Dynamic doxastic action in Doxastic Modal Logic

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Abstract. This article is a trying to combine two main traditions of belief revision: the so-called AGM-approach and Dynamic Doxatic Logic approach. We consider doxastic actions as modal operators partly like DDL-style and compare their features with AGM postulates. We construct axiomatical systems for the operator of the expansion and the operator of the contraction. Within the system based on this interpretation, we can express and prove the corresponding postulates of expansion and contraction AGM. It demonstrates that these modal operators correspond to the functions of expansion and contraction described in AGM, about that the representational theorem has been formulated.

Keywords: Belief revision, AGM, DDL, Modal logic, Dynamic modal logic, Doxastic action, Epistemic logic, Doxastic logic, Belief change.

1 Dynamic interpretation of doxastic actions

Belief revision is promising trend in modern epistemic logic deals with changes of our knowledges and beliefs. It is aimed at formal representation of the process of belief change. There are two main traditions of belief revision: the so-called *AGM-approach*, named after its initiators Carlos E. Alchourròn, Peter Gärdenfors and David Makinson (see their most seminal paper [1] and inspired reserches, e.g.: [4]), and *DDL-approach* — Dynamic Doxatic Logic based on ingenious ideas of Krister Segerberg (see, e.g.: [12]). Both approaches are very similar in a general intention and research methodology, furthermore their main purpose is an adequate representation of doxastic actions perfomed in the process of belief change. In both traditions the basic doxastic actions represented by *expansion*, *contraction* and *revision*, but each approach implements them in different ways. This ways depend on the format of main epistemic objects such as epistemic states, admissible epistemic inputs, expected epistemic results etc. Implemetation of doxastic actions also depends on specific notion of changes in belief states — it could be taken as a process (see, e.g.: [2]) or as a result of change (see, e.g.: [3]).

AGM. The main postulates of belief change were described in AGM, so this treatment sometimes is called "postulational approach". Basic properties of expansion, contraction and revision are presented in AGM as a set of formal axioms. It is commonly accepted that any interpretation of doxastic action should satisfy the suitable set of the axioms. However, main drawback of this approach is a static representation of doxastic actions. All doxastic actions in AGM factually are reduced to operations over theories.

AGM expansion: T + A (T — initial knowledge set (theory), + — the sign of expansion and A — new knowledge (or belief) added to the initial set).
AGM contraction: T ÷ A (T — initial knowledge set (theory), ÷ — the sign of contraction and A — knowledge (or belief) extracted from the initial set).

AGM considers the status of the theory before and after change, but it doesn‘t consider the process of change. Thus an underwater part of iceberg of properties of doxastic operations are remained unexplored.

DDL. The idea of constructing an axiomatic system capable to display the dynamic properties of doxastic actions is partially implemented by the dynamic direction in belief revision. The basis of this approach is the idea of representing the dynamic aspect of belief change by specific language of DDL. Doxasic actions are considered as term-formed operators +, ÷, *.

DDL expansion: $+\phi$ (means it is performing an expansion by ϕ).
DDL contraction: $\div\phi$ (means it is performing a contraction by ϕ).

These terms constitute entire formula with additional operators "[ ]", "<>". Thus, notion $\psi[+\phi]$ means "ψ regularly holds after the agent performs an expansion by ϕ" and $\psi < +\phi >$ means "perhaps ψ holds after the agent performs an expansion by ϕ"

This language has much more resources for a detailed analysis of doxastic actions, but it is too complicated and cumbersome. Whereas the postulational way allows to define doxastic action and describes the basic principles of belief change, it has too simple syntax that makes difficult further detailed researching. Therefore it is reasonable to construct an axiomatic logical system capable to combine advantages of both approaches and overcome their shortcomings. The syntax of this logic should be simple like the syntax of the AGM-system and should be flexible enough to express the process of change.

