

The Planning Investment Project with Identical Independent Jobs ^{*}

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Abstract. In the article, the investment project scheduling problem with identical technologically independent jobs is considered. The processing time of any job is a unit. A use of credit is allowed. In research of structure of the optimal solution, the continuous approximation of a problem is used. The result can then be presented in analytical view. Also, in the article, the algorithm of the solution of a problem in discrete case is offered.

Keywords: Project scheduling · NP-hardness · Pseudo-polynomial algorithm

1 Introduction

The project scheduling problem with limited resources is NP-hard in the strong sense [1]. There are few cases when the problem is polynomially solvable. This problem of minimization of the common runtime of the project in restriction for resources only the stored type [2]. But even if the resources are stored, and as criterion mean time of completion of jobs or the net present value is used, then the problem becomes strongly NP-hard [5].

Problem with the credits, when there are practically no restrictions on resources, but their using has to be paid, was considered in [3]. In [4] the strongly NP-hard of this problem was proved. The problem of finding the maximum clique in the graph was reduced to its. The proof relies heavily on the type of technological dependence of the jobs. From this fact follows that the difficulty of the problem is connected, first of all, with the partial order of performing the jobs. In such a situation, the question of the complexity of the problem with an empty partial order, when all the jobs are independent, attracts interest. At present, the computational complexity of the task of maximizing project profit at technologically independent jobs remains open.

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In this article, to investigate the structure of the solution of the problem assigned, the case of identical technologically independent jobs of unit duration is considered. In the assumption that the number of jobs is sufficiently large, the schedule can be set not by the number of jobs performed at any time, but by the fraction from their total number. Such a continuous approximation allows us to obtain analytical formulas and understand the structure of the optimal solution of the problem. In the conclusion of the article, a discrete version of the problem is considered, the pseudo-polynomial algorithm of its decision is offered, the convergence of this decision to the continuous decision experimentally is established with an increase in the number of jobs.

2 Problem Formulation

There are N technologically independent jobs of unit duration, and the only type of a resource is financial. For each job of $j = 1, 2, \dots, N$ are set necessary capital investments of k_j . Cash is invested in the moment of the beginning of the performance of a job. Upon completion of the job, we receive an income of c_j . The investor possesses an initial capital in the amount of K_0 . There is a possibility of crediting at the rate of r for the unit period of time. Reserve cash is placed on the market under an alternative risk-free rate of r_0 . For comparison of cash at different times, we use discounting operation. It is required to draw up the schedule of execution of jobs in which the net present value of the entire project will be maximum.

We will give an interpretation of this problem. The investor owns a plot of land on which he wants to build a cottage village of N houses. Each house is built independently of others, and the beginning of its construction depends solely on financing. If cash is available that materials can be bought, workers can be employed. Upon completion of construction, the house is for sale. The cash received from the sale can be invested in the construction of other houses. Use of the credits is possible. It is necessary to determine the start time of the construction of houses at which the general profit of all project will be maximum.

With single job durations, there is a schedule in which all jobs begin and end at integer instants of time. Then the schedule is determined by the sets of jobs N_t , the execution of which begins at time t , $t = 0, 1, \dots, T - 1$. T is the completion time of the entire project, which must also be found.

Let F_t is income by the time of t , $t = 0, 1, \dots, T$. Originally $F_0 = K_0$. For $t = 0, 1, \dots, T - 1$ the value of income of F_{t+1} is calculated recurrently. If there isn't enough cash of F_t on the current investments, for the amount of $\sum_{j \in N_t} k_j - F_t$ we take the credit under rate of r , and, therefore

$$F_{t+1} = (F_t - \sum_{j \in N_t} k_j)(1 + r) + \sum_{j \in N_t} c_j,$$

otherwise reserve cash in the amount of $F_t - \sum_{j \in N_t} k_j$ is placed at the rate of r_0

$$F_{t+1} = (F_t - \sum_{j \in N_t} k_j)(1 + r_0) + \sum_{j \in N_t} c_j.$$

The income F_T discounted to the initial moment of time. Subtracting the initial capital investments of K_0 we will receive value of the net present value of

$$NPV(S) = \frac{F_T}{(1 + r_0)^T} - K_0,$$

which needs to be maximized on a set of all schedules of performance of jobs. Thus, it is required to find the time of completion of the project and the partition of the set of all jobs into a sequence of subsets $(N_0, N_1, \dots, N_{T-1})$.

