Numerical Solution Tasks with Mixed Operator in Unlimited Area

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Abstract. Methods for solving problems of elliptic equations, based on the integrated Green formula, were analyzed. New methods for solving a problem with a mixed-type operator in an unbounded domain are proposed. On the basis of the proposed methods, programs for solving problems with a mixed type operator have been created. The results of computational experiments, showing the correctness of the application of methods, are presented. Moreover, the convergence of the developed iterative methods has been studied; protection of the order of the original difference scheme using the developed methods has been shown.

Keywords: Unlimited Area, Mixed Operator, Electrodynamics Accelerator, Railgun, Electromagnetic Field, Maxwell's Equations.

1 Introduction

In the study of physical phenomena, it is often necessary to perform modeling in an unlimited area, for example, in the model-roving of the electrostatic field of charges, solving the problem of thermal conductivity, etc. In the case where the phenomenon can be described by means of the simplest linear elliptic operators, the problem is solved and described, for example, in [1, 2]. The setting of non-reflective boundary conditions for wave equations is discussed in detail in [3]. However, for a wider class of problems for which the problem operator may have a mixed type but outside some finite domain, the fundamental solution of the operator is known and easily calculated, the above methods are not applicable.

In particular, problems with operators of mixed type arise when modulating the electromagnetic field in electrodynamic accelerators of the rail type. Process of current flow in conductors in this case can be described by parabolic equation and electromagnetic field in dielectric – elliptical [4, 5].

2 Methods of Solving the Problem in an Unlimited Area

In order to solve the problem, three methods of solving are proposed: the method of expansion of the area, the method of setting integral boundary conditions, three-stage method [6].

When solving problems using grid numerical methods, accuracy is determined by grid spacing and accuracy of setting boundary conditions. If you set the same boundary conditions as infinity in the original task when you solve a problem in the target area on the boundary, the error defined by the boundary condition setting will generally decrease as the calculation area increases in size. Therefore, you can use the scope extension method to solve tasks in an unlimited scope, that is, you can find the size of the calculation scope so that the effect of the artificial boundary is less than the application of the numerical method.

The method of expansion of the design area is easy to implement, but has a high computational complexity [6]. Alternative methods are methods based on the use of Green formulas.

The first is the method of defining the integral boundary condition [6]. It is based on the fact that outside some finite region there is an operator with a known fundamental solution and a trivial right part, then the solution on the boundary of the calculation region is determined by means of the basic integral Green formula. Thus, solving the problem using the method of defining integral boundary conditions requires solving a system of equations with filled lines corresponding to the boundary.

Three-stage method of solution is built in [6]. When using the three-step method, it is necessary to solve two systems, but with sparse lines related to boundary conditions. The results of calculations show that when solving problems reduced to solving linear equations with strongly filled matrices, the method is ineffective, but for problems with sparse matrix of the system, the three-step method allows to obtain the result significantly faster than the method of setting the integral boundary condition.

3 Modification of Three-Stage Method. Iterative Method of Problem Solution in Unlimited Area

Consider in more detail the three-step algorithm [6] on the example of solving the problem with an operator of mixed type:

$$\begin{cases} u_t - a^2 \Delta u = f, \ \mathbf{r} \in D_S, \\ \Delta u = 0, \ \mathbf{r} \in \mathbb{R}^2 \setminus D_S, \end{cases} \tag{1}$$

where ∂D_S is the boundary of the region D_S (Fig. 1).

After applying the finite difference method to approximate the production time instead of the system (1), it is necessary to solve the problem:

$$\begin{cases} \Delta \hat{u} - \dfrac{1}{a^2 \tau} \hat{u} = -\dfrac{1}{a^2} \hat{f} - \dfrac{u}{a^2 \tau}, & \mathbf{r} \in D_S, \\ \Delta \hat{u} = 0, & \mathbf{r} \in \mathbb{R}^2 \setminus D_S. \end{cases} \tag{2}$$

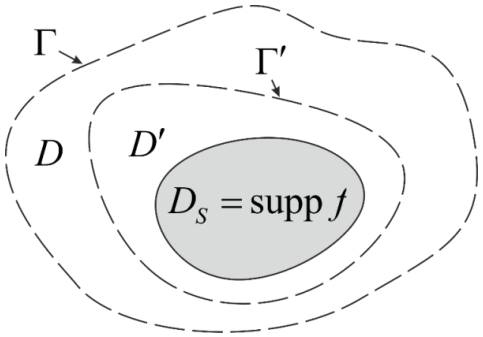

Fig. 1. Structure of area.

