An Introduction to Optimal Stable Marriage Problems and Argumentation Frameworks*

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Abstract. In the *Stable Marriage* (*SM*) problem, given two sets of individuals partitioned into men and women, a matching is stable when there does not exist any matching *man-woman* by which both *man* and *woman* would be individually better off than they are with the person to which they are currently matched. In 1995, P.M. Dung modelled stable matchings as stable extensions in *Abstract Argumentation Frameworks*. In this paper we elaborate on the original formulation by using *Weighted Abstract Argumentation* to also represent optimality criteria in *Optimal SM* problems, where some matchings are better than others: criteria may consider only the preference of either men, or women, or a more balanced view obtained by differently combining the preferences of both of them.

1 Introduction

The SM problem [6, 10] and its many variants [8] have been widely studied in the literature, because of the inherent appeal of the problem and its important practical applications. A classical instance of the problem comprises a bipartite set of n men and n women, and each person has a preference list in which they rank all members of the opposite sex in a strict total order. Then, a matching MT is simply a bijection between men and women. A man m_i and a woman w_j form a *blocking pair* for MT if m_i prefers w_j to his partner in MT and w_j prefers m_i to her partner in MT. A matching that involves no blocking pair (included in MT) is said to be *stable*, otherwise the matching is unstable. Even though the SM problem has its roots in combinatorial problems, it has also been studied in game theory, economics and operations research.

The same problem has been proposed in many variants: the *Optimal Stable Marriage* (OSM) problem [10, 8] aims to find a matching that is not only stable, but also "good" according to some criterion based on the preferences of all the individuals. Classical solutions deal only with men-optimal (or women-optimal) marriages, in which every man (woman), gets his (her) best possible partner.

P. M. Dung proposed how to encode SM problems into *Argumentation Frameworks* (*AF*) in his pioneering work [3]. In this work, we show how to encode OSM problems with *incomplete lists* of preferences and *ties* (w.r.t. preferences), that is *OSMTI*, as AFs where arguments are associated with scores.

* The work is partially supported by projects "*Argumentation 360*", "*Rappresentazione della Conoscenza e Apprendimento Automatico*" (RACRA), funded by the Department of Mathematics and Computer Science - University of Perugia, and project INdAM-GNCS "ASIA".

2 Optimal Stable Marriage Problems

The classical SM problem was extended [4] in order to find an SM problem under a more equitable measure of optimality, thus defining an *Optimal SM* problem [5, 7, 8, 10]. For example, in [7] the authors maximise the global satisfaction by summing together the preferences of both men *and* women in a matching. This sum has to be minimised, since $p(m_i, w_j)$ represents the rank of w_j in m_i's list of preferences. Therefore, we need to minimise this *egalitarian cost*[1] [7] in Eq. 1:

$$\min \left(\sum_{(m_i,w_j) \in MT} p(m_i, w_j) + \sum_{(m_i,w_j) \in MT} p(w_j, m_i) \right) \tag{1}$$

Such an optimisation problem was originally posed by D. Knuth [7]. Other optimisation criteria are represented by minimizing the *regret cost* [5], as represented in Eq. 2:

$$\min \max_{(m_i,w_j) \in MT} \max\{p(m_i, w_j), p(w_j, m_i)\} \tag{2}$$

A third criterion consists in minimising the *sex-equalness cost* [9]:

$$\min \left| \sum_{(m_i,w_j) \in MT} p(m_i, w_i) - \sum_{(m_i,w_j) \in MT} p(w_j, m_i) \right| \tag{3}$$

Finding a solution satisfying Eq. 1 and Eq. 2 already found their solution in polynomial time by using ad-hoc algorithms, such as [5] and [7], respectively. On the contrary, reaching the optimality represented by Eq. 3 was proved to be an NP-hard problem for which approximation algorithms are proposed [9].