We will write out some conditions of implementation of the project and its separate jobs.

Statement 1.1. Jobs j are performed only under the condition $\frac{c_j}{k_j} \geq 1 + r_0$.

Really, if $\frac{c_j}{k_j} < 1 + r_0$, then the shift at the beginning of job j for a later period increases the net present value of the entire project. Therefore, its implementation will postpone for an indefinitely long term.

Statement 1.2. To execute a project, it is necessary and sufficient that either $K_0 > 0$, or there exists $j \in \{1, 2, \dots, N\}$, for which $\frac{c_j}{k_j} > 1 + r$.

The proof of necessity is obvious, if there is no start-up capital and performance of job at the expense of credit leads to a negative result, then a positive balance will never be achieved. The sufficiency follows from the assumptions of the model in question about investing reserve capital under the rate r_0 . If $K_0 > 0$, then this cash is placed under the rate r_0 , and through a certain period of time, we receive the necessary sum for a performance of the project's jobs. Under the second condition, we perform the job j at the initial instant of time. Because of the $\frac{c_j}{k_j} > 1 + r$, we will gain income which can also be placed at a rate of r_0 and after a while to receive the necessary sum of cash.

Thus, to start a project, you either need the existence of seed capital or that the profitability at least of one job was above a credit rate. In this case, the profitability of all jobs must be higher than the rate r_0 .

3 Continuous Case

The continuous case for the purpose of identification of the structure of an optimal solution, research of dependence of this structure from the initial capital, the profitability of jobs and a rate of liquidity will consider.

We will assume that all jobs are identical and $N \rightarrow \infty$. In this situation the schedule is understood as a vector $(x_0, x_1, \dots, x_{T-1})$, where x_t is the share of the jobs starting in t instant. The following relation is executed $\sum_{t=0}^{T-1} x_t = 1$.

Without loss of generality, we denote by k is the common investments, and by

c is the total income. Then the investments in the year t will be $k \cdot x_t$, and the corresponding receipts in the year $t + 1$ will be $c \cdot x_t$. The schedule for which the profit is led to the initial moment it will be maximum is required to be found.

Theorem 2.1. *If $\frac{c}{k} > 1 + r$, then the shift of the part of the job performed at the expense of the credit at an earlier time increases the NPV.*

Proof. Suppose that at the moment t a credit is taken $+\varepsilon k$, at the next moment of time $t + 1$ there is an income $+c\varepsilon$ from the sale, part of which is to repay the loan taken at the previous time, namely, $-\varepsilon k(1 + r)$. Then

$$NPV = \frac{c\varepsilon - \varepsilon k(1 + r)}{(1 + r_0)^{t+1}} = \frac{\varepsilon}{(1 + r_0)^{t+1}} \cdot (c - k(1 + r)).$$

$c - k(1 + r) > 0$, because $\frac{c}{k} > 1 + r$. Therefore, if you take the credit at an earlier moment of time, then the NPV will increase at $\frac{c}{k} > 1 + r$.

Corollary. In an optimal solution, we take the credit only at the initial time.

For the finding of a share of jobs which need to be executed in time points of $t = 1, 2, \dots, T - 1$, we will use an auxiliary formula:

$$x'_t = \frac{c - kr - k}{c \left(1 - \frac{kr}{c} - \frac{k^2 r}{c^2} - \dots - \frac{k^t r}{c^t}\right)} = \frac{c - kr - k}{c \left(1 - \frac{kr}{c} \cdot \frac{1 - (\frac{k}{c})^t}{1 - \frac{k}{c}}\right)},$$

and we will calculate the fractions starting from the moment $t = T - 1$, then $t = T - 2$ and until the time $t = 1$ according to the following rule:

$$\begin{aligned} x_{T-1} &= \frac{c - kr - k}{c \left(1 - \frac{kr}{c} \cdot \frac{1 - (\frac{k}{c})^{T-1}}{1 - \frac{k}{c}}\right)}, \\ x_{T-2} &= \frac{c - kr - k}{c \left(1 - \frac{kr}{c} \cdot \frac{1 - (\frac{k}{c})^{T-2}}{1 - \frac{k}{c}}\right)} \cdot (1 - x_{T-1}), \\ x_{T-3} &= \frac{c - kr - k}{c \left(1 - \frac{kr}{c} \cdot \frac{1 - (\frac{k}{c})^{T-3}}{1 - \frac{k}{c}}\right)} \cdot (1 - (x_{T-1} + x_{T-2})), \\ &\vdots \\ x_1 &= \frac{c - kr - k}{c \left(1 - \frac{kr}{c}\right)} \cdot \left(1 - \sum_{t=2}^{T-1} x_t\right), \end{aligned}$$

