Algebras of finitary relations

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Abstract. Algebras of finitary relations naturally generalize the algebra of binary relations with the left composition. In this paper, we consider some properties of such algebras. It is well known that we can study the hypergraphs as finitary relations. In this way the results can be applied to graph and hypergraph theory, automatons and artificial intelligence.

1. Introduction

It is obvious that graphs and binary relations are closely related. We often use the facts of the binary relations theory in graph theory to solve some algorithmic problems. In the same way, we can consider hypergraphs as finitary relations. This could be a good idea for IT and AI, especially for pattern recognition and machine learning [1-13].

By now it has become common to use universal algebras [14] in various applications [15]. Algebraic methods can also be efficiently applied in graph theory. For example, the shortest path problem can be solved by transitive closure algorithm for binary relation [16].

In this way, and following by [17], we are going to study hypergraphs as elements of algebraic structures.

At first, we define a (n-uniform) hypergraph as a finitary relation on finite set U, in other words, as a subset of U^n. In case of $n = 2$ this leads to graph as a binary relation. Boolean algebras $\left\langle 2^{U\times U}, \left(\cup, \cap, \overline{}, \varnothing, U\times U\right)\right\rangle$ and $\left\langle 2^{U^n}, \left(\cup, \cap, \overline{}, \varnothing, U^n\right)\right\rangle$ are well known to us.

It is less trivial to define the inverse operation and the left composition for finitary relations. We have to start from inverse operation, left and right compositions for binary relations:

$$R^{-1} = \left\{\left(u_2, u_1\right) \mid \left(u_1, u_2\right) \in R\right\}, \tag{1}$$

$$R_1 \circ R_2 = \left\{\left(u_1, u_2\right) \mid \exists u_0 \left(u_1, u_0\right) \in R_1 \wedge \left(u_0, u_2\right) \in R_2\right\}, \tag{2}$$

$$R_1 \circ R_2 = R_2 \circ R_1 = \{(u_1,u_2) \mid \exists\ u_0 (u_0,u_2) \in R_1 \wedge (u_1,u_0) \in R_2\} \tag{3}$$

Note that $\left\langle 2^{U\times U}, (\circ, I)\right\rangle \sim \left\langle 2^{U\times U}, (\bullet, I)\right\rangle$ are isomorphic monoids, where I is identity relation on U. By the way, we can define operations

$$R_1 \circ_1 R_2 = R_1^{-1} \circ R_2 , \tag{4}$$

$$R_1 \bullet R_2 = R_2 \circ R_1 , \tag{5}$$

$$R_1 \circ_2 R_2 = R_1 \circ R_2^{-1} , \tag{6}$$

$$R_1 \bullet_2 R_2 = R_2 \circ R_1^{-1}. \tag{7}$$

$$R_1 \circ_3 R_2 = R_1^{-1} \circ R_2^{-1}, \tag{8}$$

$$R_1 \bullet_3 R_2 = R_2^{-1} \circ R_1^{-1}. \tag{9}$$

This makes it possible to set the following pairs of isomorphic magmas. $\left\langle 2^{U\times U}, (\circ_1, I)\right\rangle \cong \left\langle 2^{U\times U}, (\bullet_1, I)\right\rangle$ are isomorphic magmas with left identity elements. $\left\langle 2^{U\times U}, (\circ_2, I)\right\rangle \cong \left\langle 2^{U\times U}, (\bullet_2, I)\right\rangle$ are isomorphic magmas with right identity elements. $\left\langle 2^{U\times U}, (\circ_3)\right\rangle \cong \left\langle 2^{U\times U}, (\bullet_3)\right\rangle$ are isomorphic magmas without identity elements.

It is easy to see that in the symmetric case $R = R^{-1}$ all of monogenic monoids $\left\langle \left\{R^n\right\}_{n=0}^{\infty}, (\circ, I)\right\rangle$, $\left\langle \left\{R^n\right\}_{n=0}^{\infty}, (\bullet, I)\right\rangle$, $\left\langle \left\{R^n\right\}_{n=0}^{\infty}, (\circ_i, I)\right\rangle$, $\left\langle \left\{R^n\right\}_{n=0}^{\infty}, (\bullet_i, I)\right\rangle$ ($i \in 1..3$) are equal.

