Towards a Unified Algebraic Framework for Non-Monotonicity

Nourhan Ehab[1] and Haythem O. Ismail[2,1]

[1] German University in Cairo, Egypt
Department of Computer Science and Engineering
[2] Cairo University, Egypt
Department of Engineering Mathematics
{nourhan.ehab,haythem.ismail}@guc.edu.eg

Abstract. Tremendous research effort has been dedicated over the years to thoroughly investigate non-monotonic reasoning. With the abundance of non-monotonic logical formalisms, a unified theory that enables comparing the different approaches is much called for. In this paper, we present an algebraic graded logic we refer to as $Log_A\mathbf{G}$ capable of encompassing various non-monotonic logics. One of the most widely studied logics of uncertainty is possibilistic logic tailored for non-monotonic reasoning under incomplete and partially inconsistent knowledge. We show how to encode possibilistic theories as $Log_A\mathbf{G}$ theories, and prove that the $Log_A\mathbf{G}$ logical consequence relation subsumes its possibilistic counterpart. Since possibilistic logic subsumes any non-monotonic inference relation satisfying Makinson's rationality postulates, our results prove that $Log_A\mathbf{G}$ subsumes such inference relations as well while remaining immune to the drowning problem.

1 Introduction

Non-monotonic logics are attempts to model commonsense defeasible reasoning that allows making plausible, albeit possibly fallible, assumptions about the world in the absence of complete knowledge. The term "non-monotonic" refers to the fact that new evidence can retract previous contradicting assumptions. This contrasts with classical logics where new evidence never invalidates previous assumptions about the world. Modelling non-monotonicity has been the focus of extensive studies in the knowledge representation and reasoning community for many years giving rise to a vast family of non-monotonic formalisms. The currently existing approaches to representing non-monotonicity can be classified into two orthogonal families: fixed point logics and model preference logics [4]. Fixed point logics define a fixed point operator by which possibly multiple sets of consistent beliefs can be constructed. Typical non-monotonic logics taking the fixed point approach are Reiter's default logic [25] and Moore's autoepistemic logic [22, 19]. Model preference logics, on the other hand, define non-monotonic logical inference relations with respect to selected preferred models of the world. Typical model preference logics are probabilistic logic [1, 24], McCarthy's circumscription [20], system P proposed by Kraus, Lehmann and Magidor [15], and Pearl's system Z [23]. The wide diversity of all of these logics in addition to their non-standard semantics has rendered the task of gaining a good understanding of them quite hard. For this reason, a unified theory that enables comparing the different approaches is much called for. The purpose of this paper is to present an algebraic graded logic we refer to as $Log_A\mathbf{G}$ [14, 10, 11] capable of encompassing the previously-mentioned non-monotonic logics providing a general framework for non-monotonicity.

Another widely studied approach to non-monotonicity developed independently from the mainstream of the non-monotonic formalisms research is possibilistic logic [7]. In possibilistic logic, propositions are associated with a weight representing the degree to which these propositions are believed to be true. The possibilistic logical consequence relation is non-monotonic and can be expressed by Shoham's preferential entailment [27, 3]. In fact, in [3] it is proved that any non-monotonic inference relation satisfying some widely accepted rationality postulates according to Makinson [18] can be encoded in possibilistic logic. It was also shown how to encode default rules in possibilistic logic to yield the same logical consequences as the rational closure of system P, probabilistic logic, and system Z, proving that possibilistic logic offers a unified understanding of non-monotonic logical consequence relations.

In this paper, we will show how to encode possibilistic theories as $Log_A\mathbf{G}$ theories, and prove that the $Log_A\mathbf{G}$ logical consequence relation subsumes its possibilistic counterpart. In doing so, we prove that $Log_A\mathbf{G}$ captures any non-monotonic inference relation satisfying Makinson's rationality postulates as well, while staying immune to the drowning problem which plagues all logics based on such relations [2, 3]. This acts as the first step towards proving that $Log_A\mathbf{G}$ can be regarded as a unified framework for non-monotonic formalisms. We leave relating $Log_A\mathbf{G}$ to default logic, circumscription, and autoepistemic logic to an extended version of this paper.

In Section 2, $Log_A\mathbf{G}$ will be reviewed describing its syntax and semantics. Section 3 will briefly review possibilistic logic. In Section 4, the main results of this paper, proving that $Log_A\mathbf{G}$ subsumes possibilistic logic, will be presented. Finally, some concluding remarks are outlined in Section 5.

2 $Log_A\mathbf{G}$

$Log_A\mathbf{G}$ is a graded logic for reasoning with uncertain beliefs. "Log" stands for logic, "A" for algebraic, and "G" for grades. In $Log_A\mathbf{G}$, a classical logical formula could be associated with a grade representing a measure of its uncertainty. Non-graded formulas are taken to be certain. In this way, $Log_A\mathbf{G}$ is a logic for reasoning about graded propositions. $Log_A\mathbf{G}$ is algebraic in that it is a language of only terms, some of which denote propositions. Both propositions and their grades are taken as individuals in the $Log_A\mathbf{G}$ ontology. While some multimodal logics such as [6, 21] may be used to express graded grading propositions too, unlike $Log_A\mathbf{G}$, the grades themselves are embedded in the modal operators and are not amenable to reasoning and quantification. This makes $Log_A\mathbf{G}$ a quite expressive language that is still intuitive and very similar in syntax to first-order logic. $Log_A\mathbf{G}$ is demonstrably useful in commonsense reasoning including default reasoning, reasoning with information provided by a chain of sources of varying degrees of trust, and reasoning with paradoxical sentences as discussed in [10, 14].

While most of the graded logics we are aware of employ non-classical modal logic semantics by assigning grades to possible worlds [8], $Log_A\mathbf{G}$ is a non-modal logic with classical notions of worlds and truth values. This is not to say that $Log_A\mathbf{G}$ is a classical logic, but it is closer in spirit to classical non-monotonic logics such as default logic and circumscription. Following these formalisms, $Log_A\mathbf{G}$ assumes a classical notion of logical consequence on top of which a more restrictive, non-classical relation is defined selecting only a subset of the classical models. In defining this relation we take the algebraic, rather than the modal, route. The remaining of this section is dedicated to reviewing the syntax and semantics of $Log_A\mathbf{G}$. A sound and complete proof theory for $Log_A\mathbf{G}$ is presented in [14, 10]. In [14], it was proven that $Log_A\mathbf{G}$ is a stable and well-behaved logic observing Makinson's properties of reflexivity, cut, and cautious monotony.

2.1 $Log_A\mathbf{G}$ Syntax

$Log_A\mathbf{G}$ consists of algebraically constructed terms from function symbols. There are no sentences in $Log_A\mathbf{G}$; instead, we use terms of a distinguished syntactic type to denote propositions. Propositions are included as first-class individuals in the $Log_A\mathbf{G}$ ontology and are structured in a Boolean algebra giving us all standard truth conditions and classical notions of consequence and validity. Additionally, *grades* are introduced as first-class individuals in the ontology. As a result, propositions *about* graded propositions can be constructed, which are themselves recursively gradable.

