One Approach to Fuzzy Matrix Games

Konstantin N. Kudryavtsev[1], Irina S. Stabulit[2], and Viktor I. Ukhobotov[3]

[1] South Ural State University, Chelyabinsk, Russia
kudrkn@gmail.com
[2] South Ural State Agrarian University, Chelyabinsk, Russia
irisku76@mail.ru
[3] Chelyabinsk State University, Chelyabinsk, Russia
ukh@csu.ru

Abstract. The paper is concerned with a two-person zero sum matrix game with fuzzy payoffs, with saddle-point being defined with a classic definition in matrix game. In order to compare fuzzy numbers some different ordering operators can be used. The original game can be associated with the bimatrix game with crisp payoffs which are the operators value on a fuzzy payoff. We propose that every player use its ordering operator. The following statement can be proposed: when the ordering operators are linear, the same equilibrium strategy profile can be used for the matrix game, with fuzzy payoffs being the same for the bimatrix crisp game. We introduce and employ the algorithm of constructing a saddle-point in a two-person zero sum matrix game with fuzzy payoffs. In the instances of matrix games we use such ordering operators and construct the saddle-point.

Keywords: Fuzzy game · Matrix game · Saddle-point · Nash equilibrium

1 Introduction

Game theory is well known to take a significant part in decision making, this theory being often used to model the real world. Applied in real situations the game theory is difficult to have strict values of payoffs, because it is difficult for players to analyze some data of game. Thus, the players' information cannot be considered complete. Besides the players can have vague targets.

This uncertainty and lack of precision may be modeled as fuzzy games. Fuzzy sets are known to be initially used in non-cooperative game theory by Butnariu [3] to prove the belief of each player for strategies of other players. Since then fuzzy set theory has been applied in cooperative and non-cooperative games. The results of fuzzy games overview are given in [10]. Recently there were made

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various efforts in fuzzy bi-matrix game theory namely Maeda [11], Nayak [13], Dutta [6], Seikh [14], Verma and Kumar [16].

In [9], we represented the approach which generalizes some other ideas ([4], [5],[6] at al.).

2 Fuzzy Numbers

In this part, some basic definitions and results of fuzzy numbers and fuzzy arithmetic operations will be reminded. Here we will follow to [21].

A fuzzy set can be considered as a subset $\tilde{A}$ of universal set $X \subseteq \mathbb{R}$ by its membership function $\mu_{\tilde{A}}(\cdot)$ with a real number $\mu_{\tilde{A}}(x)$ in the interval $[0,1]$ and assigns to each element $x \in \mathbb{R}$.

Definition 2.1. *A fuzzy subset $\tilde{A}$ defined on $\mathbb{R}$, is said to be a fuzzy number if its membership function $\mu_{\tilde{A}}(x)$ comply with the following conditions:*
(1) $\mu_{\tilde{A}}(x) : \mathbb{R} \to [0,1]$ *is upper semi-continuous;*
(2) $\mu_{\tilde{A}}(x) = 0$ *for* $\forall\, x \notin [a,d]$*;*
(3) *There exist real numbers b, c such that $a \leqslant b \leqslant c \leqslant d$ and*
  (a) $x_1 < x_2 \ \Rightarrow\ \mu_{\tilde{A}}(x_1) < \mu_{\tilde{A}}(x_2) \quad \forall\, x_1, x_2 \in [a,b]$*;*
  (b) $x_1 < x_2 \ \Rightarrow\ \mu_{\tilde{A}}(x_1) > \mu_{\tilde{A}}(x_2) \quad \forall\, x_1, x_2 \in [a,b]$*;*
  (c) $\mu_{\tilde{A}}(x) = 1,\ \forall x \in [b,c]$.