The monogenic monoid $\left\langle \left\{R^n\right\}_{n=0}^{\infty}, (\circ, I)\right\rangle$ and distributive algebraic structure $\left\langle \left\{R^n\right\}_{n=0}^{\infty}, (\circ, \cup, I, \varnothing)\right\rangle$ are useful to treat all-pairs shortest path problem [16]. We are going to define and study hypergraph operations similar to (1)-(9).

2. Algebras of finitary relations

Let us consider the underlying set of finitary relations 2^{U^n}, and define the following unary and binary operations for $i \neq j$

$$R^{(\mathrm{ij})} = R^{(\mathrm{ji})} = \left\{\left(u_1,..,u_j,..,u_i,..,u_n\right) \mid \left(u_1,..,u_i,..,u_j,..,u_n\right) \in R\right\}, \tag{10}$$

$$R_1 \circ_{ij} R_2 = \left\{\left(u_1,..,u_i,..,u_j,..,u_n\right) \mid \exists u_0 \left(u_1,..,u_0,..,u_j,..,u_n\right) \in R_1 \wedge \left(u_1,..,u_i,..,u_0,..,u_n\right) \in R_2\right\}. \tag{11}$$

Obviously, the operation (10) is an involution.

$$\left(R^{(\mathrm{ij})}\right)^{(\mathrm{ij})} = R. \tag{12}$$

Moreover,

$$R_1 \circ_{ij} R_2 = R_2 \circ_{ji} R_1. \tag{13}$$

It is easy to prove that operation (11) is associative. Actually,

$$\left(u_1,..,u_i,..,u_j,..,u_n\right) \in R_1 \circ_{ij} \left(R_2 \circ_{ij} R_3\right) \Leftrightarrow \exists u_0 \left(u_1,..,u_0,..,u_j,..,u_n\right) \in R_1 \wedge \left(u_1,..,u_i,..,u_0,..,u_n\right) \in R_2 \circ_{ij} R_3 \Leftrightarrow$$
$$\Leftrightarrow \exists u_0 \left(u_1,..,u_0,..,u_j,..,u_n\right) \in R_1 \wedge \left(\exists u_0' \left(u_1,..,u_0',..,u_0,..,u_n\right) \in R_2 \wedge \left(u_1,..,u_i,..,u_0',..,u_n\right) \in R_3\right) \Leftrightarrow$$
$$\Leftrightarrow \exists u_0' \left(\exists u_0 \left(u_1,..,u_0,..,u_j,..,u_n\right) \in R_1 \wedge \left(u_1,..,u_0',..,u_0,..,u_n\right) \in R_2\right) \wedge \left(u_1,..,u_i,..,u_0',..,u_n\right) \in R_3 \Leftrightarrow$$
$$\Leftrightarrow \exists u_0' \left(u_1,..,u_0',..,u_j,..,u_n\right) \in R_1 \circ_{ij} R_2 \wedge \left(u_1,..,u_i,..,u_0',..,u_n\right) \in R_3 \Leftrightarrow$$
$$\Leftrightarrow \left(u_1,..,u_i,..,u_j,..,u_n\right) \in \left(R_1 \circ_{ij} R_2\right) \circ_{ij} R_3.$$

Then we set

$$I_{ij} = \left\{\left(u_1,..,u_i,..,u_j,..,u_n\right) \mid k \in 1..n \wedge u_k \in U \wedge u_j = u_i\right\} \in 2^{U^n}. \tag{14}$$