A $Log_A\mathbf{G}$ language is a many-sorted language composed of a set of terms partitioned into three base sorts: σ_P is a set of terms denoting propositions, σ_D is a set of terms denoting grades, and σ_I is a set of terms denoting anything else. A $Log_A\mathbf{G}$ *alphabet* includes a non-empty, countable set of constant and function symbols each having a syntactic sort from the set $\sigma = \{\sigma_P, \sigma_D, \sigma_I\} \cup \{\tau_1 \longrightarrow \tau_2 \mid \tau_1 \in \{\sigma_P, \sigma_D, \sigma_I\}\}$ and $\tau_2 \in \sigma\}$ of syntactic sorts. Intuitively, $\tau_1 \longrightarrow \tau_2$ is the syntactic sort of function symbols that take a single argument of sort σ_P, σ_D, or σ_I and produce a functional term of sort τ_2[3]. In addition, an alphabet includes a countably infinite set of variables of the three base sorts; a set of syncategorematic symbols including the comma, various matching pairs of brackets and parentheses, and the symbol $\forall$; and a set of logical symbols defined as the union of the following sets: $\{\neg\} \subseteq \sigma_P \longrightarrow \sigma_P$, $\{\wedge, \vee\} \subseteq \sigma_P \longrightarrow \sigma_P \longrightarrow \sigma_P$, $\{\prec, \doteq\} \subseteq \sigma_D \longrightarrow \sigma_D \longrightarrow \sigma_P$, and $\{\mathbf{G}\} \subseteq \sigma_P \longrightarrow \sigma_D \longrightarrow \sigma_P$. Terms involving $\Rightarrow$ [4], $\Leftrightarrow$, and $\exists$ can always be expressed in terms of the above logical operators and $\forall$.

[3] Given the restriction of the first argument of function symbols to base sorts, $Log_A\mathbf{G}$ is, in a sense, a first-order language.

[4] Through out this paper we will use $\Rightarrow$ to denote material implication.

2.2 From Syntax to Semantics

A key element in the semantics of $Log_A\mathbf{G}$ is the notion of a $Log_A\mathbf{G}$ structure.

Definition 1. *A* $Log_A\mathbf{G}$ structure *is a quintuple* $\mathfrak{S} = \langle \mathcal{D}, \mathfrak{A}, \mathfrak{g}, <, \mathfrak{e} \rangle$*, where*

- $\mathcal{D}$*, the domain of discourse, is a set with two disjoint, non-empty, countable subsets: a set of propositions* $\mathcal{P}$*, and a set of grades* $\mathcal{G}$*.*
- $\mathfrak{A} = \langle \mathcal{P}, +, \cdot, -, \perp, \top \rangle$ *is a complete, non-degenerate Boolean algebra.*
- $\mathfrak{g} : \mathcal{P} \times \mathcal{G} \longrightarrow \mathcal{P}$ *is a grading function.*
- $<: \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{P}$ *is an ordering function.*
- $\mathfrak{e} : \mathcal{G} \times \mathcal{G} \longrightarrow \{\perp, \top\}$ *is an equality function, where for every* $g_1, g_2 \in \mathcal{G}$*:* $\mathfrak{e}(g_1, g_2) = \top$ *if* $g_1 = g_2$*, and* $\mathfrak{e}(g_1, g_2) = \perp$ *otherwise.*

A *valuation* $\mathcal{V}$ of a $Log_A\mathbf{G}$ language is a triple $\langle \mathfrak{S}, \mathcal{V}_f, \mathcal{V}_x \rangle$, where $\mathfrak{S}$ is a $Log_A\mathbf{G}$ structure, $\mathcal{V}_f$ is a function that assigns to each function symbol an appropriate function on $\mathcal{D}$, and $\mathcal{V}_x$ is a function mapping each variable to a corresponding element of the appropriate block of $\mathcal{D}$. A valuation $\mathcal{V} = \langle \mathfrak{S}, \mathcal{V}_f, \mathcal{V}_x \rangle$ is called a *natural valuation* if $\mathcal{P}$ is made up of three disjoint sets: the set of base propositions $\mathcal{P}_\mathbf{B}$ forming a subalgebra that does not contain any grading proposition, the set of grading propositions $\mathcal{P}_\mathbf{G} = Range(\mathfrak{g})$, and the set of all other propositions $\overline{\mathcal{P}_\mathbf{B} \cup \mathcal{P}_\mathbf{G}}$, and $\mathcal{V}_f$ maps all propositional terms that do not contain $\mathbf{G}$ to $\mathcal{P}_\mathbf{B}$, all grading propositional terms to a proposition in $\mathcal{P}_\mathbf{G}$, and all other propositional terms to $\overline{\mathcal{P}_\mathbf{B} \cup \mathcal{P}_\mathbf{G}}$. An *interpretation* of $Log_A\mathbf{G}$ terms is given by a function $[\![\cdot]\!]^\mathcal{V}$.

Definition 2. *Let* L *be a* $Log_A\mathbf{G}$ *language and let* $\mathcal{V}$ *be a valuation of* L*. An* interpretation *of the terms of* L *is given by a function* $[\![\cdot]\!]^\mathcal{V}$*:*

- $[\![x]\!]^\mathcal{V} = \mathcal{V}_x(x)$*, for a variable* x
- $[\![c]\!]^\mathcal{V} = \mathcal{V}_f(c)$*, for a constant* c
- $[\![f(t_1, \ldots, t_n)]\!]^\mathcal{V} = \mathcal{V}_f(f)([\![t_1]\!]^\mathcal{V}, \ldots, [\![t_n]\!]^\mathcal{V})$*, for an* n*-adic* $(n \geq 1)$ *function symbol* f
- $[\![(t_1 \wedge t_2)]\!]^\mathcal{V} = [\![t_1]\!]^\mathcal{V} \cdot [\![t_2]\!]^\mathcal{V}$
- $[\![(t_1 \vee t_2)]\!]^\mathcal{V} = [\![t_1]\!]^\mathcal{V} + [\![t_2]\!]^\mathcal{V}$
- $[\![\neg t]\!]^\mathcal{V} = -[\![t]\!]^\mathcal{V}$
- $[\![\forall x(t)]\!]^\mathcal{V} = \prod_{a \in \mathcal{D}} [\![t]\!]^{\mathcal{V}[a/x]}$
- $[\![\mathbf{G}(t_1, t_2)]\!]^\mathcal{V} = \mathfrak{g}([\![t_1]\!]^\mathcal{V}, [\![t_2]\!]^\mathcal{V})$
- $[\![t_1 \prec t_2]\!]^\mathcal{V} = [\![t_1]\!]^\mathcal{V} < [\![t_2]\!]^\mathcal{V}$
- $[\![t_1 \doteq t_2]\!]^\mathcal{V} = \mathfrak{e}([\![t_1]\!]^\mathcal{V}, [\![t_2]\!]^\mathcal{V})$

2.3 Beyond Classical Logical Consequence

We define logical consequence using the familiar notion of filters from Boolean algebra [26].

Definition 3. *A filter of a boolean algebra* $\mathfrak{A} = \langle \mathcal{P}, +, \cdot, -, \perp, \top \rangle$ *is a subset* F *of* $\mathcal{P}$ *such that: (1)* $\top \in F$*; (2) If* $a, b \in F$*, then* $a \cdot b \in F$*; and (3) If* $a \in F$ *and* $a \leq b$*, then* $b \in F$*.*

A propositional term ϕ is a logical consequence of a set of propositional terms Γ if it is a member of the filter of the interpretation of Γ, denoted $F([\![\Gamma]\!]^{\mathcal{V}})$.