An SM problem formulation has incomplete lists (*SMI*) if an individual can exclude a partner whom she/he does not want to be matched with [8] (some preferences are omitted). In this case, a (perfect) matching of all men or women may not exist. A further extension is represented by problems where it is possible to express the same preference for more than one possible partner: the problem is usually named as "SM with ties" [8] (*SMT*). In this case, three stability notions are proposed in the literature [8]:

- given (m_i, w_j) and (m_k, w_z), in a *super* stable matching a pair (m_i, w_z) is blocking iff $p(m_i, w_z) \geq_{\mathbb{S}} p(m_i, w_j) \wedge p(w_z, m_i) \geq_{\mathbb{S}} p(w_z, m_k)$;[2]
- in a *strongly* stable matching a pair (m_i, w_z) is blocking iff $p(m_i, w_z) >_{\mathbb{S}} p(m_i, w_j) \wedge p(w_z, m_i) \geq_{\mathbb{S}} p(w_z, m_k)$ or $p(m_i, w_z) \geq_{\mathbb{S}} p(m_i, w_j) \wedge p(w_z, m_i) >_{\mathbb{S}} p(w_z, m_k)$;
- in a *weakly* stable matching a pair (m_i, w_z) is blocking iff $p(m_i, w_z) >_{\mathbb{S}} p(m_i, w_j) \wedge p(w_z, m_i) >_{\mathbb{S}} p(w_z, m_k)$.

The solution of *SMTI* problems is proved to be an NP-complete problem [8].

[1] Simply a cost that combines the preferences of both men and women.

[2] $\geq_{\mathbb{S}}$ means "better than", according to a c-semiring $\mathbb{S}$ (see Sect. 3).

3 Preference on Arguments

In this section we model the three optimality criteria in Sect. 2 by labelling arguments with weights as performed in [1]. We use c-semirings and their operators to model preference values and find the best stable extension.

A c-semiring is an algebraic structure to model preferences, generically defined by the tuple $\mathbb{S} = \langle S, \oplus, \otimes, \bot, \top \rangle$. The idempotency of $\oplus$ leads to the definition of a partial ordering $\leq_{\mathbb{S}}$ over the set S (S is a poset). Such a partial order is defined as $s \leq_{\mathbb{S}} t$ if and only if $s \oplus t = t$, and $\oplus$ returns the *least upper bound* of s and t. This intuitively means that t is "better" than s. $\oplus$ is used to compose values, while $\bot, \top \in S$ represent the best and worst preference respectively [1].

All criteria can be captured by using a c-semiring and a side function $f : S \times S \to S$ that we use to compose $p(m_i, w_j)$ and $p(w_j, m_i)$. We use f to find the overall preference if we match m_i with w_j, i.e., if $(m_i, w_j) \in MT$, that is the combined man-woman/woman-man preference. The general parametric formula is:

$$\bigoplus \left(\bigotimes_{(m_i, w_j) \in MT} f(p(m_i, w_j), p(w_j, m_i)) \right) \tag{4}$$

By embedding the Weighted c-semiring $\langle \mathbb{R}^+ \cup \{+\infty\}, \min, +, +\infty, 0 \rangle$ and with $f \equiv +$ (the arithmetic addition) we obtain the egalitarian cost in Eq. 1. By instead using the Fuzzy c-semiring $\langle [0,1], \max, \min, 0, 1 \rangle$ and $f \equiv \max$, we can exactly model the regret cost in Eq. 2. Moreover, we can also represent man and woman optimality (thus obtaining the best solution for either men or women) by using the Weighted c-semiring $\langle \mathbb{R}^+ \cup \{+\infty\}, \min, +, +\infty, 0 \rangle$, and a function f that always return the first (man-optimal) or second (woman-optimal) component of the considered couple of preferences. For men-optimality: $f_{mo}(p(m_i, w_j), p(w_j, m_i)) \to p(m_i, w_j)$, and for women-optimality: $f_{wo}(p(m_i, w_j), p(w_j, m_i)) \to p(w_j, m_i)$.