It is easy to see

$$\left(u_1,..,u_i,..,u_j,..,u_n\right) \in I_{ij} \circ_{ij} R \Leftrightarrow \exists u_0 \left(u_1,..,u_0,..,u_j,..,u_n\right) \in I_{ij} \wedge \left(u_1,..,u_i,..,u_0,..,u_n\right) \in R \Leftrightarrow$$
$$\Leftrightarrow \exists u_0 \left(u_1,..,u_i,..,u_0,..,u_n\right) \in R \wedge u_j = u_0 \Leftrightarrow \left(u_1,..,u_i,..,u_j,..,u_n\right) \in R,$$

and similarly

$$\left(u_1,..,u_i,..,u_j,..,u_n\right) \in R \circ_{ij} I_{ij} \Leftrightarrow \exists u_0 \left(u_1,..,u_0,..,u_j,..,u_n\right) \in R \wedge \left(u_1,..,u_i,..,u_0,..,u_n\right) \in I_{ij} \Leftrightarrow$$
$$\Leftrightarrow \exists u_0 \left(u_1,..,u_0,..,u_j,..,u_n\right) \in R \wedge u_i = u_0 \Leftrightarrow \left(u_1,..,u_i,..,u_j,..,u_n\right) \in R.$$

Thus,

$$I_{ij} \circ_{ij} R = R \circ_{ij} I_{ij} = R\,. \tag{15}$$

Hence we have just proved the

Lemma 1. $\left\langle 2^{U^n}, \left(\circ_{ij}, I_{ij}\right)\right\rangle$ is a monoid.

Note that

$$\left(u_1,..,u_i,..,u_j,..,u_n\right) \in \left(R_1 \circ_{ij} R_2\right)^{(\mathrm{ij})} \Leftrightarrow \left(u_1,..,u_j,..,u_i,..,u_n\right) \in R_1 \circ_{ij} R_2 \Leftrightarrow$$
$$\Leftrightarrow \exists u_0 \; \left(u_1,..,u_0,..,u_i,..,u_n\right) \in R_1 \wedge \left(u_1,..,u_j,..,u_0,..,u_n\right) \in R_2 \Leftrightarrow$$
$$\Leftrightarrow \exists u_0 \; \left(u_1,..,u_i,..,u_0,..,u_n\right) \in R_1^{(\mathrm{ij})} \wedge \left(u_1,..,u_0,..,u_j,..,u_n\right) \in R_2^{(\mathrm{ij})} \Leftrightarrow$$
$$\Leftrightarrow \left(u_1,..,u_i,..,u_j,..,u_n\right) \in R_1^{(\mathrm{ij})} \circ_{ji} R_2^{(\mathrm{ij})} \Leftrightarrow \left(u_1,..,u_i,..,u_j,..,u_n\right) \in R_2^{(\mathrm{ij})} \circ_{ij} R_1^{(\mathrm{ij})}\,.$$

In that way

$$\left(R_1 \circ_{ij} R_2\right)^{(\mathrm{ij})} = R_1^{(\mathrm{ij})} \circ_{ji} R_2^{(\mathrm{ij})} = R_2^{(\mathrm{ij})} \circ_{ij} R_1^{(\mathrm{ij})}\,. \tag{16}$$

Hence the bijective function $f\left(R\right) := R^{(\mathrm{ij})}$ is an isomorphism of monoids $\left\langle 2^{U^n}, \left(\circ_{ij}, I_{ij}\right)\right\rangle$ and $\left\langle 2^{U^n}, \left(\circ_{ji}, I_{ji}\right)\right\rangle$.

Moreover,

$$\left(u_1,..,u_i,..,u_k,..,u_j,..,u_n\right) \in \left(R_1 \circ_{ik} R_2\right)^{(\mathrm{ij})} \Leftrightarrow \left(u_1,..,u_j,..,u_k,..,u_i,..,u_n\right) \in R_1 \circ_{ik} R_2 \Leftrightarrow$$
$$\Leftrightarrow \exists u_0 \; \left(u_1,..,u_0,..,u_k,..,u_i,..,u_n\right) \in R_1 \wedge \left(u_1,..,u_j,..,u_0,..,u_i,..,u_n\right) \in R_2 \Leftrightarrow$$
$$\Leftrightarrow \exists u_0 \; \left(u_1,..,u_i,..,u_k,..,u_0,..,u_n\right) \in R_1^{(\mathrm{ij})} \wedge \left(u_1,..,u_i,..,u_0,..,u_j,..,u_n\right) \in R_2^{(\mathrm{ij})} \Leftrightarrow$$
$$\Leftrightarrow \left(u_1,..,u_i,..,u_j,..,u_n\right) \in R_1^{(\mathrm{ij})} \circ_{jk} R_2^{(\mathrm{ij})} \Leftrightarrow \left(u_1,..,u_i,..,u_j,..,u_n\right) \in R_2^{(\mathrm{ij})} \circ_{kj} R_1^{(\mathrm{ij})}\,.$$