Definition 4. *Let* L *be a* $Log_A\mathbf{G}$ *language. For every* $\phi \in \sigma_P$ *and* $\Gamma \subseteq \sigma_P$*,* ϕ *is a logical consequence of* Γ*, denoted* $\Gamma \models \phi$*, if, for every* L*-valuation* $\mathcal{V}$*,* $[\![\phi]\!]^{\mathcal{V}} \in F([\![\Gamma]\!]^{\mathcal{V}})$ *where* $[\![\Gamma]\!]^{\mathcal{V}} = \prod_{\gamma \in \Gamma} [\![\gamma]\!]^{\mathcal{V}}$*.*

Unfortunately, the definition of logical consequence presented in the previous section cannot address uncertain reasoning with graded propositions. To see that, consider the following situation. You see a bird from far away that looks a lot like a penguin. You know that any penguin does not fly. To make sure that what you see is indeed a penguin, you ask your brother who tells you that this bird must not be a penguin since your sister told him that she saw the same bird flying. This situation can be represented in $Log_A\mathbf{G}$ by a set of propositions $\mathcal{Q}$ as shown in Figure 1 where p denotes that the bird is a penguin, and f denotes that the bird flies. For the ease of readability of Figure 1, we write $\neg f$ instead of $-f$ and $p \Rightarrow \neg f$ instead of $-p + -f$. Since you are uncertain about whether the bird you see is a penguin, this is represented as a graded proposition $\mathfrak{g}(p, d1)$ where $d1$ is your uncertainty degree in what you see. What your brother tells you is represented by the grading chain $\mathfrak{g}(\mathfrak{g}(f, d2), d3)$ where $d2$ represents how much you trust your brother, and $d3$ represents how much you trust your sister. Now, consider an agent reasoning with the set $\mathcal{Q}$. Initially, it would make sense for the agent to be able to conclude p even if p is uncertain (and, hence, graded) since it has no reason to believe $\neg p$. The filter $F(\mathcal{Q})$, however, contains the classical logical consequences of $\mathcal{Q}$, but will never contain the graded proposition p. For this reason, we extend our classical notion of filters into a more liberal notion of *graded filters* to enable the agent to conclude, in addition to the classical logical consequences of $\mathcal{Q}$, propositions that are graded in $\mathcal{Q}$ (like p) or follow from graded propositions in $\mathcal{Q}$ (like $\neg f$). This should be done without introducing inconsistencies. Due to nested grading, graded filters come in degrees depending on the depth of nesting of the admitted graded propositions. In Figure 1, $\mathcal{F}^1(\mathcal{Q})$ is the graded filter of degree 1. $\mathcal{F}^1(\mathcal{Q})$ contains everything in $F(\mathcal{Q})$ in addition to the nested graded propositions at depth 1, p and $\mathfrak{g}(\neg q, d1)$. $\neg f$ is also admitted to $\mathcal{F}^1(\mathcal{Q})$ since it follows classically from $\{p, p \Rightarrow \neg f\}$. Hence, at degree 1, we end up believing that the bird is a penguin that does not fly. To compute the graded filter of degree 2, $\mathcal{F}^2(\mathcal{Q})$, we take everything in $\mathcal{F}^1(\mathcal{Q})$ and try to add the graded proposition f at depth 2. The problem is, once we do that, we have a contradiction with $\neg f$ (we now believe that bird flies and does not fly at the same time). To resolve the contradiction, we admit to $\mathcal{F}^2(\mathcal{Q})$ either p (and consequently $\neg f$) or f. In deciding which of p or f to kick out we will allude to their grades. The grade of p is $d1$, and f is graded in a grading chain containing $d2$ and $d3$. To get a fused grade for f, we will combine both $d2$ and $d3$ using an appropriate fusion operator. If $d1$ is less than the fused grade of f, p will not be admitted to the graded filter, together with it consequence $\neg f$. Otherwise, f will not be admitted, and p and $\neg f$ will remain. If we try to compute $\mathcal{F}^3(\mathcal{Q})$, we get everything in $\mathcal{F}^2(\mathcal{Q})$ reaching a fixed point.

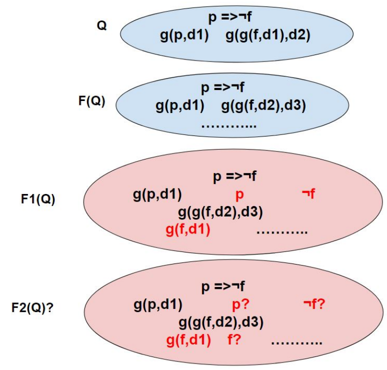

Fig. 1. Graded Filters

In general, the elements of $\mathcal{F}^i(\mathcal{Q})$ will be referred to as the graded consequences at depth i. The rest of this section is dedicated to formally defining graded filters together with our graded consequence relation based on graded filters. In the sequel, for every $p \in \mathcal{P}$ and $g \in \mathcal{G}$, $\mathfrak{g}(p, g)$ will be taken to represent a *grading proposition* that *grades* p. Moreover, if $\mathfrak{g}(p, g) \in \mathcal{Q} \subseteq \mathcal{P}$, then p is *graded* in $\mathcal{Q}$. The set of p *graders in* $\mathcal{Q}$ is defined to be the set $Graders(p, \mathcal{Q}) = \{q | q \in \mathcal{Q}$ and q grades $p\}$. Throughout, a $Log_A\mathbf{G}$ structure $\mathfrak{S} = \langle \mathcal{D}, \mathfrak{A}, \mathfrak{g}, <, \mathfrak{e}\rangle$ is assumed.

As a building step towards formalizing the notion of a graded filter, the structure of graded propositions should be carefully specified. For this reason, the following notion of an *embedded proposition* is defined.

Definition 5. *Let $\mathcal{Q} \subseteq \mathcal{P}$. A proposition $p \in \mathcal{P}$ is embedded in $\mathcal{Q}$ if (i) $p \in \mathcal{Q}$ (ii) or if, for some $g \in \mathcal{G}$, $\mathfrak{g}(p, g)$ is embedded in $\mathcal{Q}$. Henceforth, let $E(\mathcal{Q}) = \{p|p$ is embedded in $\mathcal{Q}\}$.*

Since a graded proposition p might be embedded at any depth $n \in \mathbb{N}$, the degree of embedding of a graded proposition p is defined as follows.

Definition 6. *For $\mathcal{Q} \subseteq \mathcal{P}$, let the degree of embedding of p in $\mathcal{Q}$ be a function $\delta_{\mathcal{Q}} : E(\mathcal{Q}) \longrightarrow \mathbb{N}$, where*

1. *if $p \in \mathcal{Q}$, then $\delta_{\mathcal{Q}}(p) = 0$; and*
2. *if $p \notin \mathcal{Q}$, then $\delta_{\mathcal{Q}}(p) = e + 1$, where $e = \min_{q \in Graders(p, E(\mathcal{Q}))}\{\delta_{\mathcal{Q}}(q)\}$.*

For notational convenience, we let the set of embedded propositions at depth n be $E^n(\mathcal{Q}) = \{p \in E(\mathcal{Q}) \mid \delta_{\mathcal{Q}}(p) \leq n\}$, for every $n \in \mathbb{N}$.

