# On logical and mereological renderings of the Bayes theorem

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*To the memory of Professor Helena Rasiowa (1917-1994)*

**Abstract.** [1] The Bayes theorem published posthumously as the work of Rev. Thomas Bayes (1701/2-1761) in 'Essay Towards Solving a Problem in the Doctrine of Chances' (1764) rediscovered by Lagrange, provides a foundation for some areas of Artificial Intelligence like Bayesian Reasoning, Bayesian Filtering etc. It had been reformulated in logical terms by Jan Łukasiewicz (1913). Recently, an abstract version couched in mereological terms was formulated and a strengthening of it appeared derived from the Stone representation theorem for complete Boolean algebras. It is our aim to comprehensively present those approaches with emphasis on the abstract setting of mass assignments on mereological universes endiwed with rough inclusions induced by masses of things.

## 1 The Bayes theorem original and the rendering by Łukasiewicz

Given a probability distribution on a space of events $\Omega$ (cf.[5]) one defines the conditional probability $P(E|H)$ of the event $E$ modulo the event $H$ as $P(E|H) = \frac{P(E\cap H)}{P(H)}$. From this the Bayes theorem follows in the presently used form:

$$P(E|H) = \frac{P(H|E) \cdot P(E)}{P(H)}. \tag{1}$$

It is often in use the generalization of (1) to the case when the space of events $\Omega$ is split into pairwise disjoint events $G_i, i \leq k$ with $\Omega = \bigcup_{i=1}^{k} G_i$. In this case, by axioms of probability calculus [5], $P(H) = \sum_{i=1}^{k} P(H|G_i) \cdot P(G_i)$ and the Bayes theorem is given in the form:

$$P(E|H) = \frac{P(H|E) \cdot P(E)}{\sum_{i=1}^{k} P(H|G_i) \cdot P(G_i)}. \tag{2}$$

Jan Łukasiewicz [8] (for english transl. see [1]) considered the logical framework of a collection $\Gamma$ of indefinite unary formulas over a finite universe of things $\Omega$. For a formula $\gamma(x) \in \Gamma$ and a thing $\omega \in \Omega$, one says that the thing $\omega$ satisfies the formula $\gamma(x)$, in symbols: $\omega \models \gamma(x)$ when after substitution of $\omega$ for all symbols in $\gamma(x)$ equiform with $x$, the formula $\gamma(x/\omega)$ is true (we do not discuss here the criterion of truth assuming it is set).

For a formula $\gamma(x) \in \Gamma$, Łukasiewicz had defined the *truth value* $w$:

$$w(\gamma(x)) = \frac{|\{\omega \in \Omega : \omega \models \gamma(x)\}|}{|\Omega|}, \tag{3}$$

where the symbol $|X|$ is denoting the cardinality of $X$. The function $w$ takes values in the unit interval $[0, 1]$ and it subject to the following properties (0 denotes the unsatisfiable formula and 1 denotes a tautology):

I. $(\gamma = 0) \Leftrightarrow [w(a) = 0]$.

II. $(\gamma = 1) \Leftrightarrow [w(a) = 1]$.

III. $(\gamma \Rightarrow \delta) \Rightarrow [w(\gamma) + w((\neg\gamma) \wedge \delta) = w(\delta)]$.

From postulates I, II and III, Łukasiewicz had derived, using the standard propositional calculus derivations, formulas which turn out to be the basic formulas of probability calculus like (the numeration is as in original Łukasiewicz [8], cf. [1]):

(3) $(\gamma \Leftrightarrow \delta) \Rightarrow [w(\gamma) = w(\delta)]$.

(4) $w(\gamma) + w(\neg\gamma) = 1$.

(5) $w(\gamma \wedge \delta) + w((\neg\gamma) \wedge \delta) = w(\delta)$.

(6) $w(\gamma) + w((\neg\gamma) \wedge \delta) = w(\gamma \vee \delta)$.

(7) $w(\gamma \vee \delta) = w(\gamma) + w(\delta) - w(\gamma \wedge \delta)$.

(8) $(\gamma \wedge \delta = 0) \Rightarrow [w(\gamma \vee \delta) = w(\gamma) + w(\delta)]$.

