

## ИСПОЛЬЗОВАНИЕ ПРИБЛИЖЕНИЯ СРЕДНЕГО ПОЛЯ ДЛЯ АНАЛИЗА КРУПНОМАСШТАБНЫХ СИСТЕМ МАССОВОГО ОБСЛУЖИВАНИЯ С МАЛЫМ ПАРАМЕТРОМ\*

### Аннотация

В данной статье рассматривается система массового обслуживания, которая состоит из бесконечного числа идентичных серверов FCFS с буфером, который способен хранить очередь бесконечной длины, и с пуассоновским входящим потоком заявок в систему массового обслуживания с интенсивностью  $N\lambda$ . Каждый запрос, поступающий в систему массового обслуживания, случайным образом выбирает и опрашивает два произвольных сервера этой системы, а затем мгновенно отправляется на один из них, где более короткая очередь. Динамика изменения доли серверов в системе  $u_k(t)$ , имеющих длину очередей не менее  $k$ , можно описать с помощью бесконечной системы дифференциальных уравнений. Для этой бесконечной системы дифференциальных уравнений поставлена задача Коши тихоновского типа с начальными условиями и малым параметром, который присутствует перед производными, начиная с уравнения, описывающего функцию  $u_k(t)$ . Наличие малого параметра в бесконечной системе дифференциальных уравнений позволяет описывать быстроизменяющиеся процессы в крупномасштабных системах массового обслуживания, что дает возможность анализировать переходные процессы в системах такого типа. Для рассматриваемой сингулярно возмущенной задачи Коши тихоновского типа с начальными условиями и малым параметром доказана теорема существования.

### Ключевые слова

Аналитические методы в теории массового обслуживания; системы дифференциальных уравнений бесконечного порядка; малый параметр; счетные цепи Маркова; крупномасштабные системы массового обслуживания; метод Добрушина; приближение среднего поля.

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## MEAN-FIELD APPROXIMATION FOR LARGE-SCALE QUEUEING SYSTEMS WITH A SMALL PARAMETER

### Abstract

In this paper it is considered the queueing system, consisting of an infinite number of identical servers with FCFS buffer, which can store a queue of infinite length, with Poisson input flow of requests in a queueing system with intensity  $N\lambda$ . Every request entering the queueing system, randomly selects and uses any of two servers of the system, and then is instantly sent to one of them, where a shorter queue. A share dynamics  $u_k(t)$  of servers in the system having the queue length is not less than  $k$  can be described by infinite system of differential equations. It is possible to formulate Tikhonov type Cauchy problem with initial conditions and small parameter for this infinite system of differential equations. A small parameter in the infinite system of differential equations allows describing rapidly changing processes in large-scale queueing systems. The existence theorem is proved for the considered singularly perturbed Tikhon type Cauchy problem with initial conditions and small parameter.

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## Keywords

*Analytical methods in queueing theory; systems of differential equations of infinite order; small parameter; countable Markov chains; large-scale queueing systems; Dobrushin mean-field approximation.*

## Introduction

The recent research of queueing systems with complex routing discipline in [16], [25], [26], [27] transport networks [1], [7], [8] and the asymptotic behavior of Jackson networks [21] faced with the problem of proving the global convergence of the solutions of certain infinite queueing systems of ordinary differential equations to a time-independent solution. Scattered results of these studies, however, allow a common approach to their justification. This approach will be expounded here. In work [17] the countable systems of differential equations with bounded Jacobi operators are studied and the sufficient conditions of global stability and global asymptotic stability are obtained. In [15] it was considered finite closed Jackson networks with  $N$  first come, first serve nodes and  $M$  customers. In the limit  $M \rightarrow \infty$ ,  $N \rightarrow \infty$ ,  $M/N \rightarrow \lambda > 0$ , it was got conditions when mean queue lengths are uniformly bounded and when there exists a node where the mean queue length tends to  $\infty$  under the above limit (condensation phenomena, traffic jams), in terms of the limit distribution of the relative utilizations of the nodes. It was derived asymptotics of the partition function and of correlation functions. In papers [5], [11], [20] the authors built various models of large-scale queueing systems and considered their dynamics.

Cauchy problems for the systems of ordinary differential equations of infinite order was investigated A.N.Tihonov [22], K.P.Persidsky [18], O.A.Zhautykov [28-29], Ju.Korobeinik [10], A.M.Samoilenko, Yu.V.Teplinskii [19] other researchers. For example, Markus Kreer, Aye Kzlers and Anthony W. Thomas [13] investigated fractional Poisson processes, a rapidly growing area of non-Markovian stochastic processes, that are useful in statistics to describe data from counting processes when waiting times are not exponentially distributed. They showed that the fractional Kolmogorov-Feller equations for the probabilities at time  $t$  could be represented by an infinite linear system of ordinary differential equations of first order in a transformed time variable. These new equations resemble a linear version of the discrete coagulation fragmentation equations, well-known from the non-equilibrium theory of gelation, cluster-dynamics and phase transitions in physics and chemistry.