Example 1. Consider $\mathcal{Q} = \{\mathfrak{g}(\mathfrak{g}(\mathfrak{g}(p, 2), 3), 4), \mathfrak{g}(\mathfrak{g}(\mathfrak{g}(r, 2), 3), 5),$
$\mathfrak{g}(\mathfrak{g}(\mathfrak{g}(p, 2), 3), 5), \mathfrak{g}(\mathfrak{g}(p, 2), 6), q\}$.

– $E^0(\mathcal{Q}) = \mathcal{Q}$.

- $E^1(\mathcal{Q}) = E^0(\mathcal{Q}) \cup \{\mathfrak{g}(\mathfrak{g}(p, 2), 3), \mathfrak{g}(\mathfrak{g}(r, 2), 3), \mathfrak{g}(p, 2)\}$.
- $E^2(\mathcal{Q}) = E^1(\mathcal{Q}) \cup \{\mathfrak{g}(p, 2), \mathfrak{g}(r, 2), p\}$.
- $E^3(\mathcal{Q}) = E^2(\mathcal{Q}) \cup \{r\}$.
- $E(\mathcal{Q}) = E^3(\mathcal{Q})$.
- The degree of embedding $\delta_{\mathcal{Q}}(q) = 0$.
- The degree of embedding $\delta_{\mathcal{Q}}(p) = 2$ (since the length of the minimum chain grading p is 2).
- The degree of embedding $\delta_{\mathcal{Q}}(r) = 3$. □

The key to defining graded filters is the intuition that the set of consequences of a proposition set $\mathcal{Q}$ may be further enriched by *telescoping* $\mathcal{Q}$ and accepting some of the propositions graded therein. For this, we need to define (i) the process of telescoping, which is a step-wise process that considers propositions at increasing grading depths, and (ii) a criterion for accepting graded propositions which, as mentioned before, depends on the grades of said propositions. Since the nesting of grading chains is permissible in $Log_A\mathbf{G}$, it is necessary to compute the *fused grade* of a graded proposition p in a chain C to decide whether it will be accepted or not. The fusion of grades in a chain is done according to an operator $\otimes$. Further, since a graded proposition p might be graded by more than one grading chain, we define the notion of the fused grade of p across all the chains grading it by an operator $\oplus$.

Definition 7. *Let $\mathfrak{S}$ be a $Log_A\mathbf{G}$ structure with a depth- and fan-out-bounded $\mathcal{P}$ [5]. A telescoping structure for $\mathfrak{S}$ is a quadruple $\mathfrak{T} = \langle \mathcal{T}, \mathfrak{O}, \otimes, \oplus \rangle$, where*

- *$\mathcal{T} \subseteq \mathcal{P}$, referred to as the top theory;*
- *$\mathfrak{O}$ is an ultrafilter of the subalgebra induced by $Range(<)$ (an ultrafilter is a maximal filter with respect to not including $\perp$) [26];*
- *$\otimes : \bigcup_{i=1}^{\infty} \mathcal{G}^i \longrightarrow \mathcal{G}$; and $\oplus : \bigcup_{i=1}^{\infty} \mathcal{G}^i \longrightarrow \mathcal{G}$.*

Recasting the familiar notion of a *kernel* of a belief base [13] into the context of $Log_A\mathbf{G}$ structures, we say that a $\perp$-kernel of $\mathcal{Q} \subseteq \mathcal{P}$ is a subset-minimal inconsistent set $\mathcal{X} \subseteq \mathcal{Q}$ such that $F(E(F(\mathcal{X})))$ is improper ($= \mathcal{P}$) where $E(F(\mathcal{X}))$ is the set of embedded graded propositions in the filter of $\mathcal{X}$. Let $\mathcal{Q}^{\perp\!\!\perp}\perp$ be the set of $\mathcal{Q}$ kernels that entail $\perp$. A proposition $p \in \mathcal{X}$ *survives* $\mathcal{X}$ in $\mathfrak{T}$ if p is not the weakest proposition (with the least grade) in $\mathcal{X}$. In what follows, the fused grade of a proposition p in $\mathcal{Q} \subseteq \mathcal{P}$ according to a telescoping structure $\mathfrak{T}$ will be referred to as $\mathfrak{f}_{\mathfrak{T}}(p, \mathcal{Q})$.

Definition 8. *For a telescoping structure $\mathfrak{T} = \langle \mathcal{T}, \mathfrak{O}, \otimes, \oplus \rangle$ and a fan-in-bounded [6] $\mathcal{Q} \subseteq \mathcal{P}$, if $\mathcal{X} \subseteq \mathcal{Q}$, then $p \in \mathcal{X}$* survives $\mathcal{X}$ given $\mathfrak{T}$ *if*

1. p is ungraded in $\mathcal{Q}$; or

[5] $\mathcal{P}$ is depth-bounded if every grading chain has at most d distinct grading propositions and is fan-out-bounded if every grading proposition grades at most f_{out} propositions where $d, f_{out} \in \mathbb{N}$ [10].

[6] $\mathcal{Q}$ is fan-in-bounded if every graded proposition is graded by at most f_{in} grading propositions where $f_{in} \in \mathbb{N}$ [10].

2. *there is some ungraded $q \in \mathcal{X}$ such that $q \notin F(\mathcal{T})$; or*
3. *there is some graded $q \in \mathcal{X}$ such that $q \notin F(\mathcal{T})$ and $(\mathfrak{f}_{\mathfrak{T}}(q, \mathcal{Q}) < \mathfrak{f}_{\mathfrak{T}}(p, \mathcal{Q})) \in \mathfrak{O}$.*

The set of kernel survivors of $\mathcal{Q}$ given $\mathfrak{T}$ is the set

$$\kappa(\mathcal{Q}, \mathfrak{T}) = \{p \in \mathcal{Q} \mid \text{ if } p \in \mathcal{X} \in \mathcal{Q}^{\perp\!\!\perp}\perp \text{ then } p \text{ survives } \mathcal{X} \text{ given } \mathfrak{T}\}.$$

The notion of a proposition p being *supported* in $\mathcal{Q}$ is defined as follows.

Definition 9. *Let $\mathcal{Q}, \mathcal{T} \subseteq \mathcal{P}$. We say that p is supported in $\mathcal{Q}$ given $\mathcal{T}$ if*

1. *$p \in F(\mathcal{T})$; or*
2. *there is a grading chain $\langle q_0, q_1, \ldots, q_n\rangle$ of p in $\mathcal{Q}$ with $q_0 \in F(\mathcal{R})$ where every member of $\mathcal{R}$ is supported in $\mathcal{Q}$.*

The set of propositions supported in $\mathcal{Q}$ given $\mathcal{T}$ is denoted by $\varsigma(\mathcal{Q}, \mathcal{T})$.

The $\mathfrak{T}$-induced telescoping of $\mathcal{Q}$ is defined as the set of propositions supported given $\mathcal{T}$ in the set of kernel survivors of $E^1(F(\mathcal{Q}))$.