(9) $(\sum_{i,j} gamma_i \wedge \gamma_j = 0) \Rightarrow [w(\bigvee_i \gamma_i) = \sum_i w(\gamma_i)]$.

(10) $[w(\gamma \vee \delta) = w(\gamma) + w(\delta)] \Rightarrow (\gamma \wedge \delta = 0)$.

(12) $[w(\gamma) + w((\neg\gamma) \wedge \delta) = w(\delta) \Rightarrow (\gamma \Rightarrow \delta)$.

(13) $(a\gamma \Rightarrow \delta) \Leftrightarrow [w\gamma) + w(\neg\gamma \wedge \delta) = w(\delta)]$.

'Relative truth value' is defined in Łukasiewicz [8] as follows:

$$w\gamma(\delta) = \frac{w(\gamma \wedge \delta)}{w\gamma)}. \tag{4}$$

The following formulas are consequences to definition (4).

(14) $w_1(\delta) = w(\delta)$.

(15) $w(\gamma \wedge \delta) = w(\gamma) \cdot w_\gamma(\delta) = w(\delta) \cdot w_\delta(\gamma)$.

The final thesis is a rendering of the Bayes theorem as 'a special theorem' (cf. [1] p. 31):

(22) $(\bigvee_i \gamma_i = 1) \wedge (\bigvee_{i,j} \gamma_i \wedge \gamma_j = 0) \Rightarrow [w_\delta(\gamma_m) = \frac{w(\gamma_m) \cdot w_{\gamma_m}(\delta)}{\sum_i w(\gamma_i) \cdot w_{\gamma_i}(\delta)}$.

It is manifest that after substitution of probability of an event for weight of a formula, we obtain true formulas of probability calculus on finite probability spaces. We now proceed to the meerological setting in which we give an abstract formulation for both probability calculus and Łukasiewicz's logical scheme.

## 2 Mereology and approximate (rough) mereology

The standard version of mereology had been proposed by Stanisław Leśniewski [6]. The interested reader may as well consult in addition, e.g., Casati and Varzi [2], Pietruszczak [9], or, Polkowski [11].

A mereological universe is a pair $(U, \pi)$ where $U$ is a collection of things and $\pi$ is a binary relation of *being a part of* which should satisfy the conditions:

1. *For each thing $x$, it is not true that $\pi(x, x)$.*

2. *For each triple $x, y, z$ of things, if $\pi(x, y)$ and $\pi(y, z)$, then $\pi(x, z)$.*

The relation of a part does induce the relation of an *improper part* $\Pi(x, y)$, defined as:

$$\Pi(x, y) \Leftrightarrow (\pi(x, y) \vee x = y). \tag{5}$$

When the relation $\Pi$ occurs between a pair $x, y$ then $x$ is called an *ingrediens* of $y$. Clearly, $(U, \Pi)$ is a partially ordered structure.

On basis of the relation $\Pi$, the overlap relation is defined:

$$Ov(x, y) \Leftrightarrow \exists z.\Pi(z, x) \wedge \Pi(z, y). \tag{6}$$

Helped by the relation of overlapping, we introduce the third postulate for our model of mereology:

3. *For each pair $x, y$ of things, if for each thing $z$ such that $\Pi(z, x)$ there exists a thing $w$ such that $\Pi(w, y)$ and $Ov(z, w)$, then $\Pi(x, y)$.*

The important notion in the mereology scheme is that of a class of things; the class operator converts any non-empty collection of things into a thing. The assumption is that classes always exist; by 3. they are unique.

The notion of a class was defined originally in Definition II in [7]:

*$P$ is a class of objects $a$ if and only if the following conditions are met: (i) $P$ is an object; (ii) every $a$ is an ingrediens of $P$; (iii) for any $Q$, if $Q$ is an ingrediens of the object $P$ then some ingrediens of the object $Q$ is an ingrediens of some $a$.*

Let us notice that notions of a set and of a class have been the subject of a long going philosophical dispute (cf. Pietruszczak [9] for a discussion; we owe this work the translation into English of Definition II.)