For time point of $t = 0$ share of jobs is calculated by the formula:

$$x_0 = \frac{k}{c - kr} \cdot \left(1 - \sum_{t=2}^{T-1} x_t\right).$$

The Net Present Value of the entire project have calculated by the formula:

$$NPV = \frac{c \cdot x_{T-1}}{(1+r_0)^T}.$$

We will denote by $a = \frac{k}{c}$. As $\sum_{t=0}^{T-1} x_t = 1$, the objective function takes the form

$$NPV = \frac{c}{(1+r_0)^T} \cdot \frac{1-a(1+r)}{1-ar \cdot \left(\frac{1-a^{T-1}}{1-a}\right)} \rightarrow \max_T.$$

We will find the optimal value of this function:

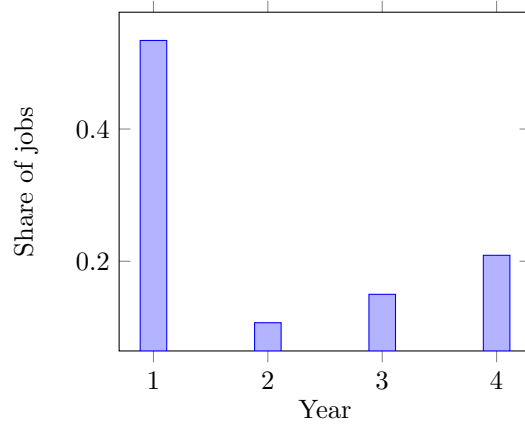
$$\begin{aligned} & \left(c(1-a(1+r)) \cdot \frac{1}{(1+r_0)^T \cdot \left(1-ar \cdot \frac{1-a^{T-1}}{1-a}\right)} \right)' = 0, \\ & c(1-a(1+r)) \cdot \left(\frac{1}{(1+r_0)^T - \frac{ar-a^T r}{1-a} \cdot (1+r_0)^T} \right)' = 0, \\ & \left(\frac{-[(1+r_0)^T \cdot \ln(1+r_0) - \frac{ar}{1-a} \cdot (1+r_0)^T \cdot \ln(1+r_0)]}{\left((1+r_0)^T - \frac{ar-a^T r}{1-a} \cdot (1+r_0)^T\right)^2} \right)' - \\ & - \left(\frac{\frac{r}{1-a} \cdot (a+ar_0)^T \ln a(1+r_0)}{\left((1+r_0)^T - \frac{ar-a^T r}{1-a} \cdot (1+r_0)^T\right)^2} \right) = 0, \\ & (1+r_0)^T \cdot \ln(1+r_0) \cdot \left(1 - \frac{ar}{1-a}\right) = -\frac{r}{1-a} \cdot (a+ar_0)^T \ln a(1+r_0), \\ & a^T = \frac{\ln(1+r_0) \cdot \left(\frac{ar}{1-a} - 1\right)}{\frac{r}{1-a} \cdot \ln a(1+r_0)}. \\ & T = \log_a \left(\frac{(a(1+r) - 1) \cdot \ln(1+r_0)}{r \cdot \ln a(1+r_0)} \right). \end{aligned}$$

Example 2.1. $k = 1$ is investments in job, $c = 1,4$ is income from job, $r = 0,2$ is rate on the credit, $r_0 = 0,1$ is discount rate. The optimal time of completion of the project is $T \approx 3,8$. Optimal integer value is $T^* = 4$. Optimal solution is $(x_0, x_1, x_2, x_3) = (0, 534; 0, 107; 0, 150; 0, 209)$