Definition 10. *Let $\mathfrak{T}$ be a telescoping structure for $\mathfrak{S}$. If $\mathcal{Q} \subset \mathcal{P}$ such that $E^1(F(\mathcal{Q}))$ is fan-in-bounded, then the $\mathfrak{T}$-induced telescoping of $\mathcal{Q}$ is given by*

$$\tau_{\mathfrak{T}}(\mathcal{Q}) = \varsigma(\kappa(E^1(F(\mathcal{Q})), \mathfrak{T}), \mathcal{T}).$$

Observation 1 *If $\mathcal{Q}$ is consistent ($F(\mathcal{Q}) \neq \mathcal{P}$), then $\tau_{\mathfrak{T}}(\mathcal{Q}) = E^1(F(\mathcal{Q}))$.*

If $F(\mathcal{Q})$ has finitely-many grading propositions, then $\tau_{\mathfrak{T}}(\mathcal{Q})$ is defined, for every telescoping structure $\mathfrak{T}$. Hence, provided that the right-hand side is defined, let

$$\tau^n_{\mathfrak{T}}(\mathcal{Q}) = \begin{cases} \mathcal{Q} & \text{if } n = 0 \\ \tau_{\mathfrak{T}}(\tau^{n-1}_{\mathfrak{T}}(\mathcal{Q})) & \text{otherwise} \end{cases}$$

Definition 11. *A* graded filter *of a top theory $\mathcal{T}$, denoted $\mathcal{F}^n(\mathfrak{T})$, is defined as the filter of the $\mathfrak{T}$-induced telescoping of $\mathcal{T}$ of degree n.*

In the following example, we now go back to the situation we introduced at the beginning of this section in Figure 1. We show how the formal construction of the graded filters matches the intuitions we pointed out earlier.

Example 2. Consider $\mathcal{Q} = \{-p + -f, \mathfrak{g}(p, 2), \mathfrak{g}(\mathfrak{g}(f, 2), 3)\}$ and $\mathfrak{T} = \langle \mathcal{Q}, \mathfrak{O}, \otimes, \oplus\rangle$ where $\oplus = max$, and $\otimes = mean$. In what follows, let $\tau^n_{\mathfrak{T}}$ be an abbreviation for $\tau^n_{\mathfrak{T}}(\mathcal{Q})$.

– $\tau^0_{\mathfrak{T}} = \mathcal{Q}$

- $\tau^1_{\mathfrak{T}} \;=\; \tau_{\mathfrak{T}}(\tau^0_{\mathfrak{T}}) = \varsigma(\kappa(E^1(F(\mathcal{Q})), \mathfrak{T}), \mathcal{T})$
  $F(\mathcal{Q}) = \mathcal{Q} \cup \{-p + -f.\mathfrak{g}(p, 2),\ \mathfrak{g}(\mathfrak{g}(f, 2), 3) + \mathfrak{g}(p, 2), ...\}$
  $E^1(F(\mathcal{Q}) = F(\mathcal{Q})\ \cup\ \{p,\ \mathfrak{g}(f, 2)\}$
  $\kappa(E^1(F(\mathcal{Q}), \mathfrak{T}) = E^1(F(\mathcal{Q}))$
  $\tau^1_{\mathfrak{T}} \;=\; \kappa(E^1(F(\mathcal{Q}), \mathfrak{T}) = \varsigma(\kappa(E^1(F(\mathcal{Q}), \mathfrak{T}), \mathcal{Q})$
  $\mathcal{F}^1(\mathfrak{T}) = F(\tau^1_{\mathfrak{T}})$
  Upon telescoping to degree 1, there are no contradictions in $E^1(F(\mathcal{Q}))$ (no $\perp-$kernel $\mathcal{X} \subseteq E^1(F(\mathcal{Q}))$. Hence, everything in $E^1(F(\mathcal{Q}))$ survives telescoping and is supported. At level 1, we believe that the bird we saw is indeed a penguin.
- $\tau^2_{\mathfrak{T}} \;=\; \tau_{\mathfrak{T}}(\tau^1_{\mathfrak{T}})$
  $E^1(F(\tau^1_{\mathfrak{T}})) \;=\; F(\tau^1_{\mathfrak{T}}) \cup \{f\}$
  $\kappa(E^1(F(\tau^1_{\mathfrak{T}})), \mathfrak{T}) = F(\tau^1_{\mathfrak{T}}) - \{p\}$
  $\tau^2_{\mathfrak{T}} = \varsigma(\kappa(E^1(F(\tau^1_{\mathfrak{T}})), \mathfrak{T}), \mathcal{Q}) = \kappa(E^1(F(\tau^1_{\mathfrak{T}})), \mathfrak{T}) - \{-f\}$
  $\mathcal{F}^2(\mathfrak{T}) = F(\tau^2_{\mathfrak{T}})$
  Upon telescoping to degree 2, there are two $\perp-$kernels $\{f, -f\}$ and $\{p, -p+ -f, f\}$. $-f$ survives the first kernel as it is not graded in $\mathcal{Q}$. f survives the first kernel as well as it is the only graded proposition in the kernel with another member $-f \notin F(\mathcal{Q})$. p does not survive the second kernel as the kernel contains another graded proposition f and the grade of p (2) is less than the fused grade of f ($mean(2, 3) = 2.5$). Accordingly, $-f$ loses its support and is not supported in the set of kernel survivors. The graded filter of degree 2 does not contain p or $-f$. At level 2, we start taking into account the information our brother told us. Since our combined trust in our brother and sister is higher that our trust in what we saw, we end up not believing that the bird we saw is a penguin since we believe that it flies.
- $\tau^3_{\mathfrak{T}} = \tau_{\mathfrak{T}}(\tau^2_{\mathfrak{T}})$
  $\tau^3_{\mathfrak{T}} = \varsigma(\kappa(E^1(F(\tau^2_{\mathfrak{T}})), \mathfrak{T}), \mathcal{Q}) = \kappa(E^1(F(\tau^2_{\mathfrak{T}})), \mathfrak{T})$
  $\mathcal{F}^3(\mathfrak{T}) = F(\tau^3_{\mathfrak{T}}) = \mathcal{F}^2(\mathfrak{T})$ reaching a fixed point. □

We use graded filters to define graded consequence as follows. Given a $Log_A\mathbf{G}$ theory $\mathbb{T} \subseteq \sigma_P$ and a valuation $\mathcal{V} = \langle \mathfrak{S}, \mathcal{V}_f, \mathcal{V}_x \rangle$, let $\mathcal{V}(\mathbb{T}) = \{[\![p]\!]^{\mathcal{V}} \mid p \in \mathbb{T}\}$. Further, for a $Log_A\mathbf{G}$ structure $\mathfrak{S}$, an $\mathfrak{S}$ grading canon is a triple $\mathcal{C} = \langle \otimes, \oplus, n \rangle$ where $n \in \mathbb{N}$ and $\otimes$ and $\oplus$ are as indicated in Definition 7.

Definition 12. *Let $\mathbb{T}$ be a $Log_A\mathbf{G}$ theory. For every $p \in \sigma_P$, valuation $\mathcal{V} = \langle \mathfrak{S}, \mathcal{V}_f, \mathcal{V}_x \rangle$ where $\mathfrak{S}$ has a set $\mathcal{P}$ which is depth- and fan-out-bounded, and $\mathfrak{S}$ grading canon $\mathcal{C} = \langle \otimes, \oplus, n \rangle$, p is a graded consequence of $\mathbb{T}$ with respect to $\mathcal{C}$, denoted $\mathbb{T} \models\!\!\!\simeq^{\mathcal{C}} p$, if $\mathcal{F}^n(\mathfrak{T})$ is defined and $[\![p]\!]^{\mathcal{V}} \in \mathcal{F}^n(\mathfrak{T})$ for every telescoping structure $\mathfrak{T} = \langle \mathcal{V}(\mathbb{T}), \mathfrak{O}, \otimes, \oplus \rangle$ for $\mathfrak{S}$ where $\mathfrak{O}$ extends $F(\mathcal{V}(\mathbb{T}) \cap Range(<))$*[7].