We formally encode OSMTI to weighted AFs in Def. 1. In Tab. 1 we represent a $\text{SMTI}_{\mathbb{S}}$ problem in tabular form, considering a (Weighted) semiring $\mathbb{S}$:

Definition 1 (OSMTI to weighted frameworks). *Given a* $\text{SMTI}_{\mathbb{S}}$*, a c-semiring* $\mathbb{S} = \langle S, \oplus, \otimes, \bot, \top \rangle$*, and preference composition operator* $f : S \times S \to S$*, a correspond-*

	w_1	w_2	w_3	w_4	w_5	w_6
m_1	1	4	-	5	5	3
m_2	3	4	6	1	5	2
m_3	1	-	4	2	3	5
m_4	6	1	3	4	2	1
m_5	3	1	2	4	5	6
m_6	3	3	1	6	5	4

	m_1	m_2	m_3	m_4	m_5	m_6
w_1	1	5	4	4	2	3
w_2	4	1	5	2	6	-
w_3	6	4	2	1	5	3
w_4	2	5	2	4	-	6
w_5	4	2	-	5	6	3
w_6	2	6	3	5	1	4

Fig. 1. An $\text{OSMTI}_{\mathbb{S}}$ problem representation in tabular form, with six men and six women. Missing preferences (of incomplete lists) are marked with "-"), while $p(w_1, m_3) = p(w_1, m_4) = 5$ is one of the five ties in this problem.

ing weighted argumentation framework is $\mathscr{A}_{rgs} = \{(m \times w) \mid m \in M, w \in W, p(m,w) \downarrow \wedge p(w,m) \downarrow\}$*, and* $R \subseteq \mathscr{A}_{rgs} \times \mathscr{A}_{rgs}$ *s.t.* $R((m_k, w_l), (m_i, w_j))$ *iff*

- $i = k$ *and* $p(m_i, w_l) \geq_{\mathbb{S}} p(m_i, w_j)$*, or*
- $j = l$ *and* $p(w_j, m_k) \geq_{\mathbb{S}} p(w_j, m_i)$*.*

and each argument (m_i, w_j) *in the obtained framework is labelled with a preference given by* $f(p(m_i, w_j), p(w_j, m_i))$*.*

Having defined a framework as in the above definition, we now declare how to reach optimality according to the different proposed criteria.

Proposition 1 (Modelling criteria with c-semirings). *The best solution according to* $\oplus$*, found by composing the preference values of arguments using* $\otimes$ *on the stable extensions found by Definition 1, optimises the*

egalitarian cost: *by using* $f \equiv +$ *and the Weighted c-semiring;*
regret cost: *by using* $f \equiv \max$ *and the Fuzzy c-semiring;*
man-optimality: *by using* $f \equiv f_{mo}$ *and the Weighted c-semiring;*
woman-optimality: *by using* $f \equiv f_{wo}$ *and the Weighted c-semiring.*

Note that sex-equalness cannot be locally modelled by costs on single arguments, since it is the global satisfaction of men and women whose difference is minimised. In this case, arguments can be labelled with a couple $\langle p(m_i, w_j), p(w_j, m_i) \rangle$. By composing such preference with a Cartesian product of two Weighted c-semiring,[3] in the end all stable extensions will have a cost of $\langle \sum_{(m_i,w_j) \in MT} p(m_i, w_j), \sum_{(m_i,w_j) \in MT} p(w_j, m_j) \rangle$, which can be finally optimised to find the sex-equalness cost.

Four weakly-stable matchings (see Sect. 3) can be found the example in Fig. 1:

- $MT_1 = \{(m_1,w_1),(m_2,w_2),(m_3,w_4),(m_4,w_5),(m_5,w_6),(m_6,w_3)\}$,
- $MT_2 = \{(m_1,w_1),(m_2,w_2),(m_3,w_4),(m_4,w_6),(m_5,w_5),(m_6,w_3)\}$,
- $MT_3 = \{(m_1,w_1),(m_2,w_2),(m_3,w_4),(m_4,w_3),(m_5,w_6),(m_6,w_5)\}$,
- $MT_4 = \{(m_1,w_1),(m_2,w_6),(m_3,w_4),(m_4,w_2),(m_5,w_5),(m_6,w_3)\}$.