From which we obtain

$$\left(R_1 \circ_{ik} R_2\right)^{(\mathrm{ij})} = R_1^{(\mathrm{ij})} \circ_{jk} R_2^{(\mathrm{ij})} = R_2^{(\mathrm{ij})} \circ_{kj} R_1^{(\mathrm{ij})}\,. \tag{17}$$

Hence we have proved the

Lemma 2. Monoids $\left\langle 2^{U^n}, \left(\circ_{ik}, I_{ik}\right)\right\rangle$ and $\left\langle 2^{U^n}, \left(\circ_{jk}, I_{jk}\right)\right\rangle$ are isomorphic, as well as monoids $\left\langle 2^{U^n}, \left(\circ_{ij}, I_{ij}\right)\right\rangle$ and $\left\langle 2^{U^n}, \left(\circ_{ji}, I_{ji}\right)\right\rangle$.

Let us set an algebraic structure $\left\langle 2^{U^n}, \left(\circ_{ij}, \circ_{ik}, I_{ij}, I_{ik}\right)\right\rangle$ and then we can write the following logical consequences:

$$\left(u_1,..,u_i,..,u_j,..,u_k,..,u_n\right) \in R_1 \circ_{ij} \left(R_2 \circ_{ik} R_3\right) \Leftrightarrow \exists u_0 \; \left(u_1,..,u_0,..,u_j,..,u_k,..,u_n\right) \in R_1 \wedge$$
$$\wedge \left(u_1,..,u_i,..,u_0,..,u_k,..,u_n\right) \in R_2 \circ_{ik} R_3 \Leftrightarrow \exists u_0 \; \exists u_0' \; \left(u_1,..,u_0,..,u_j,..,u_k,..,u_n\right) \in R_1 \wedge$$
$$\wedge \left(u_1,..,u_0',..,u_0,..,u_k,..,u_n\right) \in R_2 \wedge \left(u_1,..,u_i,..,u_0,..,u_0',..,u_n\right) \in R_3 \Leftrightarrow$$
$$\exists u_0' \; \exists u_0 \; \left(u_1,..,u_0,..,u_j,..,u_k,..,u_n\right) \in R_1 \wedge \left(u_1,..,u_0',..,u_0,..,u_k,..,u_n\right) \in R_2 \wedge$$
$$\wedge \left(u_1,..,u_i,..,u_0,..,u_0',..,u_n\right) \in R_3 \Rightarrow$$
$$\exists u_0' \left(\exists u_0 \; \left(u_1,..,u_0,..,u_j,..,u_k,..,u_n\right) \in R_1 \wedge \left(u_1,..,u_0',..,u_0,..,u_k,..,u_n\right) \in R_2\right) \wedge$$
$$\wedge \left(\exists u_0 \; \left(u_1,..,u_i,..,u_0,..,u_0',..,u_n\right) \in R_3\right) \Leftrightarrow \exists u_0' \; \left(u_1,..,u_0',..,u_j,..,u_k,..,u_n\right) \in R_1 \circ_{ij} R_2 \wedge$$
$$\wedge \left(\exists u_0 \; \left(u_1,..,u_i,..,u_0,..,u_0',..,u_n\right) \in R_3 \wedge \left(u_1,..,u_0,..,u_j,..,u_0',..,u_n\right) \in 1_R\right) \Leftrightarrow$$
$$\Leftrightarrow \exists u_0' \; \left(u_1,..,u_0',..,u_j,..,u_k,..,u_n\right) \in R_1 \circ_{ij} R_2 \wedge \left(u_1,..,u_i,..,u_j,..,u_0',..,u_n\right) \in R_3 \circ_{ij} 1_R \Leftrightarrow$$
$$\Leftrightarrow \left(u_1,..,u_i,..,u_j,..,u_k,..,u_n\right) \in \left(R_1 \circ_{ij} R_2\right) \circ_{ik} \left(R_3 \circ_{ij} 1_R\right).$$

This means that the following Lemma is true.