It is worth noting that $\models\!\!\!\simeq^{\mathcal{C}}$ reduces to $\models$ if $n = 0$ or if $F(E(\mathcal{V}(\mathbb{T})))$ does not contain any grading propositions. However, unlike $\models$, $\models\!\!\!\simeq^{\mathcal{C}}$ is non-monotonic in general.

[7] An ultrafilter U extends a filter F, if $F \subseteq U$.

3 Possibilistic Logic

Possibilistic logic is a weighted logic that handles uncertainty, in a qualitative way by associating certainty levels, to classical logic formulas [7]. At the semantic level of possibilistic logic, an epistemic state is represented by a possibility distribution π assigning to each world ω in the set of possible worlds Ω a possibility degree in the interval [0,1]. $\pi(\omega) = 0$ means that the interpretation ω is impossible, and $\pi(\omega) = 1$ means that the nothing prevents the interpretation ω to be possible in the world. The less $\pi(\omega)$, the less possible w is.

Given the possibility distribution π, the following two measures can be defined on the formulas in the language.

- The *possibility* degree $\Pi_\pi(\phi) = max\{\pi(\omega) \mid \omega \in [\phi]\}$ (where $[\phi]$ is the set of models of ϕ) evaluates the extent to which ϕ is consistent with the available information expressed by the possibility distribution π.
- The *necessity* degree $\mathbf{N}_\pi(\phi) = 1 - \Pi_\pi(\neg\phi)$ evaluates the extent to which ϕ is entailed by the available information.

The semantic determination of a belief set is defined as the set of formulas whose possibility measures are greater than the possibility of their negations.

$$BS(\pi) = \{\phi \mid \pi(\phi) > \pi(\neg\phi)\}$$

An epistemic state can also be represented syntactically in possibilistic logic. On the syntactic level, a possibilistic knowledge base is defined as a finite set of weighted formulas $\Sigma = \{(\phi_i, a_i)\}$ where a_i is a lower bound on the necessity measure of ϕ_i. A possibilistic knowledge base is said to be consistent if the classical knowledge base, obtained by omitting the weights, is consistent. Each possibilistic knowledge base is associated with an *inconsistency degree* which is defined as followed.

$$Inc(\Sigma) = \begin{cases} 0 & \text{if } \Sigma \text{ is consistent} \\ max\{a \mid \Sigma_{\geq a} \textit{ is inconsistent}\} & \text{otherwise} \end{cases}$$

where $\Sigma_{\geq a}$ are the formulas in Σ with weights greater than or equal to a.

The syntactic computation of a belief set induced by Σ is defined as the logical consequences of the formulas in Σ with weights higher than $Inc(\Sigma)$.

$$BS(\Sigma) = \{\phi \mid \Sigma_{>Inc(\Sigma)} \models \phi\}$$

4 Representing Possibilistic Logic in Log_AG

In this section, we show how to encode possibilistic theories as $Log_A\mathbf{G}$ theories, and prove that the $Log_A\mathbf{G}$ subsumes possibilistic logic. Moreover, we discuss the drowning problem that plagues possibilic logic and show that $Log_A\mathbf{G}$ does not suffer from the same problem.

In what follows, for any possibilistic knowledge base Σ_{PL}, let $\Delta(a, \Sigma_{PL}) = |A|$ where $A = \{a_i \mid (\phi_i, a_i) \in \Sigma_{PL} \textit{ and } a_i > a\}$. Intuitively, $\Delta(a, \Sigma_{PL})$ denotes the number of distinct weights appearing in a possibilistic knowledge base Σ_{PL} greater than a.

Definition 13. *Let Σ_{PL} be a possibilistic knowledge base. The corresponding $Log_A\mathbf{G}$ theory is defined as $\Sigma_{Log_A\mathbf{G}} = \{chain(\phi_i, d) \mid (\phi_i, a_i) \in \Sigma_{PL}$ and $d = \Delta(a_i, \Sigma_{PL})\}$ where chain is a function mapping a possibilistic formula (without its weight) and d to a $Log_A\mathbf{G}$ term denoting a grading proposition as follows:*

$$chain(\phi, d) = \begin{cases} \mathbf{G}(\phi, 1) & \text{if } d = 0 \\ \mathbf{G}(chain(\phi, d-1), 1) & \text{otherwise} \end{cases}$$

Example 3. Consider $\Sigma_{PL} = \{(p, 1), (p \Rightarrow b, 1), (p \Rightarrow \neg f, 0.6), (b \Rightarrow f, 0.3), (b \Rightarrow w, 0.1)\}$ where b denotes "bird", p denotes "penguin", f denotes "flies", and w denotes "has wings". The following table shows the corresponding $Log_A\mathbf{G}$ theory terms constructed according to Definition 13.

(ϕ_i, a_i)	$\Delta(a_i, \Sigma_{PL})$	$\Sigma_{Log_A\mathbf{G}}$ *term*
$(p, 1)$	0	$\mathbf{G}(p, 1)$
$(p \Rightarrow b, 1)$	0	$\mathbf{G}(p \Rightarrow b, 1)$
$(p \Rightarrow \neg f, 0.6)$	1	$\mathbf{G}(\mathbf{G}(p \Rightarrow \neg f, 1), 1)$
$(b \Rightarrow f, 0.3)$	2	$\mathbf{G}(\mathbf{G}(\mathbf{G}(b \Rightarrow f, 1), 1), 1)$
$(b \Rightarrow w, 0.1)$	3	$\mathbf{G}(\mathbf{G}(\mathbf{G}(\mathbf{G}(b \Rightarrow w, 1), 1), 1), 1)$

Observation 2 *Let Σ_{PL} be a possibilistic knowledge base with a corresponding $Log_A\mathbf{G}$ theory $\Sigma_{Log_A\mathbf{G}}$, and $\mathcal{Q} = \{[\![\gamma]\!]^{\mathcal{V}} \mid \gamma \in \Sigma_{Log_A\mathbf{G}}\}$ where $\mathcal{V}$ is a natural valuation. For any weighted formula $(\phi_i, a_i) \in \Sigma_{PL}$, the degree of embedding of of the interpretation of ϕ in $\mathcal{Q}$, $\delta_{\mathcal{Q}}([\![\phi_i]\!]^{\mathcal{V}})$, is $\Delta(a_i, \Sigma_{PL}) + 1$.*

It is worth noting that the corresponding $\Sigma_{Log_A\mathbf{G}}$ theory will always be consistent using this construction. The propositions in the corresponding $Log_A\mathbf{G}$ theory can be associated with any grade as we rely on only the degree of embedding to reflect the weight of the proposition in the original possiblistic logic knowledge base. For simplicity, we just associate the grade 1 with all graded propositions in $\Sigma_{Log_A\mathbf{G}}$. It follows directly from Observation 2 that the interpretations of all formulas with the same weight in Σ_{PL} have the same embedding degree in $\mathcal{Q}$. The less the weight a_i of a formula ϕ_i in Σ_{PL}, the higher the embedding degree of $[\![\phi_i]\!]^{\mathcal{V}}$ in $\mathcal{Q}$.