The α-cut of a fuzzy number $\tilde{A}$ plays an important role in parametric ordering of fuzzy numbers. The α-cut or α-level set of a fuzzy number $\tilde{A}$, denoted by $\tilde{A}_\alpha$, is defined as $\tilde{A}_\alpha = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) \geqslant \alpha\}$ for all $\alpha \in (0,1]$. The support or 0-cut $\tilde{A}_0$ is defined as the closure of the set $\tilde{A}_0 = \{x \in \mathbb{R} \,|\, \mu_{\tilde{A}}(x) > 0\}$. Every α-cut is a closed interval $\tilde{A}_\alpha = [g_{\tilde{A}}(\alpha), G_{\tilde{A}}(\alpha)] \subset \mathbb{R}$, where $g_{\tilde{A}}(\alpha) = \inf\{x \in \mathbb{R} \,|\, \mu_{\tilde{A}}(x) \geqslant \alpha\}$ and $G_{\tilde{A}}(\alpha) = \sup\{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) \geqslant \alpha\}$ for any $\alpha \in [0,1]$.

The sets of fuzzy number is denoted as $\mathfrak{F}$. Then, two types of fuzzy numbers are used.

Definition 2.2. *Let $\tilde{A}$ be a fuzzy number. If the membership function of $\tilde{A}$ is given by*

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a+l}{l}, & for\ \ x \in [a-l, a], \\ \frac{a+r-x}{r}, & for\ \ x \in [a, a+r], \\ 0, & otherwise, \end{cases}$$

where, a, l and r are all real (crisp) numbers, and l, r are non-negative. Then $\tilde{A}$ is called a triangular fuzzy number, denoted by $\tilde{A} = (a, l, r)$.

The sets of triangular fuzzy number are denoted as $\mathfrak{F}_3$.

Definition 2.3. *Let $\tilde{A}$ be a fuzzy number. If the membership function of $\tilde{A}$ is given by*

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a+l}{l}, & for\ \ x \in [a-l, a], \\ 1, & for\ \ x \in [a, b] \\ \frac{b+r-x}{r}, & for\ \ x \in [b, b+r], \\ 0, & otherwise, \end{cases}$$

where, a, b, l and r are all real (crisp) numbers, and l, r are non-negative. Then $\tilde{A}$ is called a trapezoidal fuzzy number, denoted by $\tilde{A} = (a, b, l, r)$. $[a, b]$ is the core of $\tilde{A}$.

The sets of trapezoidal fuzzy number are denoted as $\mathfrak{F}_4$.

If $\tilde{A} = (a_1, l_1, r_1)$ and $\tilde{B} = (a_2, l_2, r_2)$ are two triangular fuzzy numbers, arithmetic operations on $\tilde{A}$ and $\tilde{B}$ are defined as follows:

Addition: $\tilde{A} + \tilde{B} = \tilde{C} = (a_1 + a_2, l_1 + l_2, r_1 + r_2)$, $\tilde{C} \in \mathfrak{F}_3$.

Scalar multiplication: $\forall\, k > 0$, $k \in \mathbb{R}$,

$$k\tilde{A} = (ka_1, kl_1, kr_1), \quad k\tilde{A} \in \mathfrak{F}_3.$$

If $\tilde{A} = (a_1, b_1, l_1, r_1)$ and $\tilde{B} = (a_2, b_2, l_2, r_2)$ are two trapezoidal fuzzy numbers, arithmetic operations on $\tilde{A}$ and $\tilde{B}$ are defined as follows:

Addition: $\tilde{A} + \tilde{B} = \tilde{C} = (a_1 + a_2, b_1 + b_2, l_1 + l_2, r_1 + r_2)$, $\tilde{C} \in \mathfrak{F}_4$.