It was studied the singular perturbed systems of ordinary differential equations by A.N. Tihonov [23], A.B.Vasil'eva [24], S.A. Lomov [14] other researchers.

In paper [2] we investigated the singular perturbed systems of ordinary differential equations of infinite order of Tikhonov-type  $\varepsilon \dot{x} = F(x(t, g_x), y(t, g_y), t)$ ,  $\dot{y} = f(x(t, g_x), y(t, g_y), t)$  with the initial conditions  $x(t_0) = g_x$ ,  $y(t_0) = g_y$ , where  $x, g_x \in X$ ,  $X \subset l_1$  and  $y, g_y \in Y$ ,  $Y \in R^n$ ,  $t \in [t_0, t_1]$  ( $t_0 < t_1$ ),  $t_0, t_1 \in T$ ,  $T \in R$ ,  $g_x$  and  $g_y$  are given vectors,  $\varepsilon > 0$  is a small real parameter.

In this paper we apply Dobrushin mean-field approaches from [26] for the singular perturbed systems of ordinary differential equations of infinite order of Tikhonov type. We considered a system that consists of infinite number of servers with a Poisson input flow of requests of intensity  $N\lambda$ . Each request arriving to the system randomly selects two servers and is instantly sent to the one with the shorter queue. In this case a share  $u_k(t)$  of the servers that have the queues lengths with not less than  $k$  can be described using an infinite system of differential equations. It is possible to investigate Tikhonov type Cauchy problem for this system with small parameter  $\varepsilon$  and initial conditions. It is studying the singular perturbed Tikhonov systems of ordinary differential equations of infinite order  $\dot{u} = f(u(t, g_u), U(t, g_U), t)$ ,  $\varepsilon \dot{U} = F(u(t, g_u), U(t, g_U), t)$  with the initial conditions  $u(0, g_u) = g_u$ ,  $U(0, g_U) = g_U$ , where  $u, f \in X$ ,  $X \in R^n$  are  $n$ -dimensional functions;  $U, F \in Y$ ,  $Y \subset l_1$  are infinite-dimensional functions and  $t \in [0, T_0]$  ( $0 < T_0 \leq \infty$ ),  $t \in T$ ,  $T \in R$ ;  $g_u \in X$  and  $g_U \in Y$  are given vectors,  $\varepsilon > 0$  is a small real parameter. The evolution analysis of  $u_k(t)$  ( $k = 1, 2, \dots$ ) be applied to application in large-scale queueing systems.

## Large-scale queueing systems model

The basic model considered there is a queueing system  $S_N$ , with  $N$  identical infinite-buffer FCFS (First-Come, First-Served) single-servers, with a Poisson arrival flow of rate  $N\lambda$  and with i.i.d. exponential service times of mean  $1/\mu$ , where  $0 < \lambda < \mu$ . Upon its arrival each task chooses  $m$  servers at random (i.e., independently of the pre-history of the queueing system (QS) and with probability  $1/(N^m)$ ) and then selects, among the chosen ones, the server with the lowest queue-size, i.e., the lowest number of tasks in the buffer (including the task in service). If there happen to be more than one server with lowest queue-size, the task selects one of them randomly.

One is interested in the 'typical' behavior of a server in  $S_N$ , as  $N \rightarrow \infty$ . Formally, it means that  $\forall t \geq 0$  and

$k = 0, 1, \dots$ , we consider the fraction  $q_k(t) = M_k(t)/N$  where  $M_k(t)$  is the random number of servers with the queue-size  $k$  at time  $t$ . Clearly,  $0 \leq q_k(t) \leq 1$ ,  $\sum_k q_k(t) = 1$ ; and  $Q(t) = (q_k(t)), t \geq 0$ , forms a Markov process (MP). Technically, it is more convenient to pass to the tail probabilities  $r_k(t) = \sum_{j \geq k} Q_j(t)$ ; the state space of the corresponding MP  $U_N(t) = (f_k(t)), t \geq 0$ , is the set  $U_N$  of non-increasing non-negative sequences  $u = (u_k, k = 0, 1, \dots)$  with  $u_0 = 1$ ,  $\sum_{k > 1} u_k < \infty$  and with the  $u_k$ 's multiple of  $1/N$ , which implies that  $u_k = 0$  for all  $k$  large enough. It is convenient to prolong the sequences  $u \in U_N$  to the negative  $k$ 's by the value 1.