Observation 3 *Let Σ_{PL} be a possibilistic knowledge base with a corresponding $Log_A\mathbf{G}$ theory $\Sigma_{Log_A\mathbf{G}}$, and $\mathcal{Q} = \{[\![\gamma]\!]^{\mathcal{V}} \mid \gamma \in \Sigma_{Log_A\mathbf{G}}\}$ where $\mathcal{V}$ is a natural valuation. The graded filter $\mathcal{F}^n(\mathfrak{T})$ of degree $n = \Delta(Inc(\Sigma_{PL}), \Sigma_{PL})$ for some telescoping structure $\mathfrak{T}$ of $\mathcal{Q}$, is identical to $F(E^n(\mathcal{Q}))$.*

Proof. We prove this by induction on n.
Base case (n=0): By Definition 11, $\mathcal{F}^0(\mathfrak{T}) = F(\tau^0_{\mathfrak{T}}(\mathcal{Q})) = F(\mathcal{Q}) = F(E^0(\mathcal{Q}))$ since $E^0(\mathcal{Q}) = \mathcal{Q}$.
Induction Hypothesis: For $n \geq 0$, suppose that $\mathcal{F}^n(\mathfrak{T}) = F(E^n(\mathcal{Q}))$.
Induction Step: $\mathcal{F}^{n+1}(\mathfrak{T}) = F(\tau^{n+1}_{\mathfrak{T}}(\mathcal{Q})) = F(\tau_{\mathfrak{T}}(\tau^n_{\mathfrak{T}}(\mathcal{Q})))$
$= F(\varsigma(\kappa(E^1(F(\tau^n_{\mathfrak{T}}(\mathcal{Q})))) = F(\varsigma(\kappa(E^1(\mathcal{F}^n(\mathfrak{T}))))$. By the induction hypothesis, $\mathcal{F}^{n+1}(\mathfrak{T}) = F(\varsigma(\kappa(E^1(F(E^n(\mathcal{Q}))))))$. Since $\mathcal{Q}$ is consistent given it is the set

of interpretations of $\Sigma_{Log_A\mathbf{G}}$ which is consistent according to how it was constructed, then $\mathcal{F}^n(\mathfrak{T})$ must be consistent according to Definition 11. Consequently, $F(E^n(\mathcal{Q}))$ must be consistent too. Since $E^n(\mathcal{Q})$ contains only the propositions in $\mathcal{Q}$ (which are only grading propositions according to how $\Sigma_{Log_A\mathbf{G}}$ was constructed) in addition to the propositions with embedding degrees less than or equal to n, then $F(E^n(\mathcal{Q}))$ contains nothing other than the propositions in $E^n(\mathcal{Q})$ in addition to their trivial consequences. Hence, $E^1(F(E^n(\mathcal{Q}))) = F(E^1(E^n(\mathcal{Q}))) = F(E^{n+1}(\mathcal{Q}))$, and $\mathcal{F}^{n+1}(\mathfrak{T}) = F(\varsigma(\kappa(F(E^{n+1}(\mathcal{Q})))))$.
Let $(\psi, a) \in \Sigma_{PL}$ where $Inc(\Sigma_{PL}) = a$. According to Observation 2, $\delta(\psi, \mathcal{Q}) = \Delta(Inc(\Sigma_{PL}), \Sigma_{PL}) + 1 > n + 1$ (since $n + 1 = \Delta(Inc(\Sigma_{PL}), \Sigma_{PL})$). It follows then that $E^{n+1}(\mathcal{Q})$ is consistent. By Observation 1, $\varsigma(\kappa(F(E^{n+1}(\mathcal{Q})))) = F(E^{n+1}(\mathcal{Q}))$. Therefore, $\mathcal{F}^{n+1}(\mathfrak{T}) = F(F(E^{n+1}(\mathcal{Q}))) = F(E^{n+1}(\mathcal{Q}))$. □

Theorem 1. *Let Σ_{PL} be a possibilistic knowledge base and $\Sigma_{Log_A\mathbf{G}}$ be its corresponding $Log_A\mathbf{G}$ theory. For every non-grading proposition ϕ and grading canon $\mathcal{C} = \langle \otimes, \oplus, n\rangle$ with $n = \Delta(Inc(\Sigma_{PL}), \Sigma_{PL})$, $\phi \in BS(\Sigma_{PL})$ if and only if $\Sigma_{Log_A\mathbf{G}} \models^{\mathcal{C}} \phi$.*

Proof. In what follows, let $\mathcal{Q} = \{[\![\gamma]\!]^{\mathcal{V}} \mid \gamma \in \Sigma_{Log_A\mathbf{G}}\}$ be the valuation of $\Sigma_{Log_A\mathbf{G}}$ where $\mathcal{V}$ is a natural valuation.

Suppose that $\phi \in BS(\Sigma_{PL})$. Then, by definition of $BS(\Sigma_{PL})$, it must be that ϕ is logically implied by a set $\mathcal{S} = \{\psi \mid (\psi, a) \in \Sigma_{PL}$ *and* $a > Inc(\Sigma_{PL})\}$. According to how $\Sigma_{Log_A\mathbf{G}}$ was constructed by and Observation 2, for every formula $\psi \in \mathcal{S}$ in $\mathcal{Q}$, $\delta_{\mathcal{Q}}([\![\psi]\!]^{\mathcal{V}}) = \Delta(a, \Sigma_{PL}) + 1$ where a is the weight of ψ in Σ_{PL}. Since $a > Inc(\Sigma_{PL})$, it follows that $\delta_{\mathcal{Q}}([\![\psi]\!]^{\mathcal{V}}) \leq n$ and $[\![\psi]\!]^{\mathcal{V}} \in E^n(\mathcal{Q})$. Hence, by Observation 3, $[\![\phi]\!]^{\mathcal{V}} \in \mathcal{F}^n(\mathfrak{T})$ for all telescoping structures $\mathfrak{T}$ of $\mathcal{Q}$. By Definition 12, it must be that $\Sigma_{Log_A\mathbf{G}} \models^{\mathcal{C}} \phi$.

Now suppose that $\Sigma_{Log_A\mathbf{G}} \models^{\mathcal{C}} \phi$, then according to Definition 12, it must be that $[\![\phi]\!]^{\mathcal{V}} \in \mathcal{F}^n(\mathfrak{T})$ for all telescoping structures $\mathfrak{T}$ of $\mathcal{Q}$. According to Observation 3, $[\![\phi]\!]^{\mathcal{V}} \in F(E^n(\mathcal{Q}))$. Let $\mathcal{S} = \{p \mid p \in E^n(\mathcal{Q}) \cap \mathcal{P}_{\mathbf{B}}\}$. It follows that ϕ is logically implied by a set of propositional terms whose interpretations are in $\mathcal{S}$, and each $p \in \mathcal{S}$ has an embedding degree m less than or equal to n. Let ψ be the propositional term denoting p. It must be that $(\psi, a) \in \Sigma_{PL}$ since the only embedded non-grading propositions in $\Sigma_{Log_A\mathbf{G}}$ are the formulas appearing in Σ_{PL} according to Definition 13. Further, by Observation 2, $m = \Delta(a, \Sigma_{PL}) + 1$. Therefore, $m < n$ and $a > Inc(\Sigma_{PL})$. By the definition of $BS(\Sigma_{PL})$, it must be that $\phi \in BS(\Sigma_{PL})$. □