Scalar multiplication: $\forall\, k > 0$, $k \in \mathbb{R}$,

$$k\tilde{A} = (ka_1, kb_1, kl_1, kr_1), \quad k\tilde{A} \in \mathfrak{F}_4.$$

Generaly, if $\tilde{A}$ and $\tilde{B}$ are two fuzzy numbers and $\tilde{A} + \tilde{B} = \tilde{C}$, $\lambda\tilde{A} = \tilde{D}$ and $\lambda = const > 0$, then

$$\tilde{C}_\alpha = [g_{\tilde{A}}(\alpha) + g_{\tilde{B}}(\alpha), G_{\tilde{A}}(\alpha) + G_{\tilde{B}}(\alpha)],$$

and

$$\tilde{D}_\alpha = [\lambda g_{\tilde{A}}(\alpha), \lambda G_{\tilde{A}}(\alpha)]$$

for any $\alpha \in [0, 1]$.

To compare fuzzy numbers is crucial issue. There are plenty of various methods for comparing fuzzy numbers. For instance, fuzzy numbers can be ranked with the help of defuzzification methods.Defuzzification is the process of producing a real (crisp) value which correspond to a fuzzy number, the defuzzification approach being used for ranking fuzzy numbers. Fuzzy numbers are initially defuzzified and then the obtained crisp numbers are organised using the order relation of real numbers.

A function for ranking fuzzy subsets in unit interval was introduced by Yager in [17]. It was based on the integral of mean of the α-cuts. Yager index is

$$Y(\tilde{A}) = \frac{1}{2}\int_0^1 (g_{\tilde{A}}(\alpha) + G_{\tilde{A}}(\alpha))\, d\alpha.$$

Another methods for ordering fuzzy subsets in the unit interval were suggested by Jain in [8], Baldwin and Guild in [1].

The subjective approach for ranking fuzzy numbers was developed by Ibanez and Munoz in [7], the following number as the average index for fuzzy number $\tilde{A}$ beind defined in [7]

$$V_P(\tilde{A}) = \int_Y f_{\tilde{A}}(\alpha) dP(\alpha),$$

with Y is a subset of the unit interval and P is a probability distribution on Y, the definition of $f_{\tilde{A}}$ being subjective for decision maker.

The ordering operator was proposed by Ukhobotov in [15]

$$U(\tilde{A},\nu)=\int\limits_0^1 ((1-\nu)g_{\tilde{A}}(\alpha)+\nu G_{\tilde{A}}(\alpha))\,d\alpha,$$

with crisp parameter $\nu\in[0,1]$, different behavior of the decision maker being corresponded with different ν.

Other defuzzification operators are given in [2].

Definition 2.4. *Let $\tilde{A}$ and $\tilde{B}$ are a fuzzy numbers, $T:\mathfrak{F}\to\mathbb{R}$ is the operator of defuzzification ($T(\cdot)=Y(\cdot)$, $V_P(\cdot)$, $U(\cdot,\nu)$ etc.).*

We say that $\tilde{B}$ is preferable to $\tilde{A}$ by the defuzzification operator T ($\tilde{A}\preceq_T\tilde{B}$) if and only if

$$T(\tilde{A})\leqslant T(\tilde{B}).$$

The defuzzification operator T is dependant on the order relation $\preceq_T$. Then, we give the example.

Example 2.1. Let $T(\cdot)=U(\cdot,\nu)$ and $\tilde{A},\tilde{B},\tilde{C}\in\mathfrak{F}_3$, $\tilde{A}=(40,8,10)$, $\tilde{B}=$ $=(45,20,10)$, $\tilde{C}=(42,6,4)$.
If $\tilde{X}=(a,l,r)\in\mathfrak{F}_3$, then

$$U(\tilde{X},\nu)=a+\frac{\nu r-(1-\nu)l}{2}.$$

Next, if $\nu=0$, then $U(\tilde{A},0)=36$, $U(\tilde{B},0)=35$, $U(\tilde{C},0)=39$. If $\nu=\frac{1}{2}$, then $U(\tilde{A},\frac{1}{2})=40,5$, $U(\tilde{B},\frac{1}{2})=42,5$, $U(\tilde{C},\frac{1}{2})=41,5$. If $\nu=1$, then $U(\tilde{A},1)=45$, $U(\tilde{B},1)=50$, $U(\tilde{C},1)=44$.