The generator of  $\{U_N(t)\}$  is an operator  $A$  acting on functions  $f : U_N \rightarrow C^1$  and given by

$$A_N f(u) = N \sum_{k > 0} (u_k - u_{k+1}) \left[ f \left( u - \frac{e_k}{N} - f(u) \right) \right] + \lambda N \sum_{k > 0} ((u_{k-1})^2 - (u_k)^2) \left[ f \left( u + \frac{e_k}{N} - f(u) \right) \right]. \quad (1)$$

Here,  $e_k$  stands for the sequence with the  $k$ -th entry 1 and all others 0, the addition of the sequences is componentwise. Process  $\{U_N(t)\}$  is positive-recurrent and thus possess a unique invariant distribution,  $\pi_N$ ; given any initial distribution  $\varpi$ , the distribution of  $U_N(t)$  approaches  $\pi_N$  as  $t \rightarrow \infty$ . The main result of [25] is that, as  $N \rightarrow \infty$ , the expected value  $E_{\pi_N} r_k(t)$  converges to the value  $\{a_k\}$ , where

$$a_k = \left( \frac{\lambda}{\mu} \right)^{(m^k - 1)/(m-1)} \quad k \geq 0. \quad (2)$$

Pictorially speaking, it means that, as  $N \rightarrow \infty$ , an 'average' server in the QS will have  $k$  or more tasks in the buffer with probability  $a_k$ .

It is interesting to compare  $S_N$  with another queueing system  $L$ , where the arriving task chooses the server completely randomly (i.e., independently of the pre-history and with probability  $1/N$ ). Clearly,  $L$  is equivalent to an isolated  $M/M/\infty$  queue with the arrival and service rates  $\lambda$  and  $\mu$ , respectively, which justifies omitting subscript  $N$  in this notation. More precisely, the average server in  $L$  will have  $k$  or more tasks in the buffer with the geometrical probability

$$a_k^0 = \left( \frac{\lambda}{\mu} \right)^k \quad k \geq 1, \quad (3)$$

independently of  $N$ , which is much larger than  $a_k$ .

In fact, as was shown in [25], the whole process  $\{U_N(t)\}$  is asymptotically deterministic as  $N \rightarrow \infty$ . More precisely, let  $U$  denote the set of the non-increasing non-negative sequences  $u = (u_k, k \in \mathbb{Z})$  with  $u_k = 1$  for  $k \leq 0$  and  $\sum_{k \leq 0} u_k < \infty$ . Then, if the distribution  $\varpi$  of initial state  $U_N(0)$  approaches a Dirac delta-measure concentrated at a point  $g = \{g_k\} \in U$ , the distribution of  $\{U_N(t)\}$  is concentrated in the limit at the trajectory  $u(t) = u_k(t), t \geq 0$ , giving the solution to the following system of differential equations

$$\begin{cases} \dot{u}_k(t) = \mu(u_{k+1}(t) - u_k(t)) + \lambda((u_{k-1}(t))^2 - (u_k(t))^2), k \geq 1, \\ u_0(t) = 0, \\ u_k(0) = g_k \geq 0, k = 1, 2, \dots, t \geq 0. \end{cases} \quad (4)$$

Point  $a = (a_k)$  (see (2)) is a (unique) fixed point for system (4) in  $U$ .

These results illustrate the essence of the mean-field approximation for QS  $S_N$ . Equations (4) describe a 'self-compatible' evolution of vector  $u(t)$ , or, equivalently, of the probability distribution  $q(t) = \{q_k(t)\}$  defined by  $q_k(t) = u_k(t) - u_{k+1}(t)$ ,  $t \geq 0$ ,  $k = 0, 1, \dots$ . As before,  $u(t)$  is simply the sequence of the tail probabilities for  $q(t)$ .

We will compare system (4) with the linear system

$$\dot{y}_k(t) = \mu(y_{k+1}(t) - y_k(t)) + \lambda(y_{k-1}(t) - y_k(t)), k \geq 1, \quad (5)$$

describing the evolution of the probability distribution  $q^0(t) = (q_k^0(t))$ ,  $q_k^0(t) = y_k(t) - y_{k+1}(t)$  in a standard  $M/M/1/\infty$  queue with the arrival and service rates  $\lambda$  and  $\mu$ , respectively. The  $\mu$ -terms in (4) and (5) are the

