Finding and Visualizing of Limit Cycles

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Abstract

The article reflects a study aimed at using a parallel computing system for automated retrieval and visualization of cycles of a quadratic system of two differential equations. The study was conducted in the seven-dimensional space of parameters --- system coefficients and initial data of the Cauchy problem. It is very important for sustainable operation of transport systems. To implement the calculations, supercomputers of Moscow State University were remotely used. Visualization of the results carried out on Hewlett Packard personal computers. The developed software model is applicable to weaning and visualization of cycles for different systems of two differential equations.

1. Introduction

A.Poincare examined the geometric pattern solutions of the differential equation. In 1900, D.Hilbert set the task of research limit cycles (attractors) in two-dimensional quadratic systems (see [Per01], [Li03], [Leo14], [Ruz15], [Leo15], [Leo17]). In the fifties A. N. Kolmogorov suggested estimating the number of these cycles.

In his book [Arn05] V. I. Arnold wrote: "To estimate the number of limit cycles quadratic vector fields on the plane, A. N. Kolmogorov distributed several hundred such fields (with random selected coefficients of polynomials of the second degree) to several hundred students at the mechanical and mathematical Faculty of Moscow State University as a mathematical workshop. Each student had to find the number of limit cycles of their field.The results of this experiment were completely unexpected: there were no limit cycles at all."

It is well known that the definition of sustainability of various devices reduced to the definition of steady states for a certain system of differential equations. Inadequate investigation of the sustainability can lead to catastrophic consequen\-ces for designed devices and apparatuses. Hopping from a planned state of resistance to an unplanned condition repeatedly led to the destruction of bridges and structures, railway accidents, the total destruction of aircraft, etc.

Let a pair of functions x (t), y (t) be a solution to the Cauchy problem for a system of two differential equations of the form

$$x'=P(x,y) \qquad (1)$$

$$y'=Q(x,y), \qquad (2)$$

$$x(0)=x0,y(0)=y0. \qquad (3)$$

The behavior of the trajectory of the point (x (t), y (t)) is important in the plane (x, y) with increasing parameter t (t usually represents the time of the corresponding physical system). The mentioned trajectories are called phase trajectories, and the plane (x, y) is the phase plane. Steady state defined by stability points and limit cycles (the word "cycle" means a closed curve). Limit cycles (attractors) are characterized by the fact that they are approached (wind on them) by phase trajectories (i.e. they are attraction cycles). Repulsion points and cycles may also exist: phase trajectories are wound from them. Note that between two nested limit cycles (they correspond steady states) there is always a cycle repulsion (it corresponds to an unstable state).

The aforementioned tragedies show an urgent need to develop reliable methods of finding attraction cycles and repulsion cycles.

In the second half of the twentieth century, a large number of theoretical papers in which the existence of limit cycles and the boundaries of the parameters are indicated, where they should be searched (we skip the review of these works).

Unfortunately, in the vast majority of cases, there are no analytical methods (formulas) for determining these cycles. In view of this, for finding cycles have become widely used in modern computing methods and computers.

The exception of unplanned states of transport devices, and bridge and tunnel structures come down to finding all attraction and repulsion cycles for phase trajectories of the Cauchy problem

x'=x*x+x*y+y, (4)

y'=a*x*x+b*x*y+c*y*y+alpha*x+beta*y (5)

x(0)=x0, y(0)=y0. (6)

The selection of the seven parameters appearing here a, b, c, alpha,beta,x0,y0 should exclude device jump from a planned attractor to an unplanned one. Such a jump may cause the device to malfunction and even to its complete destruction. For reliable results in points of the selected region of the seven-dimensional parameter space all cycles (attractive and repulsive) have to be defined, and they have to be presented with a video monitor. Problem (4) - (6) is a special case of a more general problem (1) - (3).

The problem of finding and visualizing cycles was solved, thanks to the use of modern computing tools and high-speed computing systems. To solve this problem, Professor G.A. Leonov formed a group with researchers who conducted a series of numerical experiments using various methods on computers of various types.

Due to difficulties in processing seven-dimensional parameter spaces to these studies, the authors of this work were also involved in the organization of parallel computing on a super\-computer.

This article reflects a study aimed at using a parallel computing system for automated retrieval and visualization of cycles of a quadratic system of two differential equations in the seven-dimensional space of parameters. The parameters are the coefficients systems and initial data of the Cauchy problem for the mentioned system.

When implementing calculations, supercomputers "Chebyshev" and "Lomonosov-1" of Supercomputer Research Computing Center of Moscow State University are remotely used. The visualization of the obtained cycles is carried out on HP 27-p251 ur All-in-One and HP Pavilon x360 Convertable Notebook PC. The developed software model is applicable to the finding and visualization of cycles for different systems of two differential equations.

2 Methods and Algorithms

To solve problem (3) - (4), the authors use the Runge-Kutta method of fourth order precision with the automatic choice of step. A numerical experiment showed significant advantage of this method (relative to computational speed) compared to the high-precision Gear's method used by other researchers. In this work, the cycles of attraction and repulsion cycles are automatically determined.

Note that between every two nested attractors there is repulsion cycle (the cycle of unstable equilibrium). The definition of the location of these cycles is very important in calculating stability in the case of designing mechanisms and structures (unstable equilibrium cycle unsafe for designed devices).