Therefore,

$$\begin{array}{ccccc}\tilde{B} & \preceq_{U(\cdot,0)} & \tilde{A} & \preceq_{U(\cdot,0)} & \tilde{C},\\ \tilde{A} & \preceq_{U(\cdot,\frac{1}{2})} & \tilde{C} & \preceq_{U(\cdot,\frac{1}{2})} & \tilde{B},\\ \tilde{C} & \preceq_{U(\cdot,1)} & \tilde{A} & \preceq_{U(\cdot,1)} & \tilde{B}.\end{array}$$

Definition 2.5. *If* $\forall\,\tilde{A},\tilde{B}\in\mathfrak{F}$, $\forall\,\alpha,\beta=const$

$$T(\alpha\tilde{A}+\beta\tilde{B})=\alpha T(\tilde{A})+\beta T(\tilde{B}),$$

then the defuzzification operator $T(\cdot)$ is the linear defuzzification operator.

Clear, Yager index $Y(\cdot)$ and operator $U(\cdot,\nu)$ is linear.

3 Crisp Games

In this part, some basic definitions of non-cooperative game theory are presented.

3.1 Noncooperative N-Person Games

Let us consider a non-cooperative N-players game in the class of pure strategies

$$\Gamma = \langle \mathbf{N}, \{X_i\}_{i\in\mathbf{N}}, \{f_i(x)\}_{i\in\mathbf{N}}\rangle, \tag{1}$$

where $\mathbf{N} = 1, ..., N$ is the set of players' serial numbers; each player i chooses and applies its own pure strategy $x_i \in X_i \subseteq R^{n_i}$, with no coalition with the others, which induces a strategy profile being formed

$$x = (x_1, ..., x_N) \in X = \prod_{i\in\mathbf{N}} X_i \subset R^n \quad (n = \sum_{i\in\mathbf{N}} n_i);$$

for each $i \in \mathbf{N}$, a payoff function $f_i(x)$ is defined on the strategy profile set X, which gives the payoff of player i. $f_i(x)$ is payoff function of player i ($i \in \mathbf{N}$). In addition, denote $(x\|z_i) = (x_1, ..., x_{i-1}, z_i, x_{i+1}, ...x_N)$, $f = (f_1, ..., f_N)$.

Definition 3.1. *A strategy profile $x^e = (x_1^e, ..., x_N^e) \in X$ is called a Nash equilibrium in the game* (1) *if*

$$\max_{x_i\in X_i} f_i(x^e\|x_i) = f_i(x^e) \quad (i \in \mathbf{N}). \tag{2}$$

The set of all $\{x^e\}$ in the game (1) will be designated by X^e.

3.2 Bimatrix Games

A bimatrix game defined by a pair (A, B) of real $m\times n$ matrices being considered, matrices A and B are payoffs to play I and II, respectively. M denotes the set of pure strategies of player I (matrix rows) and N stands for the set of pure strategies of player II (columns).

$$M = (1, \ldots, m), \quad N = (1, \ldots, n).$$

The sets of mixed strategies of the two players being called X and Y, we want to write expected payoffs as matrix products xAy and xBy, for mixed strategies x and y, so that x is a row vector and y is a column vector. That is,

$$X = \{(x_1, \ldots, x_m) \mid x_i \geqslant 0\ (\forall i \in M), \sum_{i\in M} x_i = 1\}$$

and

$$Y = \{(y_1, \ldots, y_n) \mid y_j \geqslant 0\ (\forall j \in N), \sum_{j\in N} y_j = 1\}.$$

Definition 3.2. *A pair $(x^e, y^e) \in X \times Y$ is called a Nash equilibrium for the game (A, B) if*

$$\begin{aligned} x^eAy^e &\geqslant xAy^e \quad \forall x \in X, \\ x^eBy^e &\geqslant x^eBy \quad \forall y \in Y. \end{aligned}$$

The set of Nash equilibrium for a game (A, B) is proved to be non-empty in [12].