same; they correspond with the departure of the tasks and 'push' the probability mass in  $q(t)$  and  $q^{(0)}(t)$  towards  $k = 0$ . On the other hand, the  $\lambda$ -terms (different in both SQ) correspond with the arrival of the tasks; these terms shift the probability mass to larger  $k$ 's. The  $\lambda$ -term in (4) is smaller than the one in (5) when  $u_k(t)$  is small; pictorially speaking, system (4) provides (for the same values of  $\lambda$  and  $\mu$ ) more 'protection', for large  $k$ , against the shift to the right, which may lead to an 'explosion', when the relation  $\sum_{k>1} \mu_k(t) < \infty$  or  $\sum_{k>1} \gamma_k(t) < \infty$  may fail as  $t \rightarrow \infty$ . Because of this, the entries  $a_k$  of sequence  $a$  (see (2)) giving the fixed point of (4) decrease 'super-exponentially', in contrast with the exponential decay of the tail probabilities in the fixed point  $a^0 = (a_k^0)$  of (5).

### Large-scale queueing systems model with a small parameter

Let's consider a system that consists of  $N$  servers with a Poisson input flow of requests of intensity  $N\lambda$ . Each request arriving to the system randomly selects two servers and is instantly sent to the one with the shorter queue. The service time is distributed exponentially with mean  $\bar{t} = 1/\mu$ . Let  $u_k(t)$  be a share servers that have the queues lengths with not less than  $k$ . It is possible to investigate the asymptotic distribution of the queue lengths as  $N \rightarrow \infty$  and  $\lambda < 1$  [25]. The considered system of the servers is described by ergodic Markov chain. There is a stationary probability distribution for the states of the system and if  $N \rightarrow \infty$  the evolution of the values  $u_k(t)$  becomes deterministic and the Markov chain asymptotically converges to a dynamic system the evolution of which is described by infinite system of differential-difference equations for which we can formulate the Cauchy problem of such type

$$\begin{cases} \dot{u}_k(t) = \mu(u_{k+1}(t) - u_k(t)) + \lambda((u_{k-1}(t))^2 - (u_k(t))^2), \\ u_0(t) = 0, \\ u_k(0) = g_k \geq 0, k = 1, 2, \dots, t \geq 0. \end{cases} \quad (6)$$

where  $g = \{g_k\}_{k=1}^{\infty}$  is a numerical sequence ( $1 = g_1 \geq g_2, \dots$ ) [25].

We can investigate infinite system of differential-difference equations with small parameter such form

$$\begin{cases} \dot{u}_k(t) = \mu(u_{k+1}(t) - u_k(t)) + \lambda((u_{k-1}(t))^2 - (u_k(t))^2), \\ k = 0, 1, \dots, n-1, \\ \dot{u}_n(t) = \mu(U_{n+1}(t) - u_n(t)) + \lambda((u_{n-1}(t))^2 - (u_n(t))^2), \\ \varepsilon^{s_k} \dot{U}_k(t) = \mu(U_{k+1}(t) - U_k(t)) + \lambda((U_{k-1}(t))^2 - (U_k(t))^2), \\ k = n+1, n+1, \dots, \\ u_k(0) = g_k \geq 0, k = 0, 1, 2, \dots, \end{cases} \quad (7)$$

where  $\varepsilon > 0$  is a small parameter that bring a singular perturbation to the system (6), which allows us to describe the processes of rapid change of the systems, and  $s = \{s_k\}_{k=n+1}^{\infty}, (s_k > 0)$  is a numerical sequence.

Using (7) we can write Tikhonov problems for systems of ordinary differential equations of infinite order with a small parameter  $\varepsilon$  and initial conditions

$$\begin{cases} \dot{u} = f(u(t, \mu, \lambda, g_u), U(t, \lambda, g_U), t), \\ \varepsilon^{s_k} \dot{U} = F(U(t, \mu, \lambda, g_U), t); \\ u(0, \mu, \lambda, g_u) = g_u, U(0, \mu, \lambda, g_U) = g_U, \end{cases} \quad (8)$$

where  $u, f \in X$ ,  $X \in R^n$  are n-dimensional functions;  $U, F \in Y$ ,  $Y \subset l_1$  are infinite-dimensional functions and  $t \in [0, T_0]$  ( $0 < T_0 \leq \infty$ ),  $t \in T$ ,  $T \in R$ ;  $g_u \in X$  and  $g_U \in Y$  are given vectors ( $g_u = \{g_k\}_{k=0}^n, g_U = \{g_k\}_{k=n+1}^{\infty}$ ),  $\varepsilon > 0$  is a small real parameter;  $u(t, g_u) = g_u = \{u_k\}_{k=0}^n$  and  $U(t, g_U) = \{u_k\}_{k=n+1}^{\infty}$  are solutions of (8). Given functions  $f(u(t, \mu, \lambda, g_u), U(t, \mu, \lambda, g_U), t)$  and  $F(U(t, \mu, \lambda, g_U), t)$  are continuous functions for all variables

