# Completeness for the paraconsistent logic $\mathbf{CG'_3}$ based on maximal theories

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**Abstract.** $\mathbf{G'_3}$ is a recently developed three-valued logic with a sole type of true value. $\mathbf{CG'_3}$ is also a three-valued paraconsistent logic extending $\mathbf{G'_3}$ with two true values. The current state of the art of $\mathbf{CG'_3}$ comprises Kripke-type semantics. In this work, we further extend studies on the syntactic-semantic relation of $\mathbf{CG'_3}$. More precisely, we developed a Hilbert-type axiomatization inspired by the Lindenbaum-Łos technique on maximal theories applied to completeness. Furthermore, we also prove soundness.

**Keywords:** Many-valued logics · Paraconsistent logics · $\mathbf{CG'3}$

## 1 Introduction

Many-valued logics are non-classical logics. As in classical logics, many-valued logics also enjoy of the truth-functionality principle, namely, the truth value of a compound sentence is determined by the truth values of its component sentences, it remains the same when one of its component sentences is replaced by another sentence with the same truth value. However, contrastingly with the classical case, many-valued logics do not restrict the number of truth values to only two, a larger set of truth degrees is then the distinctive feature in the many-valued context. In [2], we can find a detailed analysis of many-valued logics. Some systems of many-valued logics are presented as families of uniformly defined finite-valued and infinite-valued systems, for example, Łukasiewicz logic, Gödel logic, t-Norm based systems, three-valued system, Dunn-Belnap's 4-valued system, Product systems. The main types of logical calculus for systems of Many-valued logics are Hilbert type calculus, Gentzen type sequent calculus or Tableaux [2]. The art for a wide class of infinitely valued logics is presented in [9].

In 1954, F. Asenjo in his Ph.D. dissertation proposes for the first time to use Many-valued logics to generate paraconsistent logics (logics whose semantic or proof-theoretic logical consequence relation is not explosive [7]). The many-valued approach is to drop this classical assumption and allow more than two truth values. The most common strategy is to use three truth values: true, false, and both (true and false) for the evaluations of formulas [7].

The $\mathbf{CG'_3}$ logic is a three-valued paraconsistent logic of recently developed by [5] with a many-valued semantics. Both $\mathbf{CG'_3}$ and $\mathbf{G'_3}$ are defined in terms of three-valued matrices, the unique difference among these logics is on their sets of designated values. In this paper, we further extend studies on the syntactic-semantic relation of $\mathbf{CG'_3}$. Our main contributions are briefly summarized as follows:

- A formal axiomatic theory $\mathbf{CG'_{3h}}$. This theory has four primitive connectives, twelve axioms, and Modus Ponens as the only inference rule.
- It is shown that the formal axiomatic theory $\mathbf{CG'_{3h}}$ is sound and complete with respect to $\mathbf{CG'_3}$, see Theorems 2 and Corollary 1. To prove completeness theorem for $\mathbf{CG'_{3h}}$, we use the Lindenbaum-Łos technique on maximal theories.

## 2 Background

We first introduce the syntax of the logical formulas considered in this paper. We follow common notation and basic definitions as W. Carnielli and M. Coniglio in [1].

**Definition 1 (Propositional signatures).** *A propositional signature is a set $\Theta$ of symbols called connectives, together with the information concerning to the arity of each connective.*

The following symbols will be used for logical connectives: $\wedge$ (conjunction, binary); $\vee$ (disjunction, binary); $\rightarrow$ (implication, binary); $\neg$ (weak negation, unary); $\bullet$ (inconsistency operator, unary); $\sim$ (strong negation, unary); $\perp$ (bottom formula, 0-ary).