If matrix $B = -A$, a bimatrix game is considered as a zero-sum matrix game. A solve of a zero-sum matrix game being a saddle-point.

3.3 Internally Instable Set of Nash Equilibrium

Now, consider internal instability of X^e. A subset $X^* \subset \mathbb{R}^n$ is *internally instable*, if there exist at least two strategy profiles $x^{(j)} \in X^*$ $(j = 1, 2)$ such that

$$\left[f(x^{(1)}) \ < \ f(x^{(2)})\right] \quad \Leftrightarrow \quad \left[f_i(x^{(1)}) \ < \ f_i(x^{(2)}) \ (i \in \mathbb{N})\right], \tag{3}$$

and *internally stable* otherwise.

Example 3.1. Consider a bimatrix game of the form (A, B), where

$$A = \begin{pmatrix} 2 & 5 & 1 \\ 3 & 4 & 6 \\ 6 & 7 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 8 \\ 9 & 3 & 5 \\ 2 & 7 & 6 \end{pmatrix}.$$

There are 3 Nash equilibrium strategy profiles $x^{(i)} = (x_1^{(i)}, x_2^{(i)})$ $(i = 1, 2, 3)$:

$$x^{(1)} = \left(\left(0, \frac{1}{3}, \frac{2}{3}\right), \left(0, \frac{4}{7}, \frac{3}{7}\right)\right),$$

$$x^{(2)} = \left(\left(0, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{4}{7}, 0, \frac{3}{7}\right)\right), \quad x^{(3)} = ((0, 0, 1), (0, 1, 0)).$$

Consequently, the set X^e is internally instable in the game (A, B), as for $x^{(3)}$ it follows that

$$\begin{array}{l} f_1(x^{(3)}) = x_1^{(3)} A x_2^{(3)} = 7 \ > \ f_1(x^{(1)}) = x_1^{(1)} A x_2^{(1)} = \frac{34}{7}, \\ f_2(x^{(3)}) = x_1^{(3)} B x_2^{(3)} = 7 \ > \ f_2(x^{(1)}) = x_1^{(1)} B x_2^{(1)} = \frac{17}{3}, \end{array}$$

and

$$\begin{array}{l} f_1(x^{(3)}) = x_1^{(3)} A x_2^{(3)} = 7 \ > \ f_1(x^{(2)}) = x_1^{(2)} A x_2^{(2)} = \frac{30}{7}, \\ f_2(x^{(3)}) = x_1^{(3)} B x_2^{(3)} = 7 \ > \ f_2(x^{(2)}) = x_1^{(2)} B x_2^{(2)} = \frac{11}{2}. \end{array}$$

We note that in the non-antagonistic setting of the game (1), the internal instability effect vanishes if there exists a unique Nash equilibrium strategy profile in (1).

Associate the following auxiliary N-criterion problem with the game (1):

$$\Gamma_v \ = \ \langle X^e, \{f_i(x)\}_{i \in \mathbb{N}} \rangle, \tag{4}$$

where the set X^e of *alternatives* x coincides with the set of Nash equilibrium strategy profiles x^e in the game (1), and the ith criterion $f_i(x)$ is the payoff function of player i.

Definition 3.3. *An alternative $x^P \in X^e$ is Pareto optimal (efficient) in the problem* (4), *if $\forall x \in X^e$ the system of inequalities*

$$f_i(x) \ \geqslant \ f_i(x^P) \quad (i \in \mathbb{N}),$$

is infeasible, with at least one being a strict inequality. $\{x^P\}$ $\mathbf{X}^P$.
Designate by X^P the set of all $\{x^P\}$.

According to Definition 3.3, the set X^P satisfies the inclusion$X^P \subseteq X^e$ and is *internally stable*.