Example 4. Consider the possibilistic knowledge base Σ_{PL} and its corresponding $Log_A\mathbf{G}$ theory in Example 3. The inconsistency degree $Inc(\Sigma_{PL}) = 0.3$. Therefore, $BS(\Sigma_{PL}) = \{\phi \mid \{p,\ p \Rightarrow b,\ p \Rightarrow \neg f\} \vDash \phi\}$ since $p, p \Rightarrow b$, and $p \Rightarrow \neg f$ are the formulas in Σ_{PL} with weights higher than the inconsistency degree. The graded filter of degree $n = \Delta(Inc(\Sigma_{PL}), \Sigma_{PL}) = 2$, $\mathcal{F}^2(\mathfrak{T}) = F(E^2(\mathcal{Q}))$ where $\mathfrak{T}$ is a telescoping structure of $\mathcal{Q}$ denoting the valuation of $\Sigma_{Log_A\mathbf{G}}$. The only non-grading propositions in $E^2(\mathcal{Q})$ are $p, p \Rightarrow b$, and $p \Rightarrow \neg f$. Hence, the set of propositional terms whose interpretations are in $F(E^2(\mathcal{Q}))$ is equal to $BS(\Sigma_{PL})$.

In both $BS(\Sigma_{PL})$ and $\mathcal{F}^2(\mathfrak{T})$ we end up believing that a penguin is a bird that does not fly. □

It should be obvious that the formulas in a possibilistic knowledge base Σ_{PL} with weights less than $Inc(\Sigma_{PL})$ are blocked even if they do not contribute to the inconsistency. This problem is one limitation of possibilistic logic and is referred to in the literature as the drowning problem [2]. A special effect of the drowning problem is the property inheritance blocking problem [23, 12] arising in Example 4. The rule $b \Rightarrow w$ is blocked (drowned) in both $BS(\Sigma_{PL})$ and $\mathcal{F}^2(\mathfrak{T})$ even though it has nothing to do with the inconsistency caused by $p, p \Rightarrow b, p \Rightarrow \neg f$, and $b \Rightarrow f$ because its weight (0.1) is less than $Inc(\Sigma_{PL})$ (0.3). Accordingly, the inference that a penguin has wings is blocked as well and the exceptional penguin subclass with respect to the flying property fails to inherit the property of having wings from its bird super class even though the having wings property does not contribute to the inconsistency. The drowning problem persists in all logical consequence relations observing rational monotony [3]. It turns out that $Log_A\mathbf{G}$ does not suffer from the drowning problem (and does not observe rational monotony). To see this, consider the relevant graded consequences shown in Figure 2 of the $Log_A\mathbf{G}$ theory $\Sigma_{Log_A\mathbf{G}}$ in Example 3. In what follows, let $\Sigma^n_{Log_A\mathbf{G}} = \{\phi \mid \Sigma_{Log_A\mathbf{G}} \mid\!\!\simeq^{\mathcal{C}} \phi\}$ with the grading canon $\mathcal{C} = \langle 1/sum, \oplus, n\rangle$ where $\oplus$ is any operator.

$n = 0$	0.1 $\Sigma^0_{Log_A\mathbf{G}}$
$n = 1$	1.1. $\Sigma^0_{Log_A\mathbf{G}}$
	1.2. p
	1.3. $p \Rightarrow b$
	1.4. $\mathbf{G}(p \Rightarrow \neg f, 1)$
	1.5. $\mathbf{G}(\mathbf{G}(b \Rightarrow f, 1), 1)$
	1.6. $\mathbf{G}(\mathbf{G}(\mathbf{G}(b \Rightarrow w, 1), 1), 1)$
	1.7 b
$n = 2$	2.1 $\Sigma^1_{Log_A\mathbf{G}}$
	2.2 $p \Rightarrow \neg f$
	2.3 $\mathbf{G}(b \Rightarrow f, 1)$
	2.4 $\mathbf{G}(\mathbf{G}(b \Rightarrow w, 1), 1)$
	2.5 $\neg f$
$n = 3$	3.1 $\Sigma^2_{Log_A\mathbf{G}}$
	3.2 $\mathbf{G}(b \Rightarrow w, 1)$
$n = 4$	4.1 $\Sigma^3_{Log_A\mathbf{G}}$
	4.2 $b \Rightarrow w$
	4.3 w

Fig. 2. Graded consequences of the $Log_A\mathbf{G}$ theory $\Sigma_{Log_A\mathbf{G}}$ from Example 3

Upon telescoping to degree 3, we believe that a penguin is a bird that does not fly. The rule $b \Rightarrow f$ does not survive telescoping as it has a lower grade ($\frac{1}{3}$) than p (1), $p \Rightarrow b$ (1), $p \Rightarrow \neg f$ ($\frac{1}{2}$). Upon telescoping to degree 4, we believe that a penguin has wings since the rule $b \Rightarrow w$ survives telescoping and we still believe that penguins have birds. According to the semantics of $Log_A\mathbf{G}$ and the definition of graded filters, no proposition will ever be discarded unless it directly contributes to a contradiction.

Note that the $Log_A\mathbf{G}$ logical consequence relation is only equivalent to its possibilistic counterpart for a grading canon $\mathcal{C} = \langle \otimes, \oplus, n \rangle$ with $n = \Delta(Inc(\Sigma_{PL}), \Sigma_{PL})$ as illustrated in Example 4. This is by no means the recommended use of $Log_A\mathbf{G}$. If we continue telescoping beyond $n = \Delta(Inc(\Sigma_{PL}), \Sigma_{PL})$ as illustrated in Figure 2 (which is what should naturally happen in $Log_A\mathbf{G}$), the drowning problem is avoided.

5 Conclusion

We presented an algebraic graded non-monotonic logic we refer to as $Log_A\mathbf{G}$. It is our conviction that $Log_A\mathbf{G}$ provides an interesting alternative to the currently-existing approaches for handling non-monotonicity. $Log_A\mathbf{G}$ can be regarded as a unified framework for non-monotonicity as it can capture various non-monotonic logics. In this paper, we showed how possibilistic theories can be encoded in $Log_A\mathbf{G}$, and we proved that the $Log_A\mathbf{G}$ logical consequence relation captures possibilistic inference. Since possibilistic logic is proven to capture any non-monotonic inference relation satisfying Makinson's rationality postulates, our results prove that $Log_A\mathbf{G}$ captures such non-monotonic inference relations as well while staying immune to the drowning problem. We are currently working on relating $Log_A\mathbf{G}$ to circumscription, default logic, and auto-epistemic logic. In [17], Levesque presented the logic of "only knowing" and proved that it captures auto-epistemic logic. Later, it was proved that only knowing can capture a class of default logic [16], and circumscription [5]. Our current approach is to relate $Log_A\mathbf{G}$ to Levesque's logic of only knowing. In doing so, we can prove that $Log_A\mathbf{G}$ subsumes default logic, circumscription, and autoepistemic logic. Another idea we have in mind is to relate $Log_A\mathbf{G}$ to generalized possibilistic logic [9] which is capable of encompassing KLM non-monotonic logics and a fragment of Lifschitz's logic of minimal belief and negation as failure.

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