Lemma 3. In an ordered algebra $\left\langle 2^{U^n}, \left(\circ_{ij}, \circ_{ik}, \subseteq, I_{ij}, I_{ik}, 0_R, 1_R\right)\right\rangle$, the pseudo distributive law holds

$$R_1 \circ_{ij} \left(R_2 \circ_{ik} R_3\right) \subseteq \left(R_1 \circ_{ij} R_2\right) \circ_{ik} \left(R_3 \circ_{ij} 1_R\right). \quad (18)$$

According to [17], we use the notation $1_R := U^n$ and $0_R := \varnothing$.

Then look at composition

$$\left(u_1,..,u_i,..,u_j,..,u_n\right) \in R \circ_{ij} R^{(\mathrm{ij})} \Leftrightarrow \exists u_0 \left(u_1,..,u_0,..,u_j,..,u_n\right) \in R \wedge \left(u_1,..,u_i,..,u_0,..,u_n\right) \in R^{(\mathrm{ij})} \Leftrightarrow$$
$$\Leftrightarrow \exists u_0 \left(u_1,..,u_0,..,u_j,..,u_n\right) \in R \wedge \left(u_1,..,u_0,..,u_i,..,u_n\right) \in R. \quad (19)$$

Definition 1. The finitary relation R is called a function from *i*-th to *j*-th argument if

$$\forall u_1,...,u_i,..,u_j,u_j',..,u_n \left(u_1,..,u_i,..,u_j,..,u_n\right) \in R \wedge \left(u_1,..,u_i,..,u_j',..,u_n\right) \in R \to u_j = u_j'. \quad (20)$$

We can obtain from (19) - (20) the following set inclusion

$$\left(u_1,..,u_i,..,u_j,..,u_n\right) \in R \circ_{ij} R^{(\mathrm{ij})} \Rightarrow u_i = u_j \Leftrightarrow \left(u_1,..,u_i,..,u_j,..,u_n\right) \in I_{ij} \Leftrightarrow R \circ_{ij} R^{(\mathrm{ij})} \subseteq I_{ij}. \quad (21)$$

Definition 2. The finitary relation R is called a surjection from *i*-th argument if

$$\forall u_1,...,u_{i-1},u_{i+1},..,u_j,..,u_n \; \exists u_0 \left(u_1,..,u_0,..,u_j,..,u_n\right) \in R. \quad (22)$$

From (21) - (22) we can get the reverse set inclusion

$$I_{ij} \subseteq R \circ_{ij} R^{(\mathrm{ij})}. \quad (23)$$

Thus, in the case of R is a surjective function from *i*-th to *j*-th argument we have the equality

$$R \circ_{ij} R^{(\mathrm{ij})} = I_{ij}. \quad (24)$$

Similarly, in the case of R is a surjective function from *j*-th to *i*-th argument we have the equality

$$R^{(\mathrm{ij})} \circ_{ij} R = I_{ij}. \quad (25)$$

Let us denote the set of surjective functions from both (*i*-th to *j*-th and *j*-th to *i*-th) arguments as F_{ij}. It is easy that F_{ij} is closed by $\circ_{ij}$, and hence we have proved the

Lemma 4. $\left\langle F_{ij}, \left(\circ_{ij}, I_{ij}\right)\right\rangle$ is a subgroup of the monoid $\left\langle 2^{U^n}, \left(\circ_{ij}, I_{ij}\right)\right\rangle$.