To solve this task it need to change the aspect of consideration of doxastic actions. In this connection Yaroslav Shramko makes an interesting offer, proposed a more depth consideration of the epistemological grounds of beliefs. He offers to interpret doxastic actions as pure modal operators to relegating the action directly into the center of researching [13]. The proposed idea differs from the modal interpretation of doxastic actions by K. Segerbeg (DDL-approach). If doxastic action (expansion, contraction and etc) is considered as regular modal operator, its application to a Boolean formula creates a dynamic formula. Whereas doxastic operator in DDL forms only term (see: [12], [7]) and thus it is not a purely modal operator. The representation of doxastic actions as formula-formed modal operators considerably simplify syntax and with that allows to analyze the dynamic aspect of the belief revision. It is possible to build specific dynamic modal logic on this notion.

The basic idea of this paper is to represent expansion and contraction actions as modal operators and to show the equivalence of received operators to appropriate operators AGM. We need to consider the AGM postulates for expansion and contraction and to prove that they hold doxastic actions interpreted in the modal way.

Within the AGM belief state of the subject is reduced to a set of beliefs closed under the logical consequence. After the AGM under a consequence operation we mean an operator Cn that takes sets of proposition to sets of proposition, such that three condition are satisfied, for any sets X and Y of proposition: $X \subseteq Cn(X)$, $Cn(X) = Cn(Cn(X))$, and $Cn(X) \subseteq Cn(Y)$ whenever $X \subseteq Y$ [1, p. 511]. Thus we will use the equivalent notation $y \in Cn(X)$ and $X \vdash y$.

Each set of beliefs can represent the belief state, as well as any belief state may be presented as a set of beliefs. Each proposition represents a belief and, accordingly, each belief can be represented in the form of a proposition. That is, the belief state is a set of propositions – beliefs, in which the subject is convinced of the truth. We will examine the change of a theory – consistent sets of beliefs, closed under the logical consequence (Cn).

2 AGM postulates of expansion

Expansion – the simplest doxastic operator in AGM. Its point lies in adding new information to the initial set of beliefs. Expansions is implemented by using $+$, displaying a pair (set of beliefs, statements) on the set of sets of beliefs ($K \times L \to K$). Where K is the initial belief set, the expansion of K by A is denoted by $K + A$. Expansion can be characterized by the following set of postulates [6, p. 48-51].

Closure	$K + A$ – belief set	**E 1**
Success	$A \in (K + A)$	**E 2**
Inclusion	$K \supset (K + A)$	**E 3**
Vacuity	If $A \in K$ then $(K + A) = K$	**E 4**
Monotonicity	If $K \supset H$ then $(K + A) \supset (H + A)$	**E 5**

These postulates describe a family of expansion operators. Factually, with the help of the postulates of expansion E1-E5 can be axiomatically defined the operator of expansion. That is, if the doxastic operator satisfying the postulates of expansion E1-E5, then it is equivalent to the operator expansion of AGM.

3 AGM postulates of contraction

Contraction is applied in those cases where a proposition must be removed from the belief set without giving any new information. Contraction is implemented by using $\div$, displaying a pair (set of beliefs, statements) on the set of sets of beliefs ($K \times L \to K$). Where K is the initial belief set, the contraction of K by

A is denoted by $K \div A$. Contraction can be characterized by the following set of postulates [6, p. 60-64].

Basic postulates of contraction, which factually mean the axiomatic definition contraction, were developed under the traditional approach of AGM [1]. Building a contraction operator is considered to be successfull if it exists a proving that it satisfied the postulates of contraction AGM.

Closure	$K \div A$ – belief set	**C 1**
Success	If $\nvdash A$ then $K \div A \nvdash A$	**C 2**
Inclusion	$K \div A \supset K$	**C 3**
Vacuity	If $A \notin K$ then $K \div A = K$	**C 4**
Extensionality	If $A \Leftrightarrow B$ then $K \div A = K \div B$	**C 5**
Recovery	$K \subseteq (K \div A) + A$	**C 6**

The postulate of recovery is not always achieved because it involves the composition of operators of contraction and expansion, but it leads to inconsistency in the AGM. If the contraction operator satisfies five postulates without recovery, it is named a withdrawal operator. This designation describes irreversible process and defines contraction as a singular operator, independent on the expansion.