In subsequent records of solved tasks, temporary derivatives use approximation (2).

At the first stage of the three-stage method it is proposed to solve the system with time-consuming boundary conditions:

$$\begin{cases} v_t - a^2 \Delta \hat{v} = f, \ \mathbf{r} \in D_S; \\ \Delta \hat{v} = 0, \ \mathbf{r} \in D \backslash D_S, \\ \hat{v}\big|_{\Gamma} = u\big|_{\Gamma}, \\ v(t_{i-1}) = u. \end{cases}$$

At the second stage it is assumed that the solution $\hat{v}$ in the area D_S is slightly different from the solution $\hat{u}$ of the initial problem (3), and the difference is related to the error in the boundary condition setting. Based on these considerations and Green's basic integral formula, the objective is:

$$\begin{cases} u_t - a^2 \Delta \hat{u} = f, \ \mathbf{r} \in D_S; \\ \Delta \hat{u} = 0, \ \mathbf{r} \in D' \backslash D_S, \\ \hat{u}\big|_{\Gamma'} = \hat{v}_i + \int\limits_{\Gamma} \left(\Phi_P \dfrac{\partial \hat{v}_i}{\partial \mathbf{n}} - \hat{v}_i \dfrac{\partial \Phi_P}{\partial \mathbf{n}} \right) dS, \\ u(t_{i-1}) = u. \end{cases}$$

At the third stage, according to the main integrated Green formula, the value $\hat{u}$ on the boundary Γ by the value $\hat{u}$ in the area D':

$$\hat{u}(P) = \int\limits_{\Gamma_S} \left(\Phi_P^L \frac{\partial \hat{u}}{\partial \mathbf{n}} - \hat{u} \frac{\partial \Phi_P^L}{\partial \mathbf{n}} \right) dS.$$

The main advantage of the three-stage algorithm is that without the use of an iterative process it allows to obtain results of high accuracy [6, 7]. However, the presence of the second stage deliberately creates an error that is not eliminated by iteratively repeating steps 1–3 of the algorithm and can grow with a reduction in the time integration step τ. In order to overcome this disadvantage of the method, it is proposed to delete the second step of the method, and to perform the first and third steps several times on each time layer.

The first stage of the new iterative algorithm

$$\begin{cases} u_t^k - a^2 \Delta \hat{u}^k = f, \ \mathbf{r} \in D_S; \\ \Delta \hat{u}^k = 0, \ \mathbf{r} \in D \backslash D_S, \\ \hat{u}^k \big|_\Gamma = \hat{g}^{k-1} \big|_\Gamma, \\ u(t_{i-1}) = u, \ k = 1..N. \end{cases}$$

On the first iteration, you are prompted to select a lagging boundary condition as the boundary condition

$$\hat{g}^0 \big|_\Gamma = g \big|_\Gamma .$$

The second stage of the algorithm

$$\hat{g}^k(P) = \int_{\Gamma_S} \left(\Phi_P^L \frac{\partial \hat{u}^k}{\partial \mathbf{n}} - \hat{u}^k \frac{\partial \Phi_P^L}{\partial \mathbf{n}} \right) dS.$$

4 Application of Iterative Algorithm to Solving Problems in Unlimited Area

The built algorithm is applicable to solving the following problem:

$$\begin{cases} u_t - a^2 \Delta u = f, \ r \le R_0, \\ \Delta u = 0, \ r > R_0, \end{cases} \tag{3}$$

$$f(r, \varphi, t) = a^2 \left(\sum_{k=2}^{2+m} \left(1 - r^2 \lambda_n^2 - k^2 \right) r^{k-2} y_k - C_1 r \lambda_n^2 \right) e^{-\lambda_n^2 a^2 t} \cos \varphi.$$

Initial condition ($r \le R_0$):

$$u(r, \varphi, 0) = \left(\sum_{k=2}^{m+2} \left(y_k r^k \right) + C_1 r - \frac{2c}{\lambda_n R_0^2 (J_0(R_0 \lambda_n) - J_2(R_0 \lambda_n))} J_1(r \lambda_n) \right) \cos \varphi.$$