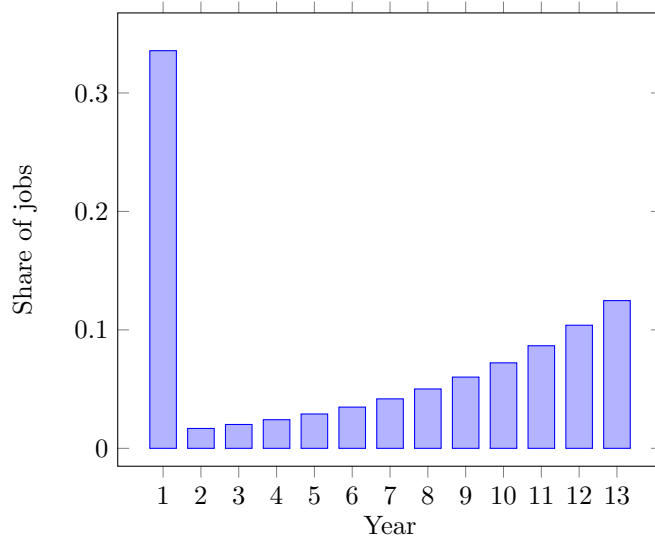
Value of profit

$T = 1$	$T = 2$	$T = 3$	$T^* = 4$	$T = 5$
0,182	0,193	0,199	0,200	0,197

The solution looks like this:



Example 2.2. $k = 2$ is investments in job, $c = 2,4$ is income from job, $r = 0,15$ is rate on the credit, $r_0 = 0,05$ is discount rate. The optimal time of completion of the project is $T \approx 12,6$. The solution looks like this:



Statement 2.1. *There will be such values of variables c , k , r provided that $\frac{c}{k} \rightarrow 1 + r$, for which $T \sim \infty$.*

Proof. The completion time of the project is calculated by the formula

$$T = \log_a \left(\frac{(a(1+r) - 1) \cdot \ln(1+r_0)}{r \cdot \ln a(1+r_0)} \right),$$

where $a = \frac{k}{c}$. On a theorem condition $\frac{c}{k} \rightarrow 1 + r$, therefore $a = \frac{k}{c} \rightarrow \frac{1}{1+r}$. Then

$$\begin{aligned} & \lim_{a \rightarrow \frac{1}{1+r}} \log_a \left(\frac{(a(1+r) - 1) \cdot \ln(1+r_0)}{r \cdot \ln a(1+r_0)} \right) = \\ & = \lim_{a \rightarrow \frac{1}{1+r}} \log_{\frac{1}{1+r}} \left(\frac{\left(\frac{1}{1+r} \cdot (1+r) - 1\right) \cdot \ln(1+r_0)}{r \cdot \ln \frac{1}{1+r} \cdot (1+r_0)} \right) = \lim_{a \rightarrow \frac{1}{1+r}} \log_{\frac{1}{1+r}} 0 = \infty, \end{aligned}$$

as $0 < \frac{1}{1+r} < 1$. Therefore $T \sim \infty$ at $\frac{c}{k} \rightarrow 1 + r$.

4 Continuous Case with Nonzero Initial Capital

In This Section, We Will Consider the Continuous Case When for Implementation of the Project There is an Initial Capital That is Not Equal to Zero.

Theorem 3.1. *If $K \geq \frac{k^2}{c+k}$, then in initial time jobs is made at the expense of the tentative capital and all of the remaining jobs is due to reinvestment of income. At the same time, all project will manage to be realized in two years.*

Proof. Really, let $K = \frac{k^2}{c+k}$. Then in an initial instant, the share of the performed jobs is equal

$$x_0 = \frac{K}{k} = \frac{\frac{k^2}{c+k}}{k} = \frac{k}{c+k}.$$

Income will be $c \cdot \frac{k}{c+k}$. It is necessary to execute

$$x_1 = 1 - \frac{k}{c+k} = \frac{c+k-k}{c+k} = \frac{c}{c+k}$$

jobs, investments into which are equal to $\frac{c}{c+k} \cdot k$.