4 Principles of Doxastic Modal Logic (DML)

DML language allows to place doxastic actions in the center of consideration. Doxastic formula DML formed by hanging doxastic operator on the Boolean formula. In fact, each doxastic formula DML is the representation of some action of belief revision at the time of conversion (see [10, p. 18]).

Let A be an usual formula of propositional calculus, + – doxastic operator expansion, $\div$ – respectively doxastic operator contraction. Consider a fixed theory. Then $+A$ will mean expansion by A. Similarly $\div A$ will mean contraction by A. In DML interpretation doxastic operators can be applied to nondoxastic (Boolean) propositiond. The result of applying these operators will be the doxastic formula. Thus, expansion and contraction are interpreted in DML as modal (single) operators in pure form. And therefore, $+A, \div A$ and similar doxastic formulas do not express a certain state theory but the transformation theory. As a result of doxastic actions a static formula can be obtained, which expresses a certain state theory, or we will need to implement the next steps of doxastic actions.

Doxastic actions of expansion and contraction over a fixed belief set will look this way.

Expansion by A: $+A$

Contraction by A: $\div A$

The general characteristics of dynamic modal logic DML will be common to all axiomatic systems built according to this approach. Basic properties of the axiomatic systems will be presented according to the work of Y. Shramko [13].

As in the language of classical logic the illogical character of a set is an infinite list of propositional variables p, q, r, s. The logical symbols are signs of the truth-functional propositional connectives: $\&, \vee, \supset, \neg$ and the symbols of doxastic operators $+$ and $\div$. The technical characters are left and right parentheses $($, $)$.

DEFINITION 1 Boolean formula
A formula is called boolean if and only if it consists of elementary propositions and propositional connectives.

In building of this logic, we assume that doxastic operators are not applicable to doxastic formulas. Iteration of doxastic operators is possible, but with additional conditions and needs a deeper study. Thus, we have the following definition of an elementary doxastic formula.

DEFINITION 2 Elementary doxastic formula
$+A$ and $\div A$ are elementary doxastic formula iff A is a boolean formula.

DEFINITION 3 Well-formed formula
Well-formed formula (WFF) is called:
(1) any Boolean formula
(2) any elementary doxastic formula
(3) if A and B are WFF, then $A\&B$, $A \vee B$, $A \to B$,$\neg A$ are WFF,
nothing else is WFF.

All doxastic modal logic which have DML syntax will be formulated as a system of axioms with the rules of substitution and inference. First of all, note that all axioms of propositional calculus are axioms of each logic DML. We have the following rule:

PC All axioms of propositional logic are an axiom of DML.

Typically inference rule is Modus Ponens. If the formula A is inferred and formula $A \to B$ is inferred, then formula B is inferred.

MP If $\vdash A$ and $\vdash A \to B$ then $\vdash B$.

To formulate the rules of substitution we will need the notion of correct substitution.

DEFINITION 4 Correct substitution Substituting of formula B instead of occurrence of some variable p in well-formed formula A is called correct if and only if the result of substitution is a WFF.

Now we can formulate a rule of admissible substitutions.

US Let A be a theorem DML, and $p_1, p_2, \ldots, p_n$ are some propositional variables that are included in A. Then formula A, obtained by simultaneous substitution of some WFF $B_1, B_2, \ldots, B_n$ instead of every occurrence of $p_1, p_2, \ldots, p_n$, is a theorem.

Factually, the dynamic modal logic DML is a propositional calculus with the alphabet extended by typing doxastic modal operators:$+$, $\div$. These modal operators are a formal interpretation of cognitive actions: expansion and contraction, respectively.