The following **statement** is obvious: if for all $x \in X^e$ we have

$$\sum_{i\in\mathbb{N}} f_i(x) \leqslant \sum_{i\in\mathbb{N}} f_i(x^P), \tag{5}$$

then x^P gives the Pareto optimal alternative in the problem (4).

Definition 3.4. [18] *A strategy profile $x^* \in X$ is called a Pareto-optimal Nash equilibrium* ($PONE$) *for the game* (1) *if x^* is*

a) *a Nash equilibrium in* (1) (*Definition* 3.1)*, and*

b) *a Pareto optimum in* (5) (*Definition* 3.3)*.*

4 Game with Fuzzy Payoffs

Further, a non-cooperative N-person game is considered

$$\widetilde{\Gamma} = \langle \mathbf{N}, \{X_i\}_{i\in\mathbf{N}}, \{\widetilde{f}_i(x)\}_{i\in\mathbf{N}}\rangle. \tag{6}$$

A game (6) differs from (1) according to payoffs functions. In (6), a payoff function of player i is $\widetilde{f}_i(x) : X \to \mathfrak{F}$. In addition, X_i involves only a finite number of elements, $\widetilde{\Gamma}$ being a finite game with fuzzy payoffs. A game(6) is a bimatrix game with fuzzy payoffs when $\mathbf{N} = \{1, 2\}$.

Determining the concept of optimality, we have to compare payoffs. We can used some defuzzification operator T ($T(\cdot) = Y(\cdot)$, $V_P(\cdot)$, $U(\cdot, \nu)$ etc.). In [9] we proposed the following definition.

Definition 4.1. [9] *A strategy profile $x^e = (x_1^e, ..., x_N^e) \in X$ is called a $T(\cdot)$-Nash equilibrium in the game* (6) *if*

$$f_i(x^e \| x_i) \preceq_T f_i(x^e) \quad (i \in \mathbf{N}).$$

We note that the solutions, which defined in [11], [5] and [6], are particular cases of Definition 4.1.

Next, we consider the associated crisp game for (6)

$$\widetilde{\Gamma}_c = \langle \mathbf{N}, \{X_i\}_{i\in\mathbf{N}}, \{T(\widetilde{f}_i(x))\}_{i\in\mathbf{N}}\rangle. \tag{7}$$

Theorem 4.1. [9] *Let x^e is a Nash equilibrium in* (7) *and $T(\cdot)$ is a linear defuzzification operator, then x^e is $T(\cdot)$-Nash equilibrium in a game* (6).

For example, we consider one bimatrix game with a triangular fuzzy payoffs.

Example 4.1. Consider a bimatrix game with fuzzy payoffs $\widetilde{\Gamma}$ of the form $(\widetilde{A}, \widetilde{B})$, where $\widetilde{A}$ and $\widetilde{B}$ are the triangular fuzzy matrixes:

$$\widetilde{A} = \begin{pmatrix} (30,6,12) & (10,8,6) & (15,10,5) \\ (20,10,5) & (22,6,10) & (30,5,10) \\ (10,8,12) & (30,20,4) & (20,8,16) \end{pmatrix}$$

and

$$\widetilde{B} = \begin{pmatrix} (10,4,6) & (15,10,5) & (20,15,4) \\ (20,10,10) & (10,6,14) & (15,10,5) \\ (12,6,10) & (15,10,10) & (10,5,15) \end{pmatrix}.$$

The operator $U(\cdot,\nu)$ is used. As a result, we obtain the associated crisp game (4).

If $\nu = 0$, then the game (4) given as follows

$$A = \begin{pmatrix} 27 & 6 & 10 \\ 15 & 19 & 27,5 \\ 6 & 20 & 16 \end{pmatrix}, \quad B = \begin{pmatrix} 8 & 10 & 12,5 \\ 15 & 7 & 10 \\ 9 & 10 & 7,5 \end{pmatrix}.$$

There are 3 pure and mixed $U(\cdot,0)$-Nash equilibrium strategy profiles. It is $x^e = (x_1^e, x_2^e)$, where
$1^0)$ $x_1^e = (0,0,1)$, $x_2^e = (0,1,0)$,
$2^0)$ $x_1^e = (\frac{10}{19}, \frac{9}{19}, 0)$, $x_2^e = (\frac{35}{59}, 0, \frac{24}{59})$,
$3^0)$ $x_1^e = (0, \frac{1}{9}, \frac{8}{9})$, $x_2^e = (\frac{1}{10}, \frac{9}{10}, 0)$.