For the universe $(U, \pi)$ of things, we define the class $V$ of all things:

$$V = Cls\{x : x \in U\}. \tag{7}$$

We call $V$ the *universal thing*.

We are now in a position to recall here two *fusion operators* due to Tarski [18]. These operators are the *sum* $x + y$ and the *product* $x \cdot y$ defined as:

$$x + y = Cls\{z : \Pi(z, x) \vee \Pi(z, y)\} \tag{8}$$

and

$$x \cdot y = Cls\{z : \Pi(z, x) \wedge \Pi(z, y)\}. \tag{9}$$

The things $x, y$ are *disjoint*, $dis(x, y)$ in symbols, whenever there is no thing $z$ such that $\Pi(z, x)$ and $\Pi(z, y)$ (a fortiori, the product of $x$ and $y$ is not defined).

$$dis(x, y) \Leftrightarrow \neg Ov(x, y). \tag{10}$$

The *difference* $x - y$ is defined as follows:

$$x - y = Cls\{z \in U : \Pi(z, x) \wedge dis(z, y)\}. \tag{11}$$

It is well-known (see [18]; cf. English transl. in [19]; cf. Pietruszczak [9]. Ch. III) that the mereological universe $(U, \pi)$ with the universal thing $V$ and operations $+, \cdot, -$ carries the structure of a complete Boolean algebra without the zero element. However, contrary to some views, the two are not identical: mereology is defined for individuals and they are defined in the Ontology by Leśniewski, so one cannot extract solely mereology from his scheme.

The *complement* $-x$ to a thing $x$ in the universe $(U, \pi)$ is the difference $V - x$:

$$-x = V - x. \tag{12}$$

We introduce finally the *mereological implication* $x \hookrightarrow y$, valued in things in the universe $U$ and defined as:

$$x \hookrightarrow y = -x + y. \tag{13}$$

The implication $x \hookrightarrow y$ is declared true if and only if $-x + y = V$.

### 2.1 Rough mereology

Approximate mereology (rough mereology, fuzzified mereology) [14], [10], [20] takes part relations of mereology and extends them to relations of *being a part to a degree* . Formally, the rendering of those relations comes in the form of the relation of rough inclusion $\mu$ (see [15] where this notion was introduced); the relation $\mu$ takes as arguments things $x, y$ in the universe $U$ and a real number $r \in [0, 1]$. The formula $\mu(x, y, r)$ reads: *the thing $x$ is a part of the thing $y$ to a degree of at least $r$.* The relation $\mu$ should obey the requirements:

4.$\mu(x, y, 1) \Leftrightarrow \Pi(x, y)$.

5. $\mu(x, y, 1) \Rightarrow \forall z \in U. \forall r \in [0, 1]. [\mu(z, x, r) \Rightarrow \mu(z, y, r)]$.

6. $[\mu(x, y, r) \wedge s < r] \Rightarrow \mu(x, y, s)$.

The reader will find a discussion of rough inclusions in [10].

## 3 Mass-based approximate mereology

By a *mass assignment* on a universe $(U, \pi)$ we understand a function $m : U \rightarrow (0, 1]$ which is subjected to postulates [12]:

7. $m(\theta) = 0$, *where $\theta$ denotes the empty thing, not in the universe $U$.*

8. $x = V \Leftrightarrow m(x) = 1$.

9. $x = \theta \Leftrightarrow m(x) = 0$.

10. $(x \hookrightarrow y) \Rightarrow [m(y) = m(x) + m(-x \cdot y)]$.

From 7-10, we derive properties of $m$ (see [12]:

11. $(x = y) \Rightarrow [m(x) = m(y)]$.
12. $m(x + y) = m(x) + m(-x \cdot y)$.
13. $(x \cdot y = \theta) \Rightarrow [m(x + y) = m(x) + m(y)]$.
14. $m(x) + m(-x) = 1$.
15. $m(y) = m(x \cdot y) + m((-x) \cdot y)$.
16. $m(x + y) = m(x) + m(y) - m(x \cdot y)$.
17. $\Pi(x, y) \Rightarrow [m(x) \leq m(y)]$.
18. $[m(x + y) = m(x) + m(y)] \Rightarrow (x \cdot y = \theta)$.
19. $\Pi(x, y) \Leftrightarrow m(x \hookrightarrow y) = 1$.
20. $\Pi(y, x) \Rightarrow y \cdot (-x) = \theta$.
21. $\Pi(y, x) \Rightarrow m(x \hookrightarrow y) = 1 - m(x) + m(y)$.
22. $m(x \hookrightarrow y) = 1 - m(x - y)$.

