } a \in A\} = \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \varphi(c_a) \text{ for some } a \in A\} =$$
$$= \bigcup_{a \in A} \{\mathfrak{B} \in K_0 \mid \mathfrak{B} \vDash \varphi(c_a)\} = \bigcup_{a \in A} \tau(\varphi(c_a)).$$

The lemma is proved. ■

Notice that the atoms of the Boolean algebra $\mathbb{B} = \langle\wp(K_0); \cup, \cap, -, \emptyset, K_0\rangle$ are exactly the one-element subsets of the set K_0. Hence, $At(\mathbb{B}) = \{\{\mathfrak{B}\} \mid \mathfrak{B} \in K_0\}$.

Consider a formal context $\langle At(\mathbb{B}), S(\sigma_A), I_\tau\rangle$. Let $\mathfrak{C} \in K_0$ and $\varphi \in S(\sigma_A)$. Then we have

$$\{\mathfrak{C}\}\ I_\tau\ \varphi \iff \{\mathfrak{C}\} \subseteq \tau(\varphi) \iff \{\mathfrak{C}\} \subseteq \{\mathfrak{B} \in K \mid \mathfrak{B} \vDash \varphi\} \iff \mathfrak{C} \vDash \varphi.$$

Thus, we have $\{\mathfrak{C}\}\ I_\tau\ \varphi \iff \mathfrak{C} \vDash \varphi$. Recall that $K \subseteq K_0 = \mathbb{K}(\sigma_A)$ and $T = Th(K)$.

Denote $\tilde{K} = \{\{\mathfrak{B}\} \mid \mathfrak{B} \in K\} \subseteq At(\mathbb{B})$. Then

$$\tilde{K}' = \{\varphi \in S(\sigma_A) \mid \{\mathfrak{B}\}\ I_\tau\ \varphi \quad \text{for any } \{\mathfrak{B}\} \in \tilde{K}\}$$
$$= \{\varphi \in S(\sigma_A) \mid \mathfrak{B} \vDash \varphi \quad \text{for any } \mathfrak{B} \in K\} = Th(K) = T.$$

Therefore, $T = T''$ and the pair (T', T) is a formal concept of the formal context $\langle At(\mathbb{B}), S(\sigma_A), I_\tau \rangle$, i.e., $(T', T) \in \underline{\mathfrak{B}}(At(\mathbb{B}), S(\sigma_A), I_\tau)$.

($\Leftarrow$) Let $\mathfrak{A}_{\mathbb{B}} = \langle A, \sigma_a, \tau \rangle$ be a Boolean-valued model and a pair $(T', T) \in \underline{\mathfrak{B}}(At(\mathbb{B}), S(\sigma_A), I_\tau)$.`

Denote $C = T'$. Then $C \subseteq At(\mathbb{B})$ and $C' = T$. Consider a class $K = \{\mathfrak{A}_b \mid b \in C\}$. We have proved above in the proof of Proposition 9 that in this case $\mathfrak{A}_b \vDash T$ holds for any $b \in C$.

Therefore, for any $\mathfrak{B} \in K$ we have $\mathfrak{B} \vDash T$. It means that $K \vDash T$. Hence, $T \subseteq Th(K) = \{\varphi \in S(\sigma_A) \mid K \vDash \varphi\}$.

Let us show that $Th(K) \subseteq T$. Let $\varphi \in Th(K)$. Then $K \vDash \varphi$, which means that $\mathfrak{B} \vDash \varphi$ for any $\mathfrak{B} \in K$. Consequently, for any $b \in C$ we have $\mathfrak{A}_b \vDash \varphi$. By Proposition 7, $\mathfrak{A}_b \vDash \varphi$ implies that $b \leq \tau(\varphi)$, hence, $b\ I_\tau\ \varphi$ holds.

Thus, for any $b \in C$ we have $b\ I_\tau\ \varphi$, then $\varphi \in C' = T'' = T$, i.e., $\varphi \in T$. Therefore, for any $\varphi \in Th(K)$ we have $\varphi \in T$, hence $Th(K) \subseteq T$, and so, $T = Th(K)$.

The theorem is proved. ∎

5 Conclusion

In the present paper we investigate the mathematical foundations of ontological modeling of the domain of mobile networks. We use the four-level model of knowledge representation to formalize this domain. We construct the case model at the third level of ontological model creation, when we formalize the empirical knowledge. The case model is presented by a set of countable algebraic systems. In the ontological model, the high-level characteristics of mobile network subscribers are represented with the help of first order theories of classes of algebraic systems of the special kind.

Next, we describe a method of constructing a formal context for a Boolean-valued model which represents the case model. We show that, without loss of generality, we may consider only object-clarified formal contexts corresponding to the Boolean-valued models.

We prove that the intents of formal concepts of formal contexts corresponding to Boolean-valued models are first order theories of the signature under consideration. In the end, we solve the following problem: What are the theories of the classes $K \subseteq \mathbb{K}(\sigma_A)$? We obtain a description of theories of the classes of domain cases in the language of formal concepts of the formal contexts corresponding to the Boolean-valued models.

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