$$\left\{ \begin{array}{l} f_k(u(t, \mu, \lambda, g_u), t) = \mu(u_{k+1}(t) - u_k(t)) + \\ + \lambda((u_{k-1}(t))^2 - (u_k(t))^2), k = 0, 1, \dots, n-1, \\ f_n(u(t, \mu, \lambda, g_u), U(t, \mu, \lambda, g_U), t) = \mu(U_{n+1}(t) - u_n(t)) + \\ + \lambda((u_{n-1}(t))^2 - (u_n(t))^2), \\ F_k(U(t, \mu, \lambda, g_U), t) = \mu(U_{k+1}(t) - U_k(t)) + \\ + \lambda((U_{k-1}(t))^2 - (U_k(t))^2), k = n+1, n+1, \dots \end{array} \right. \quad (9)$$

Let  $S$  is an integral manifold of the system (8) in  $X \times Y \times T$ . If any point  $t^* \in [0, T_0]$   $(u(t^*), U(t^*), t^*) \in S$  of trajectory of this system has at least one common point on  $S$  this trajectory  $(u(t, G), U(t, g), t) \in S$  belongs the integral manifold  $S$  totally.

If we assume in (8) that  $\varepsilon = 0$  than we have a degenerate system of the ordinary differential equations and a problem of singular perturbations

$$\left\{ \begin{array}{l} \dot{u} = f(u(t, \mu, \lambda, g_u), U(t), t), \\ 0 = F(u(t, \lambda, g_u), U(t, \mu, \lambda), t); \\ u(0, \lambda, g_u) = g_u, \end{array} \right. \quad (10)$$

where the dimension of this system is less than the dimension of the system (8), since the relations  $F(u(t, \lambda), U(t, \lambda), \lambda, t) = 0$  in the system (10) are the algebraic equations (not differential equations). Thus for the system (9) we can use limited number of the initial conditions then for system (8). Most natural for this case we can use the initial conditions  $u(0, \lambda, g_u) = g_u$  for the system (10) and the initial conditions  $U(0, \lambda, U_y) = g_U$  disregard otherwise we get the overdefined system. We can solve the system (10) if the equation  $F(u(t, \lambda), U(t, \lambda), \lambda, t) = 0$  has roots. If it is possible to solve we can find a finite set or countable set of the roots  $U_q(t, \lambda, g_u) = u_q(u(t, \lambda, g_u), t)$  where  $q \in N$ . If the implicit function  $F(u(t, \lambda), U(t, \lambda), \lambda, t) = 0$  has not simple structure we must investigate the question about the choice of roots. Hence we can use the roots  $U_q(t, \lambda, g_u) = u_q(u(t, \lambda, g_u), t)$  ( $q \in N$ ) in (10) and solve the degenerate system

$$\left\{ \begin{array}{l} \dot{u}_d = f(u_d(t, \lambda, g_u), u_q(u_d(t, \lambda, g_u), t), \lambda, t); \\ U_d(0, \lambda, g_u) = g_u. \end{array} \right. \quad (11)$$

Since it is not assumed that the roots  $U_q(t, \lambda, g_u) = u_q(u(t, \lambda, g_u), \lambda, t)$  satisfy the initial conditions of the Cauchy problem (8) ( $U_q(0) \neq g_u$ ,  $q \in N$ ), the solutions  $U(t, \lambda, g_U)$  (8) and  $U_q(t, \lambda, g_u)$  do not close to each other at the initial moments of time  $t > 0$ . Also there is a very interesting question about behaviors of the solutions  $u(t, \lambda, g_u)$  of the singular perturbed problem (8) and the solutions  $u_d(t, \lambda, g_u)$  of the degenerate problem (10). When  $t = 0$  we have  $u(0, \lambda, g_u) = u_d(0, \lambda, g_u)$ . Do these solutions close to each other when  $t \in (0, T_0]$ ? The answer to this question depends on using roots  $U_q(t, \lambda, g_u) = u_q(u(t, \lambda, g_u), t)$  and the initial conditions, which we apply for the systems (8) and (11).

