Visualization of Set Inclusion with Gloves

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Abstract. For educational purpose, we introduce visualization media for implication in classical propositional logic and inclusion of finite sets. On the former, we propose a pictorial truth table by means of a pair of pot holders. A mitten of open (closed, respectively) state denotes a true (false) proposition. We let the left mitten dive under the right mitten. Then the left mitten totally hides behind the right mitten exactly when "the left proposition implies the right proposition." On the latter, we represent a set by a glove, where each finger denotes whether an element belongs to the set. The left glove totally hides behind the right glove exactly when the left set is a subset of the right set. A slightly more abstract diagram for set inclusion is also introduced.

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1 Introduction

In classical propositional logic, for propositions p and q, the truth table of "p implies q" (in symbols, $p \to q$) is given by Table 1. For example, refer to [1, Chapter 0]. Here, T denotes true and F denotes false. In other words, $p \to q$ has the same truth table as "(not p) or q".

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1. truth table of implication

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Given sets A and B, set inclusion is defined as follows. A is a subset of B (in symbols, $A \subseteq B$[1]) if the following holds.

$$\text{For all } x \quad (x \in A \to x \in B)$$

Here, "$\to$" is the implication of classical propositional logic.

We are interested in visualization suitable for beginner course of sets and logic. In this paper, on the implication of classical propositional logic and finite set inclusion, we are going to propose new such visualization. Our visualization media are completely different from those of Venn [4]. Instead of circles overlapping each other, we use gloves with wire in them. The left glove totally hides behind the right glove exactly when the left set is a subset of the right set.

In section 2, we give a pictorial truth table of implication. In section 3, we give pictorial expressions of finite set inclusion. In section 4, we introduce a slightly more abstract diagram for set inclusion. The ideas of visualization in this paper are developments of those in the author's previous books [2, 3].

2 Pictorial truth table of implication

In this section, for the implication of classical propositional logic, we introduce a pictorial truth table. We prepare a pair of rectangular pot holders with wire in them. For simplicity, we call them mittens. We assume that a mitten has two possible states, open state and closed state. Fig. 1 is a mitten of open state on a table, where its palm is upward. By bending the perm of the mitten inside, we get a mitten of closed state (Fig. 2).

Fig. 1. open state

Fig. 2. closed state

Suppose that ℓ and r are propositions. Table 2 is a our pictorial truth table of "ℓ implies r". The table is similar to Table 1 except that pictures are inserted into cells. In the column ℓ, we append a picture of open upward left palm (of a mitten, and so forth) to T, and append a picture of closed left upward palm to F. In the column r, we append a picture of open downward right palm to T, and append a picture of closed downward right palm to F. Now, at each row, we

[1] In some literatures, $A \subset B$ is used.

let the left mitten dive under the right mitten. In the new column "overlay", we put a resulting picture. The column of $\ell \to r$ is the same as before.

ℓ	r	overlay	$\ell \to r$
 T	 T		T
 T	 F		F
 F	 T		T
 F	 F		T

Table 2. pictorial truth table of implication

Then, open upward left palm totally hides behind open downward right palm exactly like total solar eclipse. Open upward left palm does not totally hide behind closed downward right palm. Closed upward left palm totally hides behind open downward right palm. Closed upward left palm totally hides behind closed downward right palm. To sum up, in the columns overlay and $\ell \to r$, "totally hides behind" agrees with the truth of $\ell \to r$.

3 Visualization of finite set inclusion

In this section, we are going to visualize finite set inclusion. In the case where our universal set is a singleton, the visualization is achieved as a special case of Table 2. Suppose that E is a singleton whose unique element is 1. In other words, $E = \{1\}$. In this case, E has exactly two subsets. One is E, and the other is $\emptyset$ (the empty set).

Suppose that L and R are subsets of E. Let ℓ be proposition "$1 \in L$" and r be proposition "$1 \in R$". Then, open state denotes E, and closed state denotes $\emptyset$. In other words, the four possible truth values of (ℓ, r); (T, T), (T, F), (F, T) and (F, F) mean the four possible values of (L, R); (E, E), $(E, \emptyset)$, $(\emptyset, E)$ and $(\emptyset, \emptyset)$, respectively. The proposition "$\ell \to r$" agrees with "$L \subseteq R$". Thus, "totally hides behind" agrees with "is a subset of". Therefore, Table 2 visualizes set inclusion in this case.

Next, we are going to investigate the case where our universal set E is $\{1, 2, 3, 4, 5\}$. One way to visualize this situation is to use five pairs of mittens. Instead, we use a pair of gloves. Each finger of a glove is an alternative of a mitten. Each finger has two possible states, stretched state and bent state. Suppose i is one of 1,2,3,4 and 5. Stretched state of finger i denotes proposition "i belongs to the set denoted by the glove". Bent state of finger i denotes proposition "i does not belong to the set denoted by the glove".

We put the left palm upward and the right palm downward. In the case where all the fingers are stretched in the both gloves, the set L denoted by the left glove is $\{1, 2, 3, 4, 5\}$, and the set R denoted by the right glove is the same set (Fig. 3). We let the left glove dive under the right glove. The left glove of Fig. 3 totally hides behind the right glove of Fig. 3. In this case, $\{1, 2, 3, 4, 5\}$ is a subset of $\{1, 2, 3, 4, 5\}$ (Fig. 4).

Fig. 3. $L = \{1, 2, 3, 4, 5\}$, $R = \{1, 2, 3, 4, 5\}$

Fig. 4. $L \subseteq R$

We do not observe all of the 1024 possible combinations. Instead we show some examples to illustrate our idea. We bend the 4th finger and 5th finger of the left glove. The resulting new L is $\{1, 2, 3\}$, and R is $\{1, 2, 3, 4, 5\}$ (Fig. 5). We let the left glove dive under the right glove. The left glove of Fig. 5 totally hides behind the right glove of Fig. 5. $\{1, 2, 3\}$ is a subset of $\{1, 2, 3, 4, 5\}$ (Fig. 6).