As well as binary relations, finitary relations have the following properties [17]

$$R_1 \circ_{ij} \left(R_2 \cup R_3\right) = \left(R_1 \circ_{ij} R_2\right) \cup \left(R_1 \circ_{ij} R_3\right), \quad (26)$$

$$\left(R_2 \cup R_3\right) \circ_{ij} R_1 = \left(R_2 \circ_{ij} R_1\right) \cup \left(R_3 \circ_{ij} R_1\right), \quad (27)$$

$$R_1 \circ_{ij} \left(R_2 \cap R_3\right) \subseteq \left(R_1 \circ_{ij} R_2\right) \cap \left(R_1 \circ_{ij} R_3\right), \quad (28)$$

$$\left(R_2 \cap R_3\right) \circ_{ij} R_1 \subseteq \left(R_2 \circ_{ij} R_1\right) \cap \left(R_3 \circ_{ij} R_1\right), \quad (29)$$

and so we can set an algebraic structures $\left\langle F_{ij}, \left(\circ_{ij}, I_{ij}\right)\right\rangle$, $\left\langle 2^{U^n}, \left(\cup, \cap, \circ_{ij}, \circ_{ik}, ^{(\mathrm{ij})}, \subseteq, 0_R, 1_R, I_{ij}, I_{ik}\right)\right\rangle$ that have properties (12)-(18), (24)-(29).

3. Conclusion and examples

We have defined algebraic structures of finitary relations as a common case of well-known algebraic structures of binary relations. We have considered the algebraic structures on an underlying set 2^{U^n} and sometimes called a finitary relation $R \in 2^{U^n}$ by a (*n*-uniform) hypergraph. The operation $\circ_{ij}$ can be called the “straightening the edges” or “deleting shared intermediate vertices”. Let us take an example.

Example 1 (algebraic). Let us set $U = \left\{u_0, u_1, u_2, u_3\right\}$, $\left\langle 2^{U^3}, \left(\circ_{23}, I_{23}\right)\right\rangle$, and $R = \left\{\left(u_1, u_0, u_3\right), \left(u_1, u_2, u_0\right)\right\}$. Now we can get

$$I_{23} = \begin{Bmatrix} (u_0,u_0,u_0),(u_0,u_1,u_1),(u_0,u_2,u_2),(u_0,u_3,u_3), \\ (u_1,u_0,u_0),(u_1,u_1,u_1),(u_1,u_2,u_2),(u_1,u_3,u_3), \\ (u_2,u_0,u_0),(u_2,u_1,u_1),(u_2,u_2,u_2),(u_2,u_3,u_3), \\ (u_3,u_0,u_0),(u_3,u_1,u_1),(u_3,u_2,u_2),(u_3,u_3,u_3) \end{Bmatrix},$$

$$R \circ_{23} R = \{(u_1,u_2,u_3)\},$$

$$R \circ_{23} R \circ_{23} R = \varnothing .$$

Despite its simplicity, operation $\circ_{23}$ has some interesting applications. In examples 2, 3 we are going to denote 3-tuple $(u_{i_1},u_{i_2},u_{i_3})$ as a word $u_{i_1}u_{i_2}u_{i_3}$.

Example 2 (feature selection). Let $R = \{u_1u_0u_0, u_1u_1u_0, u_1u_2u_0, u_1u_2u_1, u_1u_2u_3, u_1u_3u_0\}$ be a set of words, and $R_f = \{u_1u_0u_3\}$, $R_f^{(23)} = \{u_1u_3u_0\}$ are filters. First, apply the filter R_f

$$R_f \circ_{23} R = \{u_1u_0u_3, u_1u_1u_3, u_1u_2u_3, u_1u_3u_3\}.$$

Then apply the filter $R_f^{(23)}$

$$R_f^{(23)} \circ_{23} R_f \circ_{23} R = \{u_1u_0u_0, u_1u_1u_0, u_1u_2u_0, u_1u_3u_0\}.$$

Example 3 (crossover). Let $R = \{u_1u_0u_0, u_1u_1u_0, u_1u_2u_1, u_1u_3u_2\}$ be a population. Let us define the evolution operator $\mathrm{E}(R) = R \cup R \circ_{23} R$ and start a first step of evolution

$$\mathrm{E}(R) = \{u_1u_0u_0, u_1u_1u_0, u_1u_2u_0, u_1u_2u_1, u_1u_3u_1, u_1u_3u_2\}.$$

In example 4 we are going to denote 3-tuple $(u_{i_1},u_{i_2},u_{i_3})$ as an implies $u_{i_1} \to (u_{i_2} \to u_{i_3})$.