### Analysis of infinite order system of differential equations

We can rewrite Tikhonov problems (8) for systems of ordinary differential equations of infinite order with a small parameter  $\varepsilon$  and initial conditions in the form

$$\left\{ \begin{array}{l} \dot{v} = F_R(v(t, \mu, \lambda, \varepsilon, v^0), t), \\ v(0, \mu, \lambda, \varepsilon, v^0) = v^0, \end{array} \right. \quad (12)$$

where

$$\begin{aligned} v &= (u_0, u_1, \dots, u_n, U_{n+1}, U_{n+2}, \dots), \\ F_{Rk} &= \mu(u_{k+1}(t) - u_k(t)) + \lambda((u_{k-1}(t))^2 - (u_k(t))^2), k = 0, 1, \dots, n-1, \\ F_{Rn} &= \mu(U_{n+1}(t) - u_n(t)) + \lambda((u_{n-1}(t))^2 - (u_n(t))^2), \\ F_{Rk} &= \varepsilon^{-s_k} \mu(U_{k+1}(t) - U_k(t)) + \varepsilon^{-s_k} \lambda((U_{k-1}(t))^2 - (U_k(t))^2), k = n+1, n+1, \dots, \\ v^0 &= (g_u, g_U); (v_k^0 = g_k, k = 0, 1, \dots) \end{aligned} \quad (13)$$

Using methods from [19], [28-19] we can consider Tikhonov-type problems (12)

$$\begin{cases} \dot{v} = F_R(v_0, v_1, \dots, v_n, \dots, \mu, \lambda, \varepsilon, t), \\ v(0, \mu, \lambda, \varepsilon, v_0) = v_0, \end{cases} \quad (14)$$

*Definition.* A function  $F_R(v_0, v_1, \dots, v_n, \dots, \mu, \lambda, \varepsilon, t)$  is called strongly continuous if for any  $\varepsilon_0 > 0$ , there exist  $N_0$  and  $\delta_0 > 0$  such that the inequality  $|v_i' - v_i''| < \delta_0, i = 0, 1, 2, \dots, N_0$ , implies the estimate for any  $\mu \geq 0, \lambda \geq 0, \varepsilon > 0$

$$|F_R(v_0', v_1', \dots, \mu, \lambda, \varepsilon) - F_R(v_0'', v_1'', \dots, \mu, \lambda, \varepsilon)| < \varepsilon_0. \quad (15)$$

**Theorem.** Assume that the right-hand sides of the system of equations (14)

- are defined for any  $v_i(\mu, \lambda, \varepsilon, t) \in R^1, i = 0, 1, 2, \dots, \mu \geq 0, \lambda \geq 0, \varepsilon > 0$  and all  $t \in T_0 = [0, \Delta t] \subset R^1$ ;
- are strongly continuous in  $v_0, v_1, \dots$  for fixed  $t \in T_0, \mu \geq 0, \lambda \geq 0, \varepsilon > 0$  and measurable in  $t \in T_0$  for fixed  $v_i(\mu, \lambda, \varepsilon, t), i = 0, 1, 2, \dots$ ;
- satisfy the inequalities

$$|F_{Ri}(t, v_0, v_1, \dots, \mu, \lambda, \varepsilon)| < M_i(t) \quad (16)$$

for all  $i = 0, 1, 2, \dots$ , where  $M_i(t)$  are functions summable on the segment  $T_0$  and for any  $\mu \geq 0, \lambda \geq 0, \varepsilon > 0$ .

Then, for any vector  $(v_0^0, v_1^0, \dots)$  with real coordinates, there exists at least one solution  $(v_0(\mu, \lambda, \varepsilon, t), v_1(\mu, \lambda, \varepsilon, t), \dots)$  of the system of equations (14) such that  $v_i(0) = v_i^0, i = 0, 1, 2, \dots$

*Proof.* We replace the system of equations (14) by the following system of integral equations:

$$v_i(t) = v_i^0 + \int_0^t F_{Ri}(t, v_0(t), v_1(t), \dots, \mu, \lambda, \varepsilon) dt, i = 0, 1, 2, \dots, \quad (17)$$

and consider a mapping (A)

$$z_i(t) = v_i^0 + \int_0^t F_{Ri}(t, v_0(t), v_1(t), \dots, \mu, \lambda, \varepsilon) dt, i = 0, 1, 2, \dots, \quad (18)$$

which establishes a correspondence between an arbitrary countable system of continuous functions  $\{v_i(t)\}_{i=0}^\infty$  and another system of this sort  $\{z_i(t)\}_{i=0}^\infty$ . Note that if  $F_R(t, v_0, \dots, v_n, \mu, \lambda, \varepsilon)$  is a continuous function of finitely many variables  $\{v_i(t)\}_{i=0}^n$  measurable with respect to  $t$  for fixed  $v_i, i = \overline{0, n}$ , then the function

$$\Phi(t) = F_R(t, \phi_0(t), \dots, \phi_n(t), \mu, \lambda, \varepsilon)$$

is measurable if  $\phi_i(t), i = \overline{0, n}$ , are measurable.