If $\nu = \frac{1}{2}$, then the associated crisp game (4) given as follows

$$A = \begin{pmatrix} 31,5 & 9,5 & 13,75 \\ 18,75 & 23 & 31,25 \\ 11 & 26 & 22 \end{pmatrix}, \quad B = \begin{pmatrix} 10,5 & 13,75 & 17,25 \\ 20 & 12 & 13,75 \\ 13 & 15 & 12,5 \end{pmatrix}.$$

There are 3 $U(\cdot,\frac{1}{2})$-Nash equilibrium strategy profiles. It is $x^e = (x_1^e, x_2^e)$, where
$1^0)$ $x_1^e = (0,0,1)$, $x_2^e = (0,1,0)$,
$2^0)$ $x_1^e = (\frac{25}{52}, \frac{27}{52}, 0)$, $x_2^e = (\frac{70}{121}, 0, \frac{51}{121})$,
$3^0)$ $x_1^e = (0, \frac{1}{5}, \frac{4}{5})$, $x_2^e = (\frac{12}{43}, \frac{31}{43}, 0)$.

If $\nu = 1$, then the game (4) given as follows

$$A = \begin{pmatrix} 36 & 13 & 17,5 \\ 22,5 & 27 & 35 \\ 16 & 32 & 28 \end{pmatrix}, \quad B = \begin{pmatrix} 13 & 17,5 & 22 \\ 25 & 17 & 17,5 \\ 17 & 20 & 17,5 \end{pmatrix}.$$

There are 3 $U(\cdot,1)$-Nash equilibrium strategy profiles. It is $x^e = (x_1^e, x_2^e)$, where
$1^0)$ $x_1^e = (0,0,1)$, $x_2^e = (0,1,0)$,
$2^0)$ $x_1^e = (\frac{5}{11}, \frac{6}{11}, 0)$, $x_2^e = (\frac{35}{62}, 0, \frac{27}{62})$,
$3^0)$ $x_1^e = (0, \frac{3}{11}, \frac{8}{11})$, $x_2^e = (\frac{10}{23}, \frac{13}{23}, 0)$.

Another example: one zero-sum matrix game with a trapezoidal fuzzy payoffs is considered .

Example 4.2. Let $\widetilde{A}$ be the trapezoidal fuzzy payoff matrixes of the fuzzy zero-sum matrix game $\widetilde{\Gamma}$, which is given as follows:

$$\widetilde{A} = \begin{pmatrix} (20,30,12,8) & (1,5,8,4) \\ (5,9,20,4) & (10,26,8,12) \end{pmatrix}.$$

The operator $Y(\cdot)$ is used. As a result, we obtain the associated crisp game (4)

$$A = \begin{pmatrix} 24 & 2 \\ 3 & 19 \end{pmatrix}.$$

The mixed $Y(\cdot)$-Nash equilibrium is $x^e = (x_1^e, x_2^e)$, where $x_1^e = (\frac{8}{19}, \frac{11}{19})$, $x_2^e = (\frac{17}{38}, \frac{21}{38})$.

5 Fuzzy Matrix Game with Different Preferences

In this section, we consider a two-person zero sum game

$$\widetilde{\Gamma}_a = \langle \{1,2\}, \{X_i\}_{i=1,2}, \widetilde{f}_1(x) \rangle, \tag{8}$$

where $\{1,2\}$ is the set of players' serial numbers; each player i chooses and applies his own pure strategy $x_i \in X_i \subseteq R^{n_i}$ $(i = 1,2)$, a strategy profile is $x = (x_1, x_2) \in X = X_1 \times X_2 \subset R^n$ $(n = n_1 + n_2)$; a payoff function of player 1 is $\widetilde{f}_1(x) : X \to \mathfrak{F}$. A payoff function of player 2 is $\widetilde{f}_2(x) = -\widetilde{f}_1(x)$.

