Probability-Time Characteristics of $M|G|1|\infty$ Queueing System with Renovation

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The queueing system in which the losses of incoming customers (tasks) are possible due to the introduction of a special renovation mechanism is under consideration. The renovation mechanism means that the task at the moment of the end of its service with some probability may empty the buffer and leave the system, or with an additional probability may just leave the system without emptying the buffer. The queueing system consists of the server with general service time distribution and the buffer of unlimited capacity. The incoming flow of tasks is a Poisson one. The embedded upon the end of service times Markov chain is constructed and under the assumption of the existence of a stationary regime for the embedded Markov chain the formula for the probability generation function is derived. In addition, the next probability characteristics (based on the embedded Markov chain) are obtained: the probability of the system being empty, the probability of a task in the buffer to be dropped (not to be dropped), the probability distribution of served (dropped) tasks. Also the average numbers of customers in the system, dropped customers and served customers (based on the embedded Markov chain) are derived as the service waiting time distribution for non-dropped tasks and the average service waiting time for non-dropped tasks.

Key words and phrases: queueing system, renovation, general service time distribution, embedded Markov chain, time-probability characteristics.

1. Introduction

In this work we will consider the well-known $M|G|1|\infty$ system [1, 2] with renovation mechanism [3] and try to derive time-probability characteristics.

The renovation mechanism as some other mechanisms (failure of an unreliable server [15], arrival of some «viral» applications [13, 14, 16] or some disasters [17, 18]) allows us to describe the system with the possibility of losing data. The idea or renovation is that the task at the moment of the end of its service with some probability $0 \leqslant q < 1$ may drop from the buffer all other tasks and leave the system, or with additional probability $p = 1 - q$ just leaves the system (without dropping other tasks) [3, 4]. The more general variant of renovation — general renovation (when with probability $q(i), i \leqslant 1$ if there were i and more tasks in the buffer the exactly i tasks are dropped by the served one) — is described in [5–7, 11]. In [9] is shown that the renovation mechanism may used for the analysis of RED-type active queue management algorithms [8, 10].

In [4, 9] the queueing system $M|M|n|r$ was considered, in [5–7, 11] — the $G|M|n|r$ system. Only in [12] the queueing system with general service time distribution and Poisson income flow was under investigation. In this paper we will use some the results from [12] such as the construction of embedded Markov chain and probability generation function (pgf) derivation.

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2. System description and stationary distribution of the embedded Markov chain, probability generation function

The queueing system $M|G|1|\infty$ with the general service time distribution $B(x)$ and Poisson incoming flow with arrival rate λ and renovation is considered.

The renovation mechanism, due to [3–5,7,12], operates as follows. At the end of each service completion the customer leaving the server with the some probability $0 \leqslant q < 1$ empties the buffer and leaves the system. With the additional probability $p = 1 - q$ it leaves the system without having any effect on the buffer contents.

If $p = 1$ one obtains the well-known $M|G|1|\infty$ queue [1,2].

As usual, if one considers the total number of customers $\{\nu_i, i \geqslant 0\}$ in the system just after i-th service completion, then $\{\nu_i, i \geqslant 0\}$ is the embedded Markov chain of the queue-length process $\{\nu(t), t \geqslant 0\}$. The state set of the embedded Markov is denoted by$\mathcal{X} = \{0, 1, \dots\}$.

If we introduce the probability $\beta_i = \int_0^\infty \frac{(\lambda x)^i}{i!} e^{-\lambda x} dB(x)$ $(i \geqslant 0)$ that during service time exactly i $(i \geqslant 0)$ other customers may enter the system, then we obtain the matrix of transition probabilities for the embedded Markov:

$$\begin{pmatrix} \beta_0 + \sum_{i=1}^\infty \beta_i q & \beta_1 p & \beta_2 p & \beta_3 p & \beta_4 p & \beta_5 p & \dots \\ \beta_0 + \sum_{i=1}^\infty \beta_i q & \beta_1 p & \beta_2 p & \beta_3 p & \beta_4 p & \beta_5 p & \dots \\ \sum_{i=0}^\infty \beta_i q & \beta_0 p & \beta_1 p & \beta_2 p & \beta_3 p & \beta_4 p & \dots \\ \sum_{i=0}^\infty \beta_i q & 0 & \beta_0 p & \beta_1 p & \beta_2 p & \beta_3 p & \dots \\ \sum_{i=0}^\infty \beta_i q & 0 & 0 & \beta_0 p & \beta_1 p & \beta_2 p & \dots \\ \sum_{i=0}^\infty \beta_i q & 0 & 0 & 0 & \beta_0 p & \beta_1 p & \dots \\ \sum_{i=0}^\infty \beta_i q & 0 & 0 & 0 & 0 & \beta_0 p & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1}$$

The matrix of transition probabilities is a stochastic one.