5 Minimal logic of expansion

Consider the axiomatic construction of doxastic modal logic of expansion based on the interpretation of a cognitive action of expansion as a modal operator [10]. For this reason we need a DML language and series of axioms that define the specific nature of the operator of expansion. Consider the basic properties of the expansion used in the construction of epistemical theories. AGM define it by using the apparatus of set theory [8, p. 4]:

$K + A = Cn(K \cup A)$, where K is the initial belief set and Cn is the operation of closure under the logical consequence.

Thus, if we want to expand the theory by A, we must mechanically add A to the original belief set using the set-theoretic union operation and close the obtained set under the logical consequence.

By completeness of the belief set, each deducible formula of propositional calculus will be added to the resulting theory. Obviously the next property, if A is a theorem of propositional calculus, we must add it to our theory, which is similar to the well-known in modal logic rule of hanging of necessity. In the DML language this rule will take the following form: if A is a theorem of propositional calculus, then $+A$ is a theorem of minimal logic of extension.

N If $\vdash A$, then $\vdash +A$.

In this situation doxastic operator of expansion behaves like a normal modal operator. Remarkable, such behavior should not follow the properties of the operator, but from the definition of cognitive operator, which is represented by this modal operator. It indicates that the chosen method of presentation can actually be used.

The following important property is that the operator extension is closed under Modus Ponens. This characteristic, as the previous one, also follows the definition of expansion adopted in the AGM. By analogy with the principle of bringing of modality for normal modal systems, this principle can be expressed as distributivity operator of expansion with respect to the implication.

K $+(A \to B) \to (+A \to +B)$

Based on the previous note, the operator of expansion can be interpreted as a normal modal operator, which owns the properties of positive modality. Rules as PC, MP, US, N and K are the base of the most weak logic of extension of beliefs.

6 Dynamic modal logic of extension DML_E

The expressive power of DML language is sufficient to build a stronger logic of expansion, which will examine in details the properties of AGM expansion. In building this logic we need a criteria of rationality AGM, taken in the concept:

- Requirement of minimal changes in initial beliefs,
- Priority of new information,
- Consistency.

To the list of criteria of rationality also can be added the principle of categorical matching, which is usually introduced implicitly, but is one of the most important when performing any operation. According to the principle of categorical matching (see [8, p. 5]) the result of the operation must be represented in the same form as the original data. Thus, the execution of any doxastic operation on the belief set should guarantee the result as the belief set and nothing else.

The criterion of priority of the new information requires the presence of expression in the belief set after expansion of the theory by this statement. In the AGM postulates for expansion is mentioned a criterion expressed as a postulate of success. Taking a statement to the belief set, the agent elevates it to the rank of belief and commits to adopt it.

T $\quad +A \to A$

If we consider konsequent of T as an expression of truth, that "if A is added to the theory, then A is true", we get a very strong statement. Given the properties of the agent of belief (at least, he may be wrong), we can say that this statement is quite often incorrect. However, if we represent the right side of the formula as an approval A (according to Frege), not requiring a mandatory truth — no factual or logical [13]. Then the formula can be considered as an expression of the sequence of doxastic action which forces an agent to accept all of his beliefs. Factually, if the agent adds a statement to his belief set, then he is obliged to assert it. This expression describes agent's doxastic commitments, that he imposes on himself by adding a statement to the belief set.

Logic defines a set of rules PC, MP, US, N, K and T is *dynamic modal logic of expansion* DML_E.

7 AGM postulates of expansion within DML_E

Expressive capabilities of DML_E allow to formalize the postulates of expansion AGM: closure, success, inclusion, vacuity, monotonicity and minimality. According to the theorem AGM [1, p. 513], the operator which satisfies the given postulates is equivalent to the operator of expansion AGM.