Where m and n — parameters; J_j — j order Bessel function ($j = 0,1,2$);

$\lambda_n = \dfrac{\mu_n}{R_0}$, μ_n — root of equation $J_1(\mu_n) = 0$; $y_k = \dfrac{(-1)^k C_m^{k-2}}{R_0^{k-2}\left(k^2 - 1\right)}$; $\gamma = 2$;

$C_1 = -\dfrac{1}{1+\gamma} \sum_{k=2}^{m+2} \left((k\gamma + 1) y_k R_0^{k-1} \right)$; $C_2 = \sum_{k=2}^{m+2} \left(y_k R_0^{k+1} \right) + C_1 R_0^2$; $c = C_2 \left(1 - \dfrac{1}{\gamma} \right)$.

The analytical solution is as follows:

$$u(r, \varphi, t) = \begin{cases} \left(\sum_{k=2}^{m+2} \left(y_k r^k \right) + C_1 r + \dfrac{2c}{\lambda_n R_0^2 (J_0(R_0 \lambda_n) - J_2(R_0 \lambda_n))} J_1(r \lambda_n) \right) e^{-\lambda_n^2 a^2 t} \cos \varphi, \ r \le R_0; \\ \dfrac{C_2}{r} e^{-\lambda_n^2 a^2 t} \cos \varphi, \ r > R_0. \end{cases}$$

Let's set $m=1,\ n=2,\ a=0.1,\ R_0=1.0.$ Solve the problem with a new iterative method and a three-step algorithm. Table 1 shows the calculation results. We see that all the methods considered lead to the correct result.

Table 1. Task (3) results. $R_0=1.0,\ \ R=2.0,\ \ \tau=0.001,\ \ h=0.2,\ \ T=0.008.$

h^* τ	N	$\frac{\|u_h-u\|_C}{\|u\|_C}$ analytical definition of boundary conditions	p	$\frac{\|u_h-u\|_C}{\|u\|_C}$ application of a three-stage algorithm	p	$\frac{\|u_h-u\|_C}{\|u\|_C}$ application of iterative algorithm (1 iter)	p	$\frac{\|u_h-u\|_C}{\|u\|_C}$ application of iterative algorithm (2 iter)	p
h τ	8	$3.15114\cdot10^{-3}$ (t = 0.211 s.)		$3.90629\cdot10^{-3}$ (t = 0.489 s.)		$3.20485\cdot10^{-3}$ (t = 0.267 s.)		$3.25578\cdot10^{-3}$ (t = 0.555 s.)	
h/2 τ/4	32	$4.79565\cdot10^{-4}$ (t = 1.153 s.)	2.7	$5.05743\cdot10^{-4}$ (t = 2.322 s.)	3.0	$4.50521\cdot10^{-4}$ (t = 1.367 s.)	2.8	$4.57497\cdot10^{-4}$ (t = 2.660 s.)	2.8
h/4 τ/16	128	$6.82737\cdot10^{-5}$ (t = 8.141 s.)	2.8	$6.8335\cdot10^{-5}$ (t = 15.62 s.)	2.9	$8.60114\cdot10^{-5}$ (t = 9.013 s.)	2.4	$7.39768\cdot10^{-5}$ (t = 17.71 s.)	2.6
h/8 τ/64	512	$1.75524\cdot10^{-5}$ (t = 114.5 s.)	2.0	$1.75524\cdot10^{-5}$ (t = 233.4 s.)	2.0	$2.65021\cdot10^{-5}$ (t = 133.1 s.)	1.7	$2.32554\cdot10^{-5}$ (t = 260.5 s.)	1.7

Here $p=\log_2\left(\left(\|u_h-u\|_C/\|u\|_C\right)\big|_{h^*}\Big/\left(\|u_h-u\|_C/\|u\|_C\right)\big|_{h^*/2}\right)$ — value characterizing the order of the method, $(\ \cdot\ )\big|_{h^*}$ — value calculated on a grid with step h^*.