If we denote by p_i $(i \geqslant 0)$ the probability, that there are i customers in the system upon service completion, and suppose that the stationary probability distribution of embedded Markov chain exists, then from (1) we obtain the following system for steady-state probabilities p_i:

$$p_0 = \left(\beta_0 + \sum_{i=1}^\infty \beta_i q\right) p_0 + \left(\beta_0 + \sum_{i=1}^\infty \beta_i q\right) p_1 + \sum_{k=2}^\infty \left(\sum_{i=0}^\infty \beta_i q\right) p_k, \tag{2}$$

$$p_i = \beta_i p p_0 + \sum_{k=1}^{i+1} p\beta_{i+1-k} p_k, \quad i \geqslant 1, \tag{3}$$

with normal condition $\sum\limits_{i=0}^{\infty} p_i = 1$.

In order to obtain the formula for the probability p_0 of system being empty just after the end of the service the probability generation function $P(z)$ is introduced:

$$P(z) = \sum_{i=0}^{\infty} p_i z^i,$$

and can be written by multiplying (2) by z^0 and (3) by z^i $(i \geqslant 1)$ in following form:

$$P(z) = \frac{(1-z)pp_0\beta(\lambda - \lambda z) - zq}{p\beta(\lambda - \lambda z) - z}, \tag{4}$$

where

$$\beta(\lambda - \lambda z) = \sum_{i=0}^{\infty} \beta_i z^i = \sum_{i=0}^{\infty} z^i \int\limits_0^{\infty} \frac{(\lambda x)^i}{i!} e^{-\lambda x} dB(x).$$

If $p = 1$ then (4) coincides with the Pollaczek-Khinchin formula for classic $M|G|1|\infty$ queue [1, 2].

By using the $P(z)$ analyticity property [1, 2] we can derive the expression for p_0 probability. Consider the equation

$$p\beta(\lambda - \lambda z) - z = 0.$$

It has the unique solution $0 < z_0 < 1$ for $z \in [0, 1]$. As the denominator of (4) vanishes at point $z = z_0$ then the numerator of (4) must also vanish at this point. Thus

$$(1 - z_0)pp_0\beta(\lambda - \lambda z_0) - z_0 q = 0,$$

where from it follows that

$$p_0 = \frac{qz_0}{(1 - z_0)p\beta(\lambda - \lambda z_0)}. \tag{5}$$

Also, if we consider $p = 1$ $(q = 1 - p = 0)$ then we will get the well-known expression $p_0 = 1 - \lambda b$ [1, 2], where $b = \int\limits_0^{\infty} x dB(x)$ is the mean service time (if $B(x)$ — continuous function).

3. Probability-time characteristics

In this part we will present the distribution of served (lost) tasks, stationary waiting time distribution for served (lost) task, and average numbers of tasks (served or lost) in the system based on embedded Markov chain probability distribution p_i $(i \geqslant 0)$ and pgf (4).

First of all, from (4) we may derive the analytical expression for the average number N of tasks in the system upon service completion:

$$N = P'(1) = \frac{p}{q}\left(p_0 + \lambda b - 1\right). \tag{6}$$

Here λb – is the average number of tasks arriving into the system during a service time of a single task (single service time).

Let us introduce the probability $p^{(\text{serv})}$ that a task from the buffer will not be dropped at the moment just after the end of the service and will be eventually serviced (of course, if the system is not empty):

$$p^{(\text{serv})} = \frac{1}{1-p_0}\sum_{i=1}^{\infty} p_i p^{i-1} = \frac{(P(p) - p_0)}{p(1-p_0)}.$$

The $P(p)$ — is the value of probability generation function (4) $P(z)$ when $z = p$, $1 - p_0$ — the probability that the system is not empty, p_0 is obtained in (5).

The probability $p^{(\text{loss})}$ that even one task being in the buffer at the moment just after the end of the service will be dropped later is:

$$p^{(\text{loss})} = \frac{1}{1-p_0}\sum_{i=2}^{\infty} p_i \sum_{k=0}^{i-2} q p^k = 1 - \frac{(P(p) - p_0)}{p(1-p_0)}.$$

The probability distribution $p_i^{(\text{serv})}$ that exactly i $(i \geqslant 0)$ tasks from the buffer will be served later if at initial moment just after the end of the service there were at least i tasks in the buffer:

$$\begin{cases} p_0^{(\text{serv})} = p_0, \\ p_i^{(\text{serv})} = p_i p^{i-1} + q p^{i-1} \displaystyle\sum_{k=i+1}^{\infty} p_k, \quad i \geqslant 1. \end{cases} \tag{7}$$

It' easy to prove than the normalization requirement $\sum_{i=0}^{\infty} p_i^{(\text{serv})} = 1$ is valid.