7.1 Closure

The resulting belief set obtained from the expansion of the initial theory by some proposition A, should be closed under the logical consequence. Formally, if K is a belief set, then K + A is a belief set. This postulate expresses the principle of categorical matching whereby the representation of a belief after performing some operations must be the same as the way of presenting the initial state. In DML_E the postulate of closure is expressed in rule K.

K $\quad +(A \to B) \to (+A \to +B)$

7.2 Success

After the expansion of the theory by some proposition A, it must belong to the theory. This postulate expresses the criterion of priority a new information, whereby the input information must be accepted. Formally: $A \in K + A$. In DML_E the success postulate is expressed by axiom T.

T $\quad +A \to A$

7.3 Inclusion

The initial belief set must be always a subset of the expanded belief set. This is a kind of assertion of ,,purity" of the operator of expansion. According to the postulate of inclusion, making the expansion, we combine the initial belief set and the set, which consists of the added proposition, while no proposition is removed from the initial set. Closure of resulting belief set must include the original belief set. Formally: $K \subset K + A$. In DML_E postulate of inclusion can be expressed the following:

$B\& + A \to B$

The postulate of inclusion can be obtained from the theorem of the propositional calculus by the substitution rules, so it is deducable in DML_E.

7.4 Vacuity

Expansion of the theory by proposition A which is already presented in it, does not change the theory. Formally: if $A \in K$, then $K + A = K$. In DML_E the postulate of vacuity can be expressed as follows:

$A \to (+A \to A)$

The above mentioned postulate AGM expansion may be withdrawn within DML_E using an axiom of propositional calculus and the rules of substitution, so it is a theorem of logic of expansion.

7.5 Monotonicity

Operation of expansion is monotonous under the set-theoretic operation of inclusion. If $K \subseteq N$, then $K + A \subseteq N + A$. The same property retains the operator of expansion with respect to implication in DML_E.

$(B \to C) \to ((B\& + A) \to (C\& + A))$

The postulate of monotonicity of expansion is also inferred in DML_E by the rules of substitution.

7.6 Minimality

For any belief set K and proposition A, $K + A$ is the lowest set of beliefs which satisfies the postulates of closure, success and inclusion. This postulate expresses the criterion of minimality, whereby the modified belief set must be strictly regulated. Expanding the theory by some proposition A we should add only this proposition A and nothing more to the belief set. Closure of the received belief set provides a presence added proposition in the belief set. Thus, if a proposition B does not follow from A, the expanding of the theory by A should not lead expansion of a theory by B.

Theorem. Minimality DML$_E$. $\neg(A \to B) \to \neg(+A \to +B)$

Proof.
1 $\neg(A \to B) \to \neg + (A \to B)$ US, T, contraposition
2 $\neg + (A \to B) \to \neg(+A \to +B)$ 1,K
3 $\neg(A \to B) \to \neg(+A \to +B)$ 1, 2, transitivity

If the implication $(A \to B)$ exists in the initial belief set, then when adding A, proposition B appears in the theory as a result of keeping closure. If B does not follow from A, but was still present in the initial theory, its adding does not change the theory, according to the postulate of vacuity.

Thus, the formalization of operation of expansion allows to express the definition of expansion, which moves in the AGM, the basic properties of this operation and criteria of rationality applicable to the expansion. The postulates AGM, which define the basic properties of operation of expansion, can be formalized and proved in dynamic doxastic logic of expansion. Thus, the logic given by the PC, US, N, K and T is a dynamic modal logic of extension DML_E and represents the function of expansion AGM.

8 Dynamic modal logic of contraction DML_C

Let us consider the operator of contraction. Any logical system based only on the operator of contraction, without expansion, allows us to express the properties of the withdrawal function [11], a specific type of contraction, which does not satisfy the postulate of recovery. Notably, while all the other AGM postulates of contraction are satisfied. It seems inappropriate to build a logical system based on an operator, which does not satisfy all AGM postulates of contraction, however, to express the complex operator, can be based even on withdrawal operator.