Comment. 22 is a general formula which generalizes the Łukasiewicz implication formula [4] to the case of partially ordered sets.

23. $m(x \cdot y) = m(x) \cdot m(y) \Leftrightarrow m((-x) \cdot y) = m(-x) \cdot m(y)$.

It is clear that masses $m(x)$ generalize Łukasiewicz's $w(q)$'s.

## 4 Rough inclusions in mass-based mereological universe and the Bayes theorem

We continue with the mereological universe $(U, \pi)$, augmented with a mass assignment $m$. We define a rough inclusion $\mu \subseteq U \times U \times [0, 1]$ with triples of the form $(x, y, r)$, where $x, y \in U$ and $r \in [0, 1]$ as arguments:

$$\mu(x, y, r) \Leftrightarrow \frac{m(x \cdot y)}{m(y)} \geq r. \tag{14}$$

We define in addition the inclusion function $\mu_1 : U^2 \to [0, 1]$ returning for each pair of things $x, y$ the maximal degree $r$ such that $\mu(x, y, r)$ holds true, i.e., the maximal degree of inclusion of $x$ into $y$:

$$\mu_1(x, y) = \frac{m(x \cdot y)}{m(y)}. \tag{15}$$

The rough inclusion $\mu$ and the inclusion function $\mu_1$ satisfy the following formulas.

24. $\Pi(y, x) \Rightarrow [\mu_1(x, y) = 1]$.
25. $[\mu_1(x, y) = 1] \Rightarrow \Pi(y, x)$.
26. $\Pi(y, x) \Leftrightarrow \mu_1(y, x) = 1 \Leftrightarrow y \hookrightarrow x$.
27. $[\mu(x, y, 1) \Rightarrow \forall z.\mu_1(x, z) \geq \mu_1(y, z)$.
28. $\mu_1(-y, x) = 1 - \mu_1(y, x)$.
29. $[m(x) + m(-x \cdot y) = m(y)] \Rightarrow \Pi(x, y)$.
30. $\mu_1(x, y) = \frac{m(x) \cdot \mu_1(y, x)}{m(y)}$ [the Bayes formula].

31. $\frac{\mu_1(x,y)}{\mu_1(y,x)} = \frac{m(x)}{m(y)}$.

32. $\Pi(x,y) \Rightarrow [\mu_1(y,x) = \frac{m(y)}{m(x)}]$.

33. $\frac{\mu_1(x,y)}{\mu_1(y,x)} \cdot \frac{\mu_1(y,z)}{\mu_1(z,y)} = \frac{\mu_1(x,z)}{\mu_1(z,x)}$.

The general form of the Bayes theorem is as follows. The notation $+Y$ for a finite collection of things $Y$ denotes the result of mereological addition of things in $Y$.

34. $[(+_{i \neq j} y_i \cdot y_j = \theta) \wedge (+_i y_i = V)] \Rightarrow [\mu_1(z,x) = \frac{m(z) \cdot \mu_1(x,z)}{\sum_{i=1}^{k} m(y_i) \cdot \mu_1(x,y_i)}]$.

We now pass to a discusssion of existence of a system $\{y_i\}$ satisfying the premises of formula 34. We prove (ineffecively) the existence of such a system with use of the Stone duality theorem (Stone [17]).