Example 4 (AI). Let $R = \{u_1 \to (u_1 \to u_1), u_1 \to (u_1 \to u_2)\}$ be a base set of AI premises. Let us define the semantic closure of R as $[R] = \bigcup_{k=1}^{\infty} \left(R \cup R^{(23)}\right)^k$, where $R^{k+1} = R^k \circ_{23} R$ and $R^1 = R$. By definition we have

$$[R] = \{u_1 \to (u_1 \to u_1), u_1 \to (u_1 \to u_2), u_1 \to (u_2 \to u_1), u_1 \to (u_2 \to u_2)\}.$$

Note that $\left((u_1 \to (u_0 \to u_3)) \wedge (u_1 \to (u_2 \to u_0))\right) \to (u_1 \to (u_2 \to u_3))$ is tautology, so the inference rule $u_1 \to (u_0 \to u_3), u_1 \to (u_2 \to u_0) \vdash u_1 \to (u_2 \to u_3)$ preserves truth.

We also note that $\left(((u_1 \to u_0) \to u_3) \wedge ((u_1 \to u_2) \to u_0)\right) \to ((u_1 \to u_2) \to u_3)$ is tautology, too.

It makes perfect sense to use an indicator function $\chi_R : U^n \mapsto \{0,1\}$ for $R \in 2^{U^n}$, that is defined as

$$\chi_R(u_1,..,u_n) = \begin{cases} 1, (u_1,..,u_n) \in R \\ 0, (u_1,..,u_n) \notin R \end{cases}.$$

In the case of finite set $U = \{u_1,..,u_m\}$, we can use this function to define a join-vertices logical array $\psi^R : (1..m)^n \mapsto \{false, true\}$ for (n-uniform) hypergraph. Let $f : 1..m \mapsto U$ be a total bijection and $R \in 2^{U^n}$. We define

$$\psi^R_{k_1,..,k_n} = \psi^R(k_1,..,k_n) = \begin{cases} true, & \chi_R\left(f^{-1}(k_1),..,f^{-1}(k_n)\right) = 1 \\ false, & \chi_R\left(f^{-1}(k_1),..,f^{-1}(k_n)\right) = 0 \end{cases}.$$

Let us denote $\{false, true\}$ as D and a set of logical array defined above as $\mathsf{D}^{(1..m)^n}$.

We also can set a logical algebra that generalized adjacency matrices algebra. In this way we define a binary operation $*_{ij}$ on $\mathsf{D}^{(1..m)^n}$

$$\psi^1_{k_1,..,k_n} *_{ij} \psi^2_{k_1,..,k_n} = \bigvee_{s=1}^{m} \psi^1_{k_1,..,k_{i-1},s,k_{i+1},..,k_j,..,k_n} \wedge \psi^2_{k_1,..,k_i,..,k_{j-1},s,k_{j+1},..,k_n}.$$

By our construction semigroups $\left\langle 2^{U^n}, \left(\circ_{ij}\right)\right\rangle$ and $\left\langle \mathsf{D}^{(1..m)^n}, \left(*_{ij}\right)\right\rangle$ are isomorphic.

More interesting is the case of algebraic structures on an underlying set $\bigcup_{n=1}^{\infty} 2^{U^n}$ and operations from 2^{U^n} to 2^{U^m}. For example, let us define the operations “gluing edges” $\circ_g$ and “replacing chains” $\circ_r$.

$$R_1 \circ_g R_2 = \left\{\left(u_1,..,u_{m-1},u_2',..,u_n'\right) \mid \exists u_0 \left(u_1,..,u_{m-1},u_0\right) \in R_1 \wedge \left(u_0,u_2',..,u_n'\right) \in R_2\right\},$$

$$R_1 \circ_r R_2 = \left\{\left(u_1,..,u_{i-1},u_{j+1}',..,u_n'\right) \mid \exists u_0 \exists i \exists j \left(u_1,..,u_{i-1},u_0,..,u_m\right) \in R_1 \wedge \left(u_1',..,u_0,u_{j+1}',..,u_n'\right) \in R_2\right\}.$$