The net present value of the project

$$\begin{aligned} NPV &= -K + \frac{\frac{k}{c+k} \cdot c - \frac{c}{c+k} \cdot k}{1+r_0} + \frac{\frac{c}{c+k} \cdot c}{(1+r_0)^2} = -\frac{k^2}{c+k} + \frac{c^2}{(1+r_0)^2 \cdot (c+k)} = \\ &= \frac{c^2 - k^2(1+r_0)^2}{(c+k) \cdot (1+r_0)^2} \end{aligned}$$

Now $K > \frac{k^2}{c+k}$, then $x_0 = \frac{K}{k}$, $x_1 = 1 - \frac{K}{k}$. The profit of the project in this case will be

$$\begin{aligned} NPV &= -K + \frac{\frac{K}{k} \cdot c - \left(1 - \frac{K}{k}\right) \cdot k}{1+r_0} + \frac{\left(1 - \frac{K}{k}\right) \cdot c}{(1+r_0)^2} = \\ &= \frac{-Kk(1+r_0)^2 + Kc(1+r_0) - k^2(1+r_0) + Kk(1+r_0) + kc - Kc}{(1+r_0)^2 k} = \end{aligned}$$

$$= \frac{k(c - k - kr_0) - Kr_0(c + k + kr_0)}{(1 + r_0)^2 k}$$

Thus, it turned out that the proceeds from the sale of the share of jobs completed at the initial time are equal to the investments in the remaining jobs. Thus, there will be enough starting capital to realize the project in two years without taking of the credit.

If $K < \frac{k^2}{c+k}$, then in an optimal solution for the finding of a share of jobs which need to be executed in instants $t = 2, 3, \dots, T - 1$ we use an auxiliary formula:

$$x'_t = \frac{K + Kr + c - k - kr}{c \left(1 - \frac{kr}{c} - \dots - \frac{k^t r}{c^t} + \frac{K}{c} \cdot \left(\frac{k^{t-1}}{c^{t-1}} + \dots + \frac{k}{c} + 1 \right) + \frac{Kr}{c} \cdot \left(1 + \dots + \frac{k^{t-1}}{c^{t-1}} \right) \right)}$$

or

$$x'_t = \frac{K + Kr + c - k - kr}{c \left(1 - \frac{kr}{c} \cdot \frac{1 - (\frac{k}{c})^t}{1 - \frac{k}{c}} + \frac{K}{c} \cdot (1 + r) \cdot \frac{1 - (\frac{k}{c})^{t-1}}{1 - \frac{k}{c}} \right)}$$

and we will calculate the fractions starting from the moment $t = T - 1$, then $t = T - 2$ and until the time $t = 2$ according to the following rule:

$$x_{T-1} = \frac{K + Kr + c - k - kr}{c \left(1 - \frac{kr}{c} \cdot \frac{1 - (\frac{k}{c})^{T-1}}{1 - \frac{k}{c}} + \frac{K}{c} \cdot (1 + r) \cdot \frac{1 - (\frac{k}{c})^{T-2}}{1 - \frac{k}{c}} \right)},$$

$$x_{T-2} = \frac{K + Kr + c - k - kr}{c \left(1 - \frac{kr}{c} \cdot \frac{1 - (\frac{k}{c})^{T-2}}{1 - \frac{k}{c}} + \frac{K}{c} \cdot (1 + r) \cdot \frac{1 - (\frac{k}{c})^{T-3}}{1 - \frac{k}{c}} \right)} \cdot (1 - x_{T-1}),$$

$$x_{T-3} = \frac{K + Kr + c - k - kr}{c \left(1 - \frac{kr}{c} \cdot \frac{1 - (\frac{k}{c})^{T-3}}{1 - \frac{k}{c}} + \frac{K}{c} \cdot (1 + r) \cdot \frac{1 - (\frac{k}{c})^{T-4}}{1 - \frac{k}{c}} \right)} \cdot (1 - (x_{T-1} + x_{T-2})),$$

$$\vdots$$

$$x_2 = \frac{K + Kr + c - k - kr}{c \left(1 - \frac{kr}{c} \cdot \frac{1 - (\frac{k}{c})^2}{1 - \frac{k}{c}} + \frac{K}{c} \cdot (1 + r) \right)} \cdot \left(1 - \sum_{t=3}^{T-1} x_t \right),$$

For time point of $t = 1$ and time point of $t = 0$ share of jobs is calculated by the formulas:

$$x_0 = \frac{k - K - Kr}{c - kr} \cdot \left(1 - \sum_{t=2}^{T-1} x_t \right), \quad x_1 = \left(1 - \frac{k - K - Kr}{c - kr} \right) \cdot \left(1 - \sum_{t=2}^{T-1} x_t \right).$$