## 5 The compactness aspect

We explore the fact that the mereological space $(U, \pi)$ carries the structure of a complete Boolean algebra without the null element and we refer to the Stone Representation Theorem [17] for complete Boolean algebras. We recall that a filter on a Boolean algebra $B$ is a collection $F$ of elements of $B$ with properties: (i) if $x, y \in F$ then $x \cdot y \in F$; (ii) if $x \in F$ and $x \leq y$ then $y \in F$; (iii) the null element not in $F$. A filter maximal with respect to containment in the collection of all filters is called an *ultrafilter*. It is well-known that each ultrafilter $H$ has the following defining properties (see [**?**]): (iv) for each thing $x \in U$, either $x \in F$ or $-x \in F$; (v) $H$ is *prime*, i.e., if $x + y \in H$ then either $x \in H$ or $y \in H$.

The Stone theorem states that a complete Boolean algebra $B$ is isomorphic to the space of closed - and - open sets in a compact 0-dimensional Hausdorff space. This space is the space of all ultrafilters on $B$ (the Stone space $St(B)$) and it is topologized by admitting sets $S(x)$ for $x \in B$ as the closed-and-open base, where $S(x)$ is the set of all ultrafilters in $St(B)$ which contain $x$.

It has been shown as the consequence to the Stone theorem cf. [13], that:

*In the mereological space $(U, \pi)$, there exists a finite set of elements $\{x_1, x_2, \ldots, x_k\}$ for some $k$ with the property that each thing $x \in U$ admits the representation $x = +_i x \cdot x_i$.* The set $\{x_i : i \leq k\}$ is called a *base* in the space $(U, \pi)$.

One can produce an *orthogonal base* $\{y_i : i \leq k\}$ by letting $y_i = x_i \cdot \prod_{j<i}(-x_j)$ for $i \leq k$ [13]; then:

$$y_i \cdot y_j = \theta \tag{16}$$

whenever $i \neq j$.

Accordingly, the rough inclusion $m$ has by virtue of 13 the property,

35. For each thing $x \in U$, $m(x) = \sum_{i=1}^{k} m(x \cdot y_i)$.

In consequence, the mass-based inclusion function $\mu_1$ acquires the form

$$\mu_1(x,z) = \frac{\sum_{i \leq k} m(x \cdot z \cdot y_i)}{\sum_{i \leq k} m(x \cdot y_i)}. \tag{17}$$

From (17), we obtain the ultimate form of the Bayes theorem by taking as the set $Y$ in 34 any orthogonal base $\{y_i : i \leq k\}$ and expressing terms for $\mu_1$ as in (17).

## 6 Appendix. Topological notions used and some proofs

. A topological space is a pair $(U, \tau)$, where $U$ is a set and $\tau$ is a family of subsets of $U$ which is closed on arbitrary unions and finite intersections. It follows that both the empty set $\emptyset$ and the set $U$ are members of $\tau$, whose elements are called *open sets*. Complements to open sets are *closed sets*. An *open covering* of a space $U$ is a family $\omega \subseteq \tau$ such that $\bigcup \omega = U$. A topological space $(U, \tau)$ is *Hausdorff* when for each pair $x, y$ of distinct elements of U there exist open disjoint sets $X, Y$ such that $x \in X, y \in Y$. An *open base* for a topological soace is a family $\beta \subseteq \tau$ such that each non–empty open set $X$ is a union of some sub–family $\beta_X \subseteq \beta$. A topological space $(U, \tau)$ is *zero-dimensional* when there exists in it an open base consisting of sets which are closed as well. A topological space $(U, \tau)$ is *compact* when each open covering $\beta$ contains a finite subfamily $\delta$ which is also a covering of U. A convenient paraphrase of this condition is couched in terms of closed sets: a family $K$ of closed sets is *finitely centered* when each finite non-empty sub–family of $K$ has a non–empty intersection; the compactness condition can be stated as follows: a topological space is compact when each finitely centered family of closed sets has a non–empty intersection. The reader will find a detailed discussion of topological spaces in [3].