Thus, the function

$$\Psi_n(t) = F_R(t, \phi_0(t), \dots, \phi_n(t), 0, 0, \dots, \mu, \lambda, \varepsilon)$$

is measurable and, therefore, the function

$$F_R(t, \phi_0(t), \phi_1(t), \dots, \mu, \lambda, \varepsilon) = \Psi(t, \mu, \lambda, \varepsilon)$$

is also measurable because

$$\Psi(t) = \lim_{n \rightarrow \infty} \Psi_n(t, \mu, \lambda, \varepsilon), \quad (19)$$

which readily follows from the condition of strong continuity. The requirement of summability follows from condition 3 of Theorem. We consider a system of functions  $\{v_i(t)\}_{i=0}^\infty$  as a point  $P$  of an abstract space  $R$ . If there exists a point  $P$  invariant under mapping (A) (18), then it specifies a solution of the system of equations (17) and, hence, of system (14).

Consider a set  $M_0$  formed by three points  $P$  for which  $\{v_i(t)\}_{i=0}^\infty$  satisfy the conditions

$$|v_k(t) - v_k^0| \leq \int_0^t M_k(t) dt, |v_k(t') - v_k(t'')| \leq \int_{t'}^{t''} M_k(t) dt, k = 0, 1, 2, \dots$$

It is easy to see that mapping (A) (18) maps the set  $M_0$  into itself. We now introduce mapping (B) by putting every point  $P$  in correspondence with a set of numbers

$$\frac{a_0^0}{N_0}, \dots, \frac{a_0^n}{N_0}, \dots, \quad (20)$$

$$\frac{a_n^1}{nN_n}, \dots, \frac{a_n^n}{nN_n}, \dots,$$

where  $N_i = v_i^0 + \int_0^{\Delta t} M_i(t) dt$  and the numbers  $\{a_n^r\}_{n,r=0}^{\infty}$  ( $a_n^0, \dots, a_n^n, \dots$ ) are the coefficients of the Fourier expansion of a function  $v_n(t)$  in a certain complete orthogonal system of functions on the segment  $T_0$ . By ordering the set of numbers (20), we obtain a numerical sequence  $b_0, b_1, \dots, b_n, \dots$ . Moreover, we have

$$\begin{aligned} \sum_{k=0}^{\infty} (a_n^k)^2 &= \int_0^{\Delta t} (v_n(t))^2 dt \leq \int_0^{\Delta t} \left( v_n^0 + \int_0^t M_k(t) dt \right)^2 dt \leq \\ &\leq \int_0^{\Delta t} N_n^2 dt = aN_n^2, \end{aligned} \quad (21)$$

whence it follows that

$$\sum_{i=0}^{\infty} b_i^2 = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{a_n^k}{nN_n} \right)^2 \leq a \sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{a\pi^2}{6}. \quad (22)$$

Thus, mapping (B) maps the set  $M_0$  into a subset  $M_0^*$  of the Hilbert space  $l_2$ . Therefore, mapping (A) induces a mapping (A\*) of the set  $M_0^*$  into itself. Further, if mapping (A\*) has a fixed point  $P^* \in M_0^*$ , then the corresponding point  $P^* \in M_0$  determines the solution of equation (17) and, hence, (14). To use the Schauder theorem, it suffices to show that the set  $M_0^*$  is compact and convex. If  $P^{*'} = (b_0', \dots, b_n', \dots)$  and  $P^{*''} = (b_0'', \dots, b_n'', \dots)$  are points from  $M_0^*$ , then the point

$$\alpha P^{*'} + \beta P^{*''} = (\alpha b_0' + \beta b_0'', \alpha b_1' + \beta b_1'', \dots), \alpha + \beta = 1, \alpha > 0, \beta > 0, \quad (23)$$

belongs to  $M_0^*$  because it corresponds to the system of functions

$$\alpha v_0'(t) + \beta v_0''(t), \alpha v_1'(t) + \beta v_1''(t), \dots \quad (24)$$

specifying a point from the set  $M_0$ . Indeed,

$$|\alpha v_k'(t) + \beta v_k''(t) - v_k^0| = |\alpha(v_k'(t) - v_k^0) + \beta(v_k''(t) - v_k^0)| \leq (\alpha + \beta) \int_0^t M_k(t) dt = \int_0^t M_k(t) dt, \quad (25)$$

i.e., condition 1 is satisfied. Similarly, the inequality

$$|\alpha v_k'(t') + \beta v_k''(t') - \alpha v_k'(t'') - \beta v_k''(t'')| \leq (\alpha + \beta) \int_0^t M_k(t) dt \quad (26)$$