The net present value of the entire project is calculated by the formula

$$NPV = \frac{c \cdot x_{T-1}}{(1 + r_0)^T} - K.$$

$$x_{T-1} = \frac{K + Kr + c - k - kr}{c \left(1 - \frac{kr}{c} \cdot \frac{1 - (\frac{k}{c})^{T-1}}{1 - \frac{k}{c}} + \frac{K}{c} \cdot (1+r) \cdot \frac{1 - (\frac{k}{c})^{T-2}}{1 - \frac{k}{c}} \right)}$$

We will denote by $a = \frac{k}{c}$, $b = \frac{K}{c}$. Then

$$x_{T-1} = \frac{b(1+r) + 1 - a(1+r)}{\left(1 - ar \cdot \frac{1 - a^{T-1}}{1-a} + b(1+r) \cdot \frac{1 - a^{T-2}}{1-a} \right)}.$$

As $\sum_{t=0}^{T-1} x_t = 1$, the objective function takes the form

$$NPV = \frac{c}{(1+r_0)^T} \cdot \frac{1 + (b-a)(1+r)}{1 - ar \cdot \frac{1 - a^{T-1}}{1-a} + b(1+r) \cdot \frac{1 - a^{T-2}}{1-a}} - K \rightarrow \max_T.$$

We note that this formula holds for $T \geq 2$.

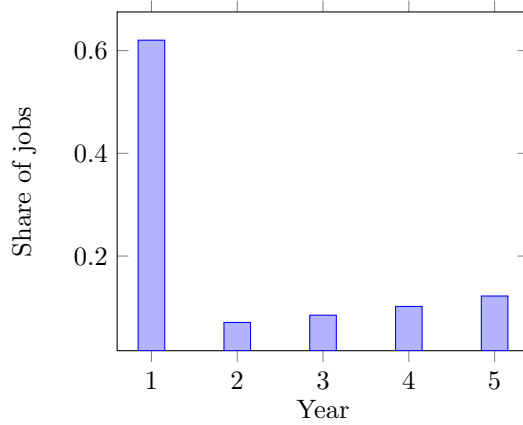
We will find the optimal value of this function:

$$\begin{aligned} & \left(c \cdot \frac{1 + (b-a)(1+r)}{(1+r_0)^T \cdot \left(1 - ar \cdot \frac{1 - a^{T-1}}{1-a} + b(1+r) \cdot \frac{1 - a^{T-2}}{1-a} \right)} \right)' = 0, \\ & \left(\frac{1}{(1+r_0)^T - \frac{ar - a^T r}{1-a} \cdot (1+r_0)^T + b(1+r) \cdot \frac{1 - a^{T-2}}{1-a} \cdot (1+r_0)^T} \right)' = 0, \\ & \left(\frac{- \left[(1+r_0)^T \ln(1+r_0) - \frac{ar}{1-a} \cdot (1+r_0)^T \ln(1+r_0) \right]}{\left((1+r_0)^T - \frac{ar - a^T r}{1-a} \cdot (1+r_0)^T + b(1+r) \cdot \frac{1 - a^{T-2}}{1-a} \cdot (1+r_0)^T \right)^2} \right) - \\ & - \left(\frac{\frac{r}{1-a} \cdot (a + ar_0)^T \ln a(1+r_0) + \frac{b(1+r)}{1-a} \cdot (1+r_0)^T \ln(1+r_0)}{\left((1+r_0)^T - \frac{ar - a^T r}{1-a} \cdot (1+r_0)^T + b(1+r) \cdot \frac{1 - a^{T-2}}{1-a} \cdot (1+r_0)^T \right)^2} \right) + \\ & + \left(\frac{\frac{b(1+r)}{1-a} \cdot [a^{T-2}(1+r_0)^T \ln a + a^{T-2}(1+r_0)^T \ln(1+r_0)]}{\left((1+r_0)^T - \frac{ar - a^T r}{1-a} \cdot (1+r_0)^T + b(1+r) \cdot \frac{1 - a^{T-2}}{1-a} \cdot (1+r_0)^T \right)^2} \right) = 0, \\ & (1+r_0)^T \ln(1+r_0) \cdot \left(1 - \frac{ar}{1-a} + \frac{b(1+r)}{1-a} \right) = \frac{b(1+r)}{a^2(1-a)} a^T (1+r_0)^T \ln a(1+r_0) - \\ & - \frac{r}{1-a} a^T (1+r_0)^T \ln a(1+r_0), \\ & a^T = \frac{\ln(1+r_0) \cdot \left(1 - \frac{ar}{1-a} + \frac{b(1+r)}{1-a} \right)}{\frac{b(1+r)}{a^2(1-a)} \ln a(1+r_0) - \frac{r}{1-a} \cdot \ln a(1+r_0)}, \\ & T = \log_a \left(a^2 \cdot \frac{(1 - (a-b)(1+r)) \cdot \ln(1+r_0)}{(b(1+r) - a^2 r) \cdot \ln a(1+r_0)} \right) = \end{aligned}$$