Initial testing of algorithms and programs was conducted in uniprocessor mode, and then with parallelization emulation with an MPI interface on a laptop and on a parallel cluster. After that, work was carried out remotely on supercomputers "Chebyshev" and "Lomonosov-1" of Supercomputer Research Computing Center of Moscow State University. The most interesting calculation results in the latter case were automatically saved on the supercomputer, then sent and autonomously visualized on the HP 27-p251ur All-in-One and on the HP Pavilon x360 Convertable Notebook (Figures 1 -- 3 show the results of some visualizations).

The results can be used in calculating and designing various devices, as well as for checking reliability of created designs. The simple modification of algorithms and programs allows you to use the program in the case of solving similar problems for other autonomous systems of differential equations.

3 Results

3.1 First Series of Values Parameter beta

Search for limit cycles (attractors) for a set of parameters a=-10.0, b=2.7, c=0.4, alpha=-473.5, beta=0.003-epsilon, epsilon=s*0.0000000001, s=0,1,…, 1000, in each of these options led to 3 attractors (the case epsilon=0 with a gradual expansion of the study area, see Fig. 1 -- 3).

3.2 Second Series of Values Parameter beta

When searching for limit cycles (attractors) in another set parameters, namely, for a=-10.0, b=2.7, c=0.4, alpha=-173.5,beta=0.004+ 0.0001*s, s=0,1,…,9950.

In each of the options listed, exactly one attractor appeared.

3.3 Pair (b, c) gets 32 Million Values

Here we consider 32 million of the pairs of parameter (b,c) according to the next formulas

a=(b-1)*(b-1)/(4*(c-1)+1), b=2.1+0.0001*s, s=0,1,…,8000, c=0.5+0.0001*p, p=0,1,…,4000, alpha=a*(2+b)/(b*c-1) +|a*(2+b)/(b*c-1)|/2, beta=0.

Here, in each variant, three attractors appeared, but for some parameters (and for small perturbation of the parameter β) a fourth attractor arose. The occurrence of the fourth attractor cannot be considered reliable because rounding errors are occurring when floating point is used. Therefore we will not discuss the fourth attractor.

3.4 Wavelet Decomposition

To speed up data transfer was considered a first-order wavelet decomposition with the following parameters a=-10.0, b=2.7, c=0.4, alpha=-173.5,beta=0.003.

1000 Cauchy problems were solved with the initial data $(x_0, y_0) = (j, 0)$; here $j = 1,2, \ldots, 1000$. The resulting sets of values were saved and then divided into the main and wavelet streams (the main stream turned out to be about 2 times less source). Next, to another computer via ssh-protocol source and main streams are transferred. Let T0 be the transmission time of the original flow, let T1 be the transmission time of the main flow, and let k be their ratio, i.e. k = T0 / T1. In the described numerical experiment, the coefficient k turned out to be 1.92. Thus this indicates savings significant resource when the wavelet decomposition is used.

4 Conclusion

This study showed that the problem of finding and visualization of cycles in multidimensional (in seven-dimensional) space of parameters can be solved using modern computational algorithms and high-speed parallel computing systems. In particular, the application of the Runge-Kutta method proved to be very effective. The usage of a parallel computing system for automated search and visualization of cycles quadratic system of two differential equations gives very precise results in the seven-dimensional space of parameters. The developed software model is obviously applicable to weaning and visualization of cycles for different systems of two differential equations.

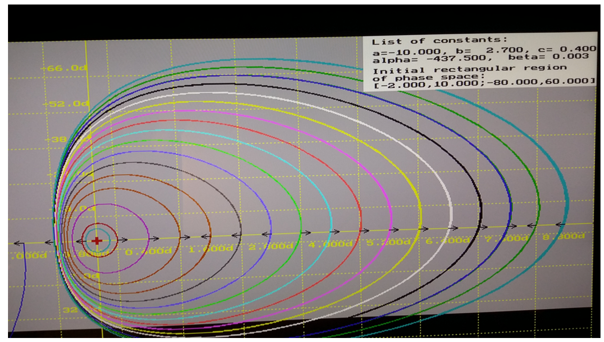

Figure 1: Visualization of wound trajectories

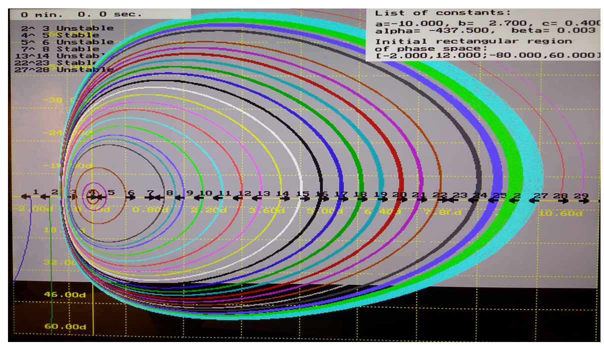

Figure 2: Expansion of the area. More wound trajectories

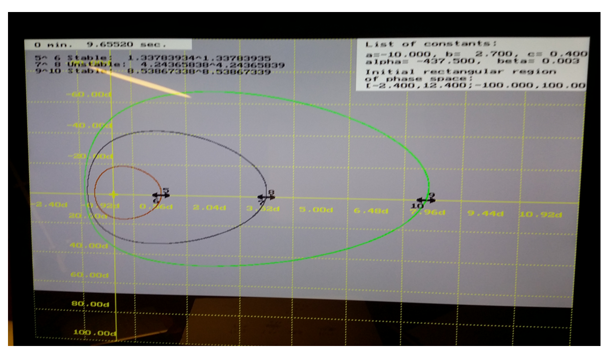

Figure 3: Visualization of three cycles

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