To build a logical system based on the operator of contraction, use the alphabet of dynamic modal logic DML with the rules of PC, MP, US. In addition, we introduce some specific rules in a logical system, that determine the properties of contraction, defined by the axiomatic definition of the AGM contraction functions.

First of all, consider the introduction of the operator of contraction. The next rule says that when we contract the theory by proposition B, we should remove all propositions A, which inferred B. Factually, if $A \to B$ is a theorem, then $\div B \to \div A$ is a theorem.

C If $\vdash A \to B$ and $A \to B$ is a boolean formula, then $\vdash \div B \to \div A$.

Note, the same principle, expressed by a formula $(A \to B) \to (\div B \to \div A)$ can not be accepted as an axiom of DML_C, because if $(A \to B)$ is not a tautology, an alternative for removal A (due to contraction by B) could be generally removal formula $(A \to B)$.

The operation of contraction should provide consistency of belief set. If the belief set contains some proposition A, it should not contain a denial of this proposition $\neg A$. Contraposition of these statements is more convenient: if a belief set contains $\neg A$, then in order to preserve consistency of the set, we need to delete A.

This doxastic commitment can be justified in another way, if we believe in a false proposition A (this is possible if A is contrary to our initial belief set and initial set contains $\neg A$), we must reject the false proposition. Indeed, we are trying to get rid of false propositions in the change of belief, no matter for what reason — as a result of obtaining some new information, or, for example, due to review the existing belief set in search of contradictions.

This principle of consistency of a belief set reflects the mechanism of contraction, and therefore should be accepted as an axiom DML_C and can be stated as follows:

U $\neg A \to \div A$

The following axiom expresses an important principle of partial meet contraction AGM, also called "conjunctive factorization".

DM1 $\div(A\&B) \to (\div A \vee \div B)$

It is natural to adopt this axiom. If we wish to contract the belief set by a conjunction and there exists some preference between the conjuncts, then this contraction is equivalent to contraction by the non-preferred conjuncts. If the both conjuncts are equal, then we will have to remove both [5, p. 12]. There are three ways to contract initial belief set by conjunction $A\&B$, such us: remove the sentence A but leave the sentence B, remove the sentence B but leave the sentence A, and remove both.

9 Doxastic modal logic of contraction DML_C and AGM postulates

The most convenient kind of contraction is represented in AGM by so-calling „partial meet contraction". The basic postulates of contraction are developed in AGM (six basic and two additional postulates)[1]. All the postulates of contraction can be expressed in the language of DML_C, moreover almost all postulates of contraction can be proved in the doxastic modal logic of contraction.

9.1 Closure

According to the principle of categorial matching, the representation of a belief state after a belief change should have the same form as the representation of the belief state before the change [8, p. 5]. Hence, if the initial theory K is logically closed, then theory $K \div A$ is closed too. Let us contracting the set K by proposition A. Thus, before contraction we need to make sure that the original set K is closed under logical consequense, and after a contraction we must to implement the closure outcome set $K \div A$.

On the expansion postulate of closure acquires the form

$$+(A \to B) \to (+A \to +B)$$

On the contraction this postulate points out that if the contraction the set K by proposition A was done correctly, then closure of the outcome set $K \div A$ must not contain A. The proposal should be deleted explicitly and implicitly. To do this we need to remove all statements that lead A otherwise the closure operation again returns it to the theory. In the language of DML_C it will look as follows.

Theorem 1. ***DML1*** $\quad \vdash (A \to B) \Rightarrow\ \vdash (\div B \to \div A)$

Proof. If B follows from A, then, if we remove B, then we must remove A. This is a inference rule **C**.

9.2 Inclusion

$K \div A \subseteq K$

A theory obtained by contraction of some theory must be a subset of the initial theory. It is impossible that the outcome belief set $K \div A$ was intersected with the initial belief set or K was a proper subset of $K \div A$.

Theorem 2. ***DML2*** $\quad \div B \to (\div A \to \div B)$

Proof. This principle is a special case of the consequent approval, and therefore, it is the theorem of our system.