$$= 2 + \log_a \left(\frac{((a - b)(1 + r) - 1) \cdot \ln(1 + r_0)}{(a^2 r - b(1 + r)) \cdot \ln a(1 + r_0)} \right).$$

The presented continuous solution sets the general character of the investor's behavior at the exercise of the project.

Example 3.1. $k = 2$ is investments in job, $c = 2, 4$ is income from job, $r = 0, 15$ is rate on the credit, $r_0 = 0, 05$ is discount rate, $K = 0, 1$ is starting capital. The optimal time of completion of the project is $T \approx 4, 97$. The solution looks like this:



5 Discrete Case

Now we will return to an initial problem, in which there are N identical works of unit duration. Investments in each job will be k , and receipts will be c .

We use the scheme of dynamic programming. To implement the scheme, it is necessary to construct an upper estimate of the completion time of the project. On condition of $\frac{c}{k} > 1 + r$ it is enough to take $T_{max} = N$. $R(t, n)$ is the maximum income received by the moment t , if by that time n jobs were completed. The variable x_t designates quantity of jobs which begin realization during t . We will write down the Bellman's equation

$$R(t + 1, n) = \max_{x_t=0,1,\dots,n} \{(R(t, n - x_t) - kx_t)(1 + s) + cx_t\},$$

where $s = r$, if $R(t, n - x_t) - kx_t < 0$, and $s = r_0$ otherwise.

For realization of an algorithm for each $n = 1, 2, \dots, N$ we establish starting conditions of $R(1, n) = (K_0 - nk)(1 + s) + nc$, and we will organize a double cycle concerning $t = 1, 2, \dots, T_{max}$ and $n = 1, 2, \dots, N$, we calculate the value $R(t, n)$ on the recursion formula which is written out above.

For the following values of the project $N = 5$, $k = 3$, $c = 5$, $r = 0, 2$, $r_0 = 0, 1$, $K_0 = 0$, the optimal solution is reached in $T = 3$ and quantity of the executed jobs $(x_0, x_1, x_2) = (3, 1, 1)$. The net present value of such a project is

$NPV = 6,5$. If completely to execute all five jobs during the first period of time at the expense of the credits, profit will be 6,36 units.

The complexity of an algorithm does not exceed $O(N^3)$ of operations. We note that this algorithm is a pseudo-polynomial as the problem entrance in a case with jobs of unit duration has a logarithmic dependence on N .

The program was written. The experimental calculations show that at $N \rightarrow \infty$ the discrete decision converges to the continuous optimum which is analytically received above.

Statement 3.1. There will be such values of variables c , k , r provided that $\frac{c}{k} \rightarrow 1 + r$, for which $T \sim N$.

The illustration is given in the table:

N	k	c	r	T
100	5	$6 + 10^{-1}$	0,2	14
100	5	$6 + 10^{-2}$	0,2	25
100	5	$6 + 10^{-3}$	0,2	39
100	5	$6 + 10^{-4}$	0,2	50
100	5	$6 + 10^{-5}$	0,2	64
100	5	$6 + 10^{-6}$	0,2	76
100	5	$6 + 10^{-7}$	0,2	87
100	5	$6 + 10^{-8}$	0,2	94
100	5	$6 + 10^{-9}$	0,2	95

Thus, we receive that at $N = 100$, $r = 0,2$, $k = 5$, $c = 6 + \varepsilon$, $\varepsilon \rightarrow 0$, $T \sim N$.

In connection with a statement 3.1, existence of a polynomial algorithm for problem with identical jobs under doubt, since the output of the problem $(x_0, x_1, \dots, x_{T-1})$ longwise is comparable with N and from the length of the input this dependence is pseudo-polynomial. The problem of the complexity of the problem with arbitrary k_j and c_j remains open.

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