Table 2. The convergence of iterative methods in the solution of task (3) in case of selection at each step as the initial approximation of the zero vector $\tau=1.5625\cdot10^{-5},\ \ h=0.025,$ $T_0=0.000125.$ Error in analytical method of setting boundary conditions is equal to $8.45\cdot10^{-6}$

Iteration number	$\frac{\|u_h-u\|_C}{\|u\|_C}$ (two-step iterative method)	$\frac{\|u_h-u\|_C}{\|u\|_C}$ (iterative process based on a three-step method)
1	$3.82473\cdot10^{-1}$	$1.39064\cdot10^{-1}$
2	$1.25412\cdot10^{-1}$	$1.65792\cdot10^{-2}$
4	$1.34993\cdot10^{-2}$	$2.42915\cdot10^{-4}$
8	$1.80879\cdot10^{-4}$	$1.07342\cdot10^{-5}$
16	$2.53095\cdot10^{-5}$	$1.07263\cdot10^{-5}$
32	$2.53009\cdot10^{-5}$	—
64	$2.53275\cdot10^{-5}$	—

Note that the iterative process could also be built on the basis of a three-stage algorithm [6]. However, according to the results in Table 2, this will not help to get the answer with the required accuracy in less time, since one iteration of the three-step method contains almost twice as many deductive operations as the number of operations of the new iterative method.

By studying Tables 1 and 2, it can be observed that the iterative methods constructed have a high convergence rate as long as the distance from the approximation to the true solution is large. In the vicinity of the solution, iterative projects do not converge, which is confirmed by the results of numerical calculations (Table 3).

Table 3. Convergence of two-step iterative method.

$$R_0 = 1.0, \quad R = 2.0, \quad \tau = \frac{0.001}{64}, \quad h = \frac{0.2}{8}, \quad T = 0.008$$

	$\frac{\|u_h - u\|_C}{\|u\|_C}$ analytical definition of boundary conditions	$\frac{\|u_h - u\|_C}{\|u\|_C}$ application of iterative algorithm (1 iter)	$\frac{\|u_h - u\|_C}{\|u\|_C}$ application of iterative algorithm (2 iter)	$\frac{\|u_h - u\|_C}{\|u\|_C}$ application of iterative algorithm (3 iter)	$\frac{\|u_h - u\|_C}{\|u\|_C}$ application of iterative algorithm (4 iter)
Error	$1.75524 \cdot 10^{-5}$	$2.65021 \cdot 10^{-5}$	$2.32554 \cdot 10^{-5}$	$2.45878 \cdot 10^{-5}$	$4.61259 \cdot 10^{-5}$
Time (s.)	114.472	133.145	260.535	359.556	486.5

One way to improve convergence rate in the vicinity of the solution (Table 4) is to change the second step of the iterative method, for example:

$$\hat{g}^k(P) = \frac{1}{2}\left(\hat{g}^{k-1}(P) + \int_{\Gamma_S} \left(\Phi_P^L \frac{\partial \hat{u}^k}{\partial \mathbf{n}} - \hat{u}^k \frac{\partial \Phi_P^L}{\partial \mathbf{n}} \right) dS \right).$$

It is also worth noting that the newly built method is less demanding to the size of the auxiliary sub area $D \backslash D_S$, which allows reducing its time-measures, and thus reducing the computational complexity of the task.

Table 4. Convergence of modified two-step iterative method.

$$R_0 = 1.0, \quad R = 2.0, \quad \tau = \frac{0.001}{64}, \quad h = \frac{0.2}{8}, \quad T = 0.008$$

	$\frac{\|u_h - u\|_C}{\|u\|_C}$ analytical definition of boundary conditions	$\frac{\|u_h - u\|_C}{\|u\|_C}$ application of iterative algorithm (1 iter)	$\frac{\|u_h - u\|_C}{\|u\|_C}$ application of iterative algorithm (2 iter)	$\frac{\|u_h - u\|_C}{\|u\|_C}$ application of iterative algorithm (4 iter)	$\frac{\|u_h - u\|_C}{\|u\|_C}$ application of iterative algorithm (8 iter)
Error	$1.75524 \cdot 10^{-5}$	$2.65021 \cdot 10^{-5}$	$2.51139 \cdot 10^{-5}$	$2.4427 \cdot 10^{-5}$	$2.42558 \cdot 10^{-5}$
Time (s.)	114.472	133.145	236.651	468.127	925.113

5 Conclusion

On the basis of the method of setting integral boundary conditions and the three-stage method, computational algorithms are built and software implemented to solve a number of problems in an unlimited area. The results of computational experiments confirm the correctness of the methods. They also show that if the order of accuracy of quadrature formulas used in the implementation of methods is consistent with the order of the difference scheme, the order of the scheme is preserved.

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