Fig. 5. $L = \{1, 2, 3\}$, $R = \{1, 2, 3, 4, 5\}$

Fig. 6. $L \subseteq R$

For the next example, we bend the 1st, 4th and 5th finger of the right glove. Thus $L = \{1, 2, 3\}$ and $R = \{2, 3\}$ (Fig. 7). The left glove does not totally hide behind the right glove. $\{1, 2, 3\}$ is not a subset of $\{2, 3\}$ (Fig. 8).

Fig. 7. $L = \{1, 2, 3\}$, $R = \{2, 3\}$

Fig. 8. $L \nsubseteq R$

In the case where $L = \emptyset$ and $R = \{2, 3\}$ (Fig. 9), the left glove hides behind the right glove. $\emptyset$ is a subset of $\{2, 3\}$ (Fig. 10).

Fig. 9. $L = \emptyset$, $R = \{2, 3\}$

Fig. 10. $L \subseteq R$

In the case where our universal set is $\{1, 2, \ldots, 10\}$, we can carry out the same thing as above. We use two pairs of gloves, say pair A and pair B (Fig. 11). We let the left glove of pair A dive under the right glove of pair A, and let the left glove of pair B dive under the right glove of pair B (Fig. 12).

Fig. 11. $L = \{1, 2, \cdots, 10\}$, $R = \{1, 2, \cdots, 10\}$

Fig. 12. $L \subseteq R$

4 Discussion

The following are examples of cases where we need extra caution.

- The case where the cardinality of the universal set is not a multiple of 5.
- The case where elements of the universal set is not a number, say fruits.
- The case where the universal set is an infinite set.

In order to adapt to these cases, we introduce a slightly more abstract visualization. We draw a baseline. Under the baseline, we write labels for elements, for example 1,2,3,4 and 5. A square above label n denotes an upright nth finger. If there is only the baseline above label n then this denotes a bent nth finger.

For example, counterpart to Fig. 7 is given by Fig. 13.

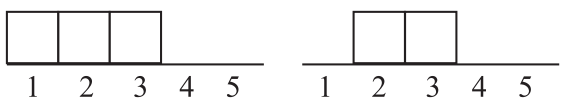

Fig. 13. $L = \{1, 2, 3\}$, $R = \{2, 3\}$

In Fig. 13, we may take the universal set as $\{1, 2, 3\}$. It is no longer unnatural to shorten the baseline and erase labels 4 and 5. Fig. 14 denotes the same sets $\{1, 2, 3\}$ and $\{2, 3\}$ as Fig. 13. The only difference is what set is considered as the universal set.

Fig. 14. $L = \{1, 2, 3\}$, $R = \{2, 3\}$

In the case where $L = \{$ apple, lemon, orange $\}$ and $R = \{$ lemon, orange $\}$, we replace labels 1,2 and 3 by apple, lemon and orange respectively (Fig. 15). Another order, for example lemon, orage, apple is possible, provided that the same order of labels are adopted in the two sets.

Fig. 15. $L = \{$ apple, lemon, orange $\}$, $R = \{$ lemon, orange $\}$

Next, we observe the cases of inifinite sets. For transfinite ordinals and axiom of choice, consult an introductory textbook on set theory [1].

We look at the case where the universal set is von Neumann ordinal $\omega + 4 = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3\}$ and a subset $L = \{1, 3, \ldots, 2n + 1, \ldots, \omega + 1, \omega + 3\}$ is given. Then under the baseline, we write elements of the universal set in order from the smallest. In Fig. 16, we draw L only. If another subset R is given, we draw a similar diagram for R.

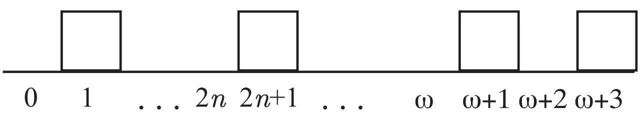

Fig. 16. $\{1, 3, \ldots, 2n + 1, \ldots, \omega + 1, \omega + 3\}$

In general, if the universal set is a von Neumann ordinal, we may imagine a diagram similar to Fig. 16 in our mind. Under axiom of choice, for any set X, there is a bijection from X to an ordinal, thus if once we fix such a bijection we may imagine a similar diagram for X. However, the general case of an infinite set X is not straightforward yet.

5 Concluding remarks

Beginners of sets and logic sometimes confuse exact definition of "implies" with meaning that the phrase "if $\cdots$ then" have in everyday conversation. "F implies T" would sound strange for them. Similar confusion would occur between the technical term "is a subset of" and the phrase "is included by". "The empty set is a subset of a given set" would sound strange for beginners.

Advanced learners can understand the truth table (Table 1) by means of the Boolean algebra of exactly two elements. Nevertheless, such understanding is appropriate only for those who are already familiar with sets and logic. Thus, visualization is valuable for beginners of sets and logic.

In our pictorial truth table of implication (Table 2), "totally hides behind" coincides with the truth of implication. By substituting gloves for mittens, we visualized finite set inclusion. We visualized the empty set by a closed glove.

As a by-product of our visualization media, we can visualize difference between "is an element of" with "is a subset of". Number i is an element of a given set exactly in the case where finger i is stretched in the glove expressing the set. On the other hand, set L is a subset of set R exactly in the case where the upward left palm expressing L totally hides behind the downward right palm expressing R.

In order to see for which type of learners our approach is effective, feedback from various learners and teachers would be helpful.

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