Table 1 shows the egalitarian, regret, sex-equalness, man-optimal, and woman-optimal costs obtained for the four matchings on the example in Fig. 1. These values were obtained by using a Python script that besides building the .apx file with the framework (as given in Definition 1), also runs the dockerised[4] version of ConArg[5]. In this way, it is possible to run ConArg on any platform (Linux, Mac OSX, Windows) and use a Python library[6] to enumerate all the stable extensions in the given framework.

The best matching for both the egalitarian and sex-equalness criteria is MT_1, while according to the regret cost, all the four matchings optimise this criterion. The best matching according to only men's preference (i.e., man-optimality) is MT_4, and for women's preference is MT_3. As expected, man-optimality corresponds to the worst preference possible for women, and vice-versa.

[3] The Cartesian product of c-semirings is still a c-semiring.

[4] Docker: `https://www.docker.com`.

[5] The Docker image of ConArg is downloadable from the repository with command `docker pull iccma19/conarg`.

[6] Docker SDK for Python: `https://docker-py.readthedocs.io`.

	egalitarian	regret	sex-equalness	man-optimality	woman-optimality
MT_1	**29**	**6**	**3**	16	13
MT_2	32	**6**	4	14	18
MT_3	30	**6**	12	21	**9**
MT_4	32	**6**	8	**12**	20

Table 1. The egalitarian, regret, and sex-qualness cost obtained for the four matchings. We report the man-optimal and woman-optimal costs. In bold, the best solution.

4 Conclusion

In this paper we elaborated one of the directions pioneered in the '95 seminal work by P.M. Dung [3]: the strong ties between Abstract Argumentation (and better, the stable semantics) and the Stable Marriage problem. As far as we know, except for Dung's paper, this topic was not investigated in successive studies.

In the future we plan to study closely related problems: examples are the *Stable Roommates* (all participants belong to a single pool, not divided into different sexes), the *hospitals/residents* (a hospital can accept more than one resident), or the *assignment* problem, which consists of finding, in a weighted bipartite graph, a matching in which the sum of weights of the edges is as large as possible. A connection to previous works on the formation of argument coalitions is also possible [2].

References

1. Bistarelli, S., Pirolandi, D., Santini, F.: Solving weighted argumentation frameworks with soft constraints. In: Recent Advances in Constraints, Revised Selected Papers. LNCS, vol. 6384, pp. 1–18. Springer (2009)
2. Bistarelli, S., Santini, F.: Coalitions of arguments: An approach with constraint programming. Fundam. Inform. **124**(4), 383–401 (2013)
3. Dung, P.M.: On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. Artif. Intell. **77**(2), 321–357 (1995)
4. Gale, D., Shapley, L.S.: College admissions and the stability of marriage. The American Mathematical Monthly **69**(1), 9–15 (1962)
5. Gusfield, D.: Three fast algorithms for four problems in stable marriage. SIAM J. Comput. **16**(1), 111–128 (1987)
6. Gusfield, D., Irving, R.W.: The stable marriage problem: structure and algorithms. MIT Press, Cambridge, MA, USA (1989)
7. Irving, R.W., L., P., Gusfield, D.: An efficient algorithm for the "optimal" stable marriage. J. ACM **34**(3), 532–543 (1987)
8. Iwama, K., Miyazaki, S.: A survey of the stable marriage problem and its variants. In: International Conference on Informatics Education and Research for Knowledge-Circulating Society (icks 2008). pp. 131–136. IEEE Computer Society (2008)
9. Iwama, K., Miyazaki, S., Yanagisawa, H.: Approximation algorithms for the sex-equal stable marriage problem. In: Algorithms and Data Structures, 10th International Workshop, WADS. LNCS, vol. 4619, pp. 201–213. Springer (2007)
10. Knuth, D.E.: Stable marriage and its relation to other combinatorial problems: An introduction to the mathematical analysis of algorithms. American Mathematical Society (1997)