For a Boolean algebra (see Sikorski [16]) $B$, by a *filter* on $B$ we understand a collection $F$ of elements of $B$ such that (i) if $x, y \in F$ then $x \cdot y \in F$, (ii) if $x \in F$ and $x \leq y$) then $y \in F$, (iii) $\emptyset \notin F$. For a mereological space $(U, \pi)$, condition (ii) translates as (ii)' if $x \in F$ and $\Pi(x, y)$ then $y \in F$ and condition (iii) translates as (iii)' $\theta \notin F$. An *ultrafilter* is a filter which is not contained properly in any other filter. A Boolean algebra $B$ is *complete* when each subset $C \subseteq B$ has the least upper bound $L$, i.e. (i) $x \leq L$ for each $x \in C$ and (ii) if an element $M$ satisfies (i) then $L \leq M$. It is well known that the mereological space $(U, \pi)$ with mereological operations of sum, product and complement, the unit element $V$ and augmented with $\theta$ is a complete Boolean algebra (see [18], [19]).

For a Boolean algebra $B$, the *Stone space* $S(B)$ consists of ultrafilters on $B$. The *Stone topology* $st(B)$ on the Stone space $S(b)$ is induced by the open base consisting of sets $S(x) = \{F :$ F an ultrafilter on B and $x \in F\}$ for all $x \in B$.

The fundamental Stone theorem [17] states that

**Theorem 1 (M. Stone [17]).** *The Stone space $S(B)$ with the Stone topology $st(B)$ on a complete Boolean algebra $B$ is a compact Hausdorff zero-dimensional topological space.*

We recall a proof, for completeness' sake.

By Definition of a filter $S(x) \cap S(y) = S(x \cdot y)$, hence, $st(B)$ has properties of a base. Each set $S(x)$ is clopen: $S(x) = S(U) \setminus S(-x)$. $S(U)$ is compact: let $B$, a collection of sets of the form of $S(x)$ for $x \in W \subseteq U$, be finally centered, i.e., for each finite sub-collection $X=\{x_1, x_2, \ldots, x_k\}$ of $W$, there exists an ultrafilter $F$ with $X \subseteq F$. Let us consider a set $G = W \cup \{z \in U : \text{ there exists } x \in W \text{ with } \Pi(x, z)\}$. Then $G$ extends to an ultrafilter $H$ and $H \in \bigcap B$, i.e., $S(U)$ is compact. $S(U)$ is Hausdorff: let $F \neq G$ for ultrafilters $F, G$. Assume, for the attention sake, that $x \in F \setminus G$ for some thing $x$. Hence, $-x \in G$ and $F \in S(x)$, $G \in S(-x)$, and, $S(x) \cap S(-x) = \emptyset$.

The Stone theorem implies that there exists a finite set $\Psi = \{x_1, \ldots, x_k\}$ of elements of the space $(U, \pi)$ with the property that $S(U) = \bigcup\{S(x_i) : i = 1, \ldots, k\}$. One proves that $\Psi$ is a *base* in $U$ in the sense that for each $x \in U$, we have the representation for $x$ in the form

$$x = +_i x \cdot x_i, \tag{18}$$

where $+_i$ denotes the mereological sum of all elements $x_i \; for i = 1, \ldots, k$.

We recall a proof from [12]. Consider an arbitrary thing $y$ with $\Pi(y, x)$. Let $F(y)$ be an ultrafilter containing $y$; hence, $x \in F(y)$. Let $x_i \in K$ be such that $F(y) \in S(x_i)$. Then, $y \cdot x_i \neq \theta$ and $i \in I(x)$. As $\Pi(y \cdot x_i, x \cdot x_i)$, it follows $\Pi(x, +_i x \cdot x_i)$ by M3. Contrariwise, assume for an arbitrary thing $z$ that $\Pi(z, +_i x \cdot x_i)$, hence, $\Pi(z, x \cdot x_i)$ for some $i \in I(x)$ and thus $\Pi(z, x)$; by M3, $\Pi(+_i x \cdot x_i, x)$ and finally $x = +_i x \cdot x_i$.

We let $y_i = x_i - (+_{j<i} y_j$; then, the formula

$$x = +_i x \cdot y_i, \tag{19}$$

holds true (cf. [12]).

The collection $\{y_i : i = 1, \ldots, k\}$ is an *orthogonal base* in $U$. It serves as the collection of pairwise disjoint elements needed in the general Bayes formula.

## 7 Conclusions

We have presented a survey of some approaches to the Bayes theorem in frameworks of logic and mereology. An extended version will appear in [13].

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