For the finitary relation $R = \left\{\left(u_1,u_0,u_3\right),\left(u_1,u_2,u_0\right)\right\}$ from Example 1 we can get

$$R^{(13)} = \left\{\left(u_3,u_0,u_1\right),\left(u_0,u_2,u_1\right)\right\},$$

$$R \circ_g R^{(13)} = \left\{\left(u_1,u_0,u_0,u_1\right),\left(u_1,u_2,u_2,u_1\right)\right\},$$

$$R \circ_r R = \left\{\left(u_1\right),\left(u_0,u_3\right),\left(u_1,u_0\right),\left(u_1,u_2\right),\left(u_1,u_3\right),\left(u_2,u_0\right),\left(u_1,u_2,u_3\right)\right\}.$$

It is clear that even in the case of finite set U we would never make a finite representation for such algebraic structures. But in particular cases, maybe we can. This case is of interest.

4. References

[1] Hein M, Setzer S, Jost L and Rangapuram S S 2013 The total variation on hypergraphs-learning on hypergraphs revisited *Advances in Neural Information Processing Systems* 2427-2435

[2] Ricatte T, Gilleron R and Tommasi M 2014 Hypernode graphs for spectral learning on binary relations over sets *Joint European Conference on Machine Learning and Knowledge Discovery in Databases* 662-677

[3] Louis A 2015 Hypergraph markov operators, eigenvalues and approximation algorithms *Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing* 713-722

[4] Zhang C, Hu S, Tang Z G and Chan T H 2017 Re-revisiting Learning on Hypergraphs: Confidence Interval and Subgradient Method *Proceedings of the 34th International Conference on Machine Learning, in PMLR* **70** 4026-4034

[5] Pu L, Faltings B 2012 Hypergraph learning with hyperedge expansion *Machine Learning and Knowledge Discovery in Databases* 410-425

[6] Yu J, Tao D and Wang M 2012 Adaptive hypergraph learning and its application in image classification *IEEE Transactions on Image Processing* **21(7)** 3262-3272

[7] Panagopoulos A, Wang C, Samaras D and Paragios N 2013 Simultaneous cast shadows, illumination and geometry inference using hypergraphs *IEEE Transactions on Pattern Analysis and Machine Intelligence* **35(2)** 437-449

[8] Wang M, Wu X 2015 Visual classification by l1-hypergraph modeling *IEEE Transactions on Knowledge and Data Engineering* **27(9)** 2564-2574

[9] Zhou D, Huang J and Scholkopf B 2007 Learning with Hypergraphs: Clustering, Classification, and Embedding *Advances in Neural Information Processing Systems: Proceedings of the 2006 Conference* **19** 1601-1608

[10] Ghoshdastidar D, Ambedkar D 2014 Consistency of Spectral Partitioning of Uniform Hypergraphs under Planted Partition Model *Advances in Neural Information Processing Systems 27: Annual Conference on Neural Information Processing Systems* 397-405

[11] Ghoshdastidar D, Ambedkar D 2015 A Provable Generalized Tensor Spectral Method for Uniform Hypergraph Partitioning *Proceedings of the 32nd International Conference on Machine Learning, ICML* 400-409

[12] Ghoshdastidar D, Ambedkar D 2017 Consistency of spectral hypergraph partitioning under planted partition model *Ann. Statist.* **45(1)** 289-315

[13] Chien I, Lin C and Wang I 2018 Community Detection in Hypergraphs: Optimal Statistical Limit and Efficient Algorithms *Proceedings of the Twenty-First International Conference on Artificial Intelligence and Statistics, in PMLR* **84** 871-879

[14] Mal'tsev A I 1973 *Algebraic systems* (Springer) p 319

[15] Chernov V M 2018 Ternary number systems in finite fields *Computer Optics* **42(4)** 704-711 DOI: 10.18287/2412-6179-2018-42-4-704-711

[16] Tsvetov V P 2014 On a syntactic algorithm on graphs *Proceedings of the International Conference Advanced Information Technologies and Scientific Computing* (Samara, Russia) 235-238

[17] Tsvetov V P 2018 Dual ordered structures of binary relations *Proceedings of the International Conference Information Technology and Nanotechnology. Session Data Science* (Samara, Russia) 2635-2644