In a similar way we may define the probability distribution $p_i^{(\text{loss})}$ that exactly i $(i \geqslant 0)$ tasks from the buffer will be dropped later if at initial moment just after the end of the service there were at least i tasks in the buffer:

$$\begin{cases} p_0^{(\text{loss})} = p_0 + \displaystyle\sum_{i=1}^{\infty} p_i p^{i-1}, \\ p_i^{(\text{loss})} = q \displaystyle\sum_{k=i+1}^{\infty} p^{k-1} p_{k+i}, \quad i \geqslant 1. \end{cases} \tag{8}$$

The normalization requirement $\sum\limits_{i=0}^{\infty} p_i^{(\text{loss})} = 1$ is also valid.

For both distributions ($p_i^{(\text{serv})}$ and $p_i^{(\text{loss})}$, $i \geqslant 0$) the probabilities p_i ($i \geqslant 0$) are the stationary probabilities of embedded Markov chain distribution.

The average number $N^{(\text{serv})}$ of tasks (of remained in the buffer just after the end of the service) which will be served and the average number $N^{(\text{loss})}$ of tasks (of remained in the buffer just after the end of the service) which will be lost may be obtained from (7) and (8) by definition:

$$N^{(\text{serv})} = \sum_{i=0}^{\infty} i p_i^{(\text{serv})} = \frac{1 - P(p)}{q}, \tag{9}$$

$$N^{(\text{loss})} = \sum_{i=0}^{\infty} i p_i^{(\text{loss})} = N - \frac{1 - P(p)}{q}, \tag{10}$$

where $P(p)$ — the probability generation function (4) for $z = p$ and N — the average number of tasks (6). If $p = 1$ ($q = 0$) then $N^{(\text{serv})} = N$ for $M|G|1|\infty$ queue [1,2].

The average characteristics N is the sum of (9) and (10).

Now let us consider the waiting time distribution for served task (customer). If we define as $W^{(\text{serv})}(x)$ the probability, that the waiting time for the last task in the buffer (just after the end of the service) will be less than x, then we obtain:

$$W^{(\text{serv})}(x) = \frac{1}{(1 - p_0)p^{(\text{serv})}} \sum_{i=1}^{\infty} W_i^{(\text{serv})}(x) p_i, \tag{11}$$

here $W_i^{(\text{serv})}(x)$ is the probability, that the waiting time for the i-th customer in the buffer (just after the end of the service) will be less than x with the requirement that there were exactly i customers just after the end of service. The probability $(1 - p_0)p^{(\text{serv})}$ is the probability of condition that just after the end of the service our system is not empty and all the tasks from the buffer will be served later.

The Laplace–Stieltjes transformation (LST) of (11) is:

$$\omega^{(\text{serv})}(s) = \frac{1}{(1 - p_0)p^{(\text{serv})}} \left(p_1 + \sum_{i=2}^{\infty} p_i p^{i-1} \beta(s)^{i-1} \right) = \\ = \frac{1}{(1 - p_0)p^{(\text{serv})}} \frac{P(p\beta(s)) - p_0}{p\beta(s)}, \tag{12}$$

here βs is LST of service time distribution function $B(x)$.

The mean waiting time of the customer which received service is obtained by differentiation of (12) at $s = 0$:

$$w^{(\text{serv})} = -\left(\omega^{(\text{serv})}(s)\right)'_{s=0} = b\left(\frac{P'(p)}{(1 - p_0)p^{(\text{serv})}} - 1\right),$$

where

$$P'(p) = b \frac{p_0 \left(p\beta^2(\lambda q) - \beta(\lambda q) - \lambda q p \beta'(\lambda q)\right) + q\left(\beta(\lambda q) + \lambda p \beta'(\lambda q)\right)}{(1 - \beta(\lambda q))^2}.$$

4. Conclusions

The paper considers the queuing system with full renovation. Analytical expressions for the main performance characteristics for the embedded Markov chain are obtained. The study of $M|G|1|\infty$ queues with the general renovation as well as with renovation and repeated service (due to [19]) is an open issue.

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