implies condition 2. Hence, the set  $M_0^*$  is convex. In this set, we choose an arbitrary sequence of points  $P_i^*$ . This sequence corresponds to the sequence of points  $P_i(v_0^{(i)}(t), v_1^{(i)}(t), \dots)$  in the set  $M_0$ . According to conditions 1 and 2, the sequence  $v_0^{(i)}(t), i = 0, 1, 2, \dots$ , is uniformly bounded and equicontinuous and, consequently, it contains a subsequence  $v_0^{(\alpha_0)}(t), v_0^{(\alpha_1)}(t), \dots, v_0^{(\alpha_s)}(t), \dots$  that converges uniformly in  $t \in T_0$ . However, the sequence  $v_1^{(\alpha_h)}(t), h \rightarrow \infty$ , is also uniformly bounded and equicontinuous and, hence, it also contains a convergent subsequence

$$v_1^{(\beta_0)}(t), v_1^{(\beta_1)}(t), \dots, v_1^{(\beta_s)}(t), \dots \quad (27)$$

This process can be continued infinitely.

We compose the table

$$\begin{array}{l} v_0^{(\alpha_0)}(t) v_0^{(\alpha_1)}(t) v_0^{(\alpha_2)}(t) \dots \\ v_1^{(\beta_0)}(t) v_1^{(\beta_1)}(t) v_1^{(\beta_2)}(t) \dots \\ v_2^{(\gamma_0)}(t) v_2^{(\gamma_1)}(t) v_2^{(\gamma_2)}(t) \dots \\ \dots \dots \dots \end{array} \quad (28)$$

and rewrite the set of sequences row by row

$$v_0^{(\alpha_0)}(t) v_0^{(\beta_1)}(t) v_0^{(\gamma_2)}(t) \dots$$

$$\begin{aligned} &v_1^{(\alpha_0)}(t)v_1^{(\beta_1)}(t)v_1^{(\gamma_2)}(t)\dots \\ &v_2^{(\alpha_0)}(t)v_2^{(\beta_1)}(t)v_2^{(\gamma_2)}(t)\dots \\ &\dots\dots\dots \end{aligned} \quad (29)$$

Each of these sequences converges as a subsequence of a convergent sequence supplemented by finitely many elements. Thus, the sequence of points

$$P_{\alpha_0}, P_{\beta_1}, P_{\gamma_2}, \dots \subset M_0 \quad (30)$$

converges weakly (coordinatewise) to a point  $P_0 \in M_0$  (uniformly in  $t \in T_0$ ). For the sake of convenience, we rewrite sequence (30) as

$$P_0, P_1, P_2, \dots, P_n, \dots \quad (31)$$

Let us show that the sequence of the corresponding points  $P_0^*, P_1^*, P_2^*, \dots$  from the set  $M_0^*$  converges to the point  $P_0^* \in M_0^*$  in the norm of the Hilbert space  $l_2$ . Indeed, the distance between the points  $P^{r*}$  and  $P^{r'*}$  from  $M_0^*$  is given by the formula

$$\rho(P^{r*}, P^{r'*}) = \sqrt{\sum_{i=0}^{\infty} (b_i^r - b_i^{r'})^2} = \sqrt{\sum_{n=0}^{\infty} \frac{1}{n^2 N_n^2} \int_0^{\Delta t} (v_n^r - v_n^{r'})^2 dt}, \quad (32)$$

whence it follows that

$$\rho(P_0^*, P_k^*) \leq \sqrt{\sum_{n=0}^{n_0} \frac{1}{n^2 N_n^2} \int_0^{\Delta t} (v_n^0 - v_n^k)^2 dt + \Delta t \sum_{n=n_0}^{\infty} \frac{1}{n^2}} \quad (33)$$

is arbitrarily small for sufficiently large  $n_0$  and  $k$ . This means that the set  $M_0^*$  is compact. Note that one can easily prove that mapping (B) is a homeomorphism, i.e., the sets  $M_0$  and  $M_0^*$  are topologically equivalent. Theorem is proved.

## Conclusions

We investigated the large-scale queueing system model that consists of infinite number of servers with a Poisson input flow of requests of intensity  $N\lambda$ . We assume that each request arriving to the system randomly selects two servers and is instantly sent to the one with the shorter queue. The service time is distributed exponentially with mean  $1/\mu$ . In this case a share  $u_k(t)$  of the servers that have the queues lengths with not less than  $k$  can be described using an infinite system of differential equations. Tikhonov type Cauchy problem for this system with small parameter  $\varepsilon$  is investigated. The theorems of existence of solutions for this Cauchy problem is proved with taking into account parameters  $\lambda, \mu, \varepsilon$ .

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