A deletion any proposition should not cause an expansion of a system by means of proposition which are not exist in it. Factually, any proposition which does not belong to belief set should not appear in the theory as a result of contraction by anyone proposition.

9.3 Vacuity

$A \notin K \Rightarrow K \div A = K$

If the proposotion does not belong to the original belief set, then its removal should not cause any transformation of the system. In doxastic modal logic of contraction this postulate can be expressed by the following method:

Theorem 3. ***DML3*** $(B\&\neg A) \to (\div A \to B)$

In other words, if belief set is compatible with $\neg A$, that it does not contain statement A, then we obtain the same initial set B, when try to remove the A.

Proof.

1 $(B\&\neg A) \to B$	PC
2 $B \to (B \vee \neg \div A)$	1, PC
3 $(B \vee \neg \div A) \to (\div A \to B)$	2, PC
4 $(B\&\neg A) \to (\div A \vee B)$	1, 3, transitivity

9.4 Success

$A \notin Cn(\varnothing) \Rightarrow A \notin (K \div A)$

Reformulating this postulate by contraposition, we obtain the next note: if $A \in (K \div A)$, then $A \in Cn(\varnothing)$. That is, if a statement is present in the theory after it was removed, then the statement is a logical tautology. In the language of DML_C this postulate can be written as

Theorem 4. ***DML4*** $(\div A \to A) \to ((A \to A) \to A)$

Proof.

1 $\neg A \to \div A$	U
2 $(\neg A \to \div A) \to (\div A \vee A)$	1, PC
3 $(\neg A \vee \div A) \to ((\neg A \vee \div A)\&(A \vee \neg A))$	2, PC
4 $((\neg A \vee \div A)\&(A \vee \neg A)) \to (\div A\&\neg A) \vee (A\&\neg A) \vee A$	3, PC
5 $((\div A\&\neg A) \vee (A\&\neg A) \vee A) \to ((\neg \div A \vee A) \to ((\neg A \vee A) \to A))$	4, PC
6 $((\neg \div A \vee A) \to ((\neg A \vee A) \to A)) \to ((A \to A) \to A)$	5, PC

9.5 Extensionality

If $A \leftrightarrow B \in Cn(\varnothing)$, then $K \div A = K \div B$

Extensionality guarantees that the logic of contraction is extensionak in the sense of allowing logically equivalent sentences to be freely substituted for each other [8, p. 8].

Theorem 5. ***DML5*** $\vdash A \leftrightarrow B \Rightarrow \vdash \div A \leftrightarrow \div B$

Proof. The proof of this rule can be easily obtained using the rule **C**.

Thus, we expressed the five postulates of AGM contraction and built their proof in the system DML_C. Within the logic contraction can not formalize the postulate of recovery, because it requires a explicit use of the operator of expansion.

So, for the modal operator of the contraction imposed by axioms DML_C, performed five basic postulates of contraction AGM: closure, inclusion, vacuity, success, and extensionality. Thus, the modal operator of contraction reflects the properties of the functions of contraction AGM. Now, we can formulate appropriately *representational theorem*, analogous to [1].

Representational Theorem. *Sinse the modal operator of the contraction DML_C satisfies the properties closure, inclusion, vacuity, success, and extensionality, hence the operator of the contraction DML_C is equivalent to withdrawal function AGM.*

Thus, the representation doxastic actions as modal operators allows to construct axiomatical systems for the operator of the expansion and the operator of the contraction. Within the system based on this interpretation, we can express and prove the corresponding postulates of expansion and contraction AGM. It demonstrates that these modal operators correspond to the functions of expansion and contraction described in AGM, about that the representational theorem has been formulated. This method of representation of doxastic actions makes it possible to formulate rigorously known properties of expansion and contraction and, moreover, to trace the new properties of doxastic actions, which manifest themselves by virtue of this representation.

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