In addition, let X_i contains only a finite number of elements. In this case, $\widetilde{\Gamma}_a$ is *a two-person zero sum matrix game with fuzzy payoffs*. This game is determined by a fuzzy matrix $\widetilde{A}$.

In the last section, we considered that the players prefer the same defuzzification operator. However, the players can have the different preferences. For example, it can be caused by the various attitude to the risk. In this case, the players can use the different defuzzification operators. The main idea of this paper is following:

Suppose that the player 1 has decided to use a defuzzification operator $T_1(\cdot)$. And the player 2 chose to use a defuzzification operator $T_2(\cdot)$ $(T_1(\cdot) \neq T_2(\cdot))$.
Definition 5.1. *A strategy profile $x^* = (x_1^*, x_2^*) \in X$ is called a $T_1(\cdot)T_2(\cdot)$-saddle-point in the game* (8) *if*

$$\widetilde{f}_1(x_1, x_2^*) \preceq_{T_1} \widetilde{f}_1(x_1^*, x_2^*) \preceq_{T_2} \widetilde{f}_1(x_1^*, x_2) \quad \forall\, x_1 \in X_1,\ x_2 \in X_2.$$

Next, we consider the associated crisp game for (8)

$$\Gamma_a = \langle \{1,2\}, \{X_i\}_{i=1,2}, \{T_1(\widetilde{f}_1(x)), -T_2(\widetilde{f}_1(x))\} \rangle. \tag{9}$$

In contrast to (8), a crisp game (9) is a bimatrix game. Usually, the solution of crisp game (9) is Nash equilibrium. But, a set of Nash equilibriums X^e is internally instable. We will use PONE in this game.
Theorem 5.1. *Let x^* is a Pareto-optimal Nash equilibrium in* (9) *and $T_1(\cdot)$, $T_2(\cdot)$ are a linear defuzzification operators, then x^* is $T_1(\cdot)T_2(\cdot)$-saddle-point in a game* (8).

For example, we consider zero-sum matrix game with a trapezoidal fuzzy payoffs from Example 4.2..
Example 5.1. Let $\widetilde{A}$ be the trapezoidal fuzzy payoff matrixes of the fuzzy zero-sum matrix game $\widetilde{\Gamma}$, which is given as follows:

$$\begin{pmatrix} (20,30,12,8) & (1,5,8,4) \\ (5,9,20,4) & (10,26,8,12) \end{pmatrix}.$$

The operator $Y(\cdot)$ is used for player 1, and the operator $U(\cdot, 0)$ is used for player 2.

As a result, we obtain the associated crisp bimatrix game (9) given as follows

$$A = \begin{pmatrix} 24 & 2 \\ 3 & 19 \end{pmatrix}, \quad B = \begin{pmatrix} -14 & 3 \\ 5 & -6 \end{pmatrix}.$$

The $Y(\cdot)U(\cdot, 0)$-saddle-point is $x^* = (x_1^*, x_2^*)$, where $x_1^* = (\frac{11}{28}, \frac{17}{28})$, $x_2^* = (\frac{17}{38}, \frac{21}{38})$.

6 Conclusion

In this paper we proposed a method for formalizing and constructing equilibrium in fuzzy matrix games to generalize some already known methods. In the future, we will apply it for formalizing a Berge equilibrium [19] and a coalition equilibrium [20] in n-person games with fuzzy payoffs. The case of continuous game is also if great interest. In the case, when we construct equilibrium in pure strategies, the linearity condition of a defuzzification operator is not required. We plan to study a continuous games case.

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