**Definition 2 (Propositional language).** *Let $Var = \{p_1, p_2, \ldots\}$ be a denumerable set of propositional variables, and let $\Theta$ be any propositional signature. The propositional language generated by $\Theta$ from $Var$ will be denoted by $\mathcal{L}_\Theta$.*

**Definition 3 (Tarskian logic).** *A logic $\mathscr{L}$ defined over a language $\mathcal{L}$ which has a consequence relation $\vdash$, is Tarskian if it satisfies the following three properties, for every $\Gamma \cup \Delta \cup \{\alpha\} \subseteq \mathcal{L}$:*

*(i) if $\alpha \in \Gamma$ then $\Gamma \vdash \alpha$;*
*(ii) if $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash \alpha$;*
*(iii) if $\Delta \vdash \alpha$ and $\Gamma \vdash \beta$ for every $\beta \in \Delta$ then $\Gamma \vdash \alpha$.*
*A logic satisfying item (ii) above is called monotonic. Moreover, a logic $\mathscr{L}$ is said to be finitary if it satisfies the following:*
*(iv) if $\Gamma \vdash \alpha$ then there exists a finite subset $\Gamma_0$ of $\Gamma$ such that $\Gamma_0 \vdash \alpha$.*
*A logic $\mathscr{L}$ defined over a propositional language $\mathcal{L}$ generated by a signature from a set of propositional variables is called structural if it satisfies the following property:*
*(v) if $\Gamma \vdash \alpha$ then $\sigma[\Gamma] \vdash \sigma[\alpha]$, for every substitution $\sigma$ of formulas for variables.*
*A propositional logic is standard if it is Tarskian, finitary, and structural.*

From now on, a logic $\mathscr{L}$ will be represented by a pair $\mathscr{L} = \langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ and $\vdash$ denote the language and the consequence relation of $\mathcal{L}$, respectively. If $\mathcal{L}$ is generated by a propositional signature $\Theta$ from $Var$, this is $\mathcal{L} = \mathcal{L}_\Theta$ then we will write $\mathscr{L} = \langle \Theta, \vdash \rangle$.

Let $\mathscr{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, $\alpha$ be a formula in $\mathcal{L}$ and $X_1 \dots X_n$ be a finite sequence (for $n \geq 1$) such that each $X_i$ is either a set for formulas in $\mathcal{L}$ or a formula in $\mathcal{L}$. Then, as usual, $X_1, \dots, X_n \vdash \alpha$ will stand for $X'_1 \cup \cdots \cup X'_n \vdash \alpha$, where, for each $i$, $X'_i$ is $X_i$, if $X_i$ is a set of formulas, or $X'_i$ is $\{X_i\}$, if $X_i$ is a formula.

**Definition 4 (Paraconsistent logic).** *A Tarskian logic $\mathscr{L}$ is paraconsistent if it has a (primitive or defined) negation $\neg$ such that $\alpha, \neg\alpha \nvdash_{\mathscr{L}} \beta$ for some formulas $\alpha$ and $\beta$ in the language of $\mathscr{L}$.*

The most adequate manner to define the many-valued semantics of logic is through a matrix. We introduce the definition of the deterministic matrix, also known as the logical matrix or simply as a matrix. In [4], we can find an exhaustive discussion about many-valued logic and some examples.

**Definition 5 (Matrix).** *Given a logic $\mathscr{L}$ in the language $\mathcal{L}$, the matrix of $\mathscr{L}$ is a structure $M = \langle D, D^*, F \rangle$, where:*

*(i) $D$ is a non-empty set of truth values (domain).*
*(ii) $D^*$ is a subset of $D$ (set of designated values).*
*(iii) $F = \{f_c | c \in \mathcal{C}\}$ is a set of truth functions, with one function for each logical connective $c$ of $\mathcal{L}$.*

**Definition 6 (Interpretation).** *Given a logic $\mathscr{L}$ in the language $\mathcal{L}$, an interpretation $t$, is a function $t : Var \rightarrow D$ that maps propositional variables to elements in the domain.*

Any interpretation $t$ can be extended to a function on all formulas in $\mathcal{L}_\Sigma$ as usual, i.e. applying recursively the truth functions of logical connectives in $F$. If $t$ is a valuation in the logic $\mathscr{L}$, we will say that $t$ is an $\mathscr{L}$-valuation. Interpretations allow us to define the notion of validity in this type of semantics as follows:

**Definition 7 (Valid formula).** *Given a formula $\varphi$ and an interpretation $t$ in a logic $\mathscr{L}$, we say that the formula $\varphi$ is valid under $t$ in $\mathscr{L}$, if $t(\varphi) \in D^*$, and we denote it as $t \models_{\mathscr{L}} \varphi$.*

Let us note that validity depends on the interpretation, but if we want to talk about "logical truths" in the system, then the validity should be absolute, as stated in the next definition:

**Definition 8 (Tautology).** *Given a formula $\varphi$ in the language of a logic $\mathscr{L}$, we say $\varphi$ is a tautology in $\mathscr{L}$, if for every possible interpretation, the formula $\varphi$ is valid, and we denote it as $\models_{\mathscr{L}} \varphi$.*

If $\varphi$ is a tautology in the logic $\mathscr{L}$, we say that $\varphi$ is an $\mathscr{L}$-tautology. When logic is defined via a many-valued semantics, it is usual to define the set of theorems of $\mathscr{L}$ as the set of tautologies obtained from the many-valued semantics, i.e. $\varphi \in \mathscr{L}$ if and only if $\models_{\mathscr{L}} \varphi$.

# 3 The logic $\mathbf{CG'_3}$

In this section, we present a summary of the state of the art of logic CG'3. Starting with the many-valued semantics defined by Osorio et al. and ending with Kripke semantics of the $\mathbf{CG'_3}$ logic as well as some important results proposed by Borja and Pérez-Gaspar.

### Many-valued semantic of $\mathbf{CG'_3}$

The logic $\mathbf{CG'_3}$ was introduced in [5] the authors, defined it as a three-valued logic where the matrix is given by the structure $\mathcal{M} = \langle D, D^*, F\rangle$, where $D = \{0, 1, 2\}$, the set $\mathcal{D}^*$ of designated values is $\{1, 2\}$, and $\mathcal{F}$ is the set of truth functions defined in Table 3.

**Table 1.** Truth functions for the connectives $\vee, \wedge, \rightarrow$, and $\neg$ in $\mathbf{CG'_3}$.

| $f_\vee$ | **0** | **1** | **2** |
|---|---|---|---|
| **0** | 0 | 1 | 2 |
| **1** | 1 | 1 | 2 |
| **2** | 2 | 2 | 2 |

| $f_\wedge$ | **0** | **1** | **2** |
|---|---|---|---|
| **0** | 0 | 0 | 0 |
| **1** | 0 | 1 | 1 |
| **2** | 0 | 1 | 2 |

| $f_\rightarrow$ | **0** | **1** | **2** |
|---|---|---|---|
| **0** | 2 | 2 | 2 |
| **1** | 0 | 2 | 2 |
| **2** | 0 | 1 | 2 |

| | $f_\neg$ |
|---|---|
| **0** | 2 |
| **1** | 2 |
| **1** | 0 |

### Kripke-type semantic for $\mathbf{CG'_3}$

In [3] Borja and Pérez-Gaspar proposed Kripke-type semantics for $\mathbf{CG'_3}$. This semantics is defined in two different ways. The first one is based on the semantics of $\mathbf{G'_3}$ as follows:

**Definition 9.** *Let $\mathcal{M} = \langle W, R, v\rangle$ be a Kripke model for $\mathbf{G'_3}$, $w \in W$ and $\varphi$ a formula. We define the modeling relation (denoted as $\models_{\mathbf{CG'_3}}$) as follows:*

*$(\mathcal{M}, w) \models^*_{\mathbf{CG'_3}} \varphi$ if and only if there is $wRw'$ such that $(\mathcal{M}, w') \models^*_{\mathbf{G'_3}} \varphi$.* [1]

**Theorem 1.** *Let $\varphi$ be a formula in the language of $\mathbf{CG'_3}$, then:*
*$\models_{\mathbf{CG'_3}} \varphi$ iff for any Kripke model $\mathcal{M}$ for $\mathbf{CG'_3}$ it holds that $\mathcal{M} \models^*_{\mathbf{CG'_3}} \varphi$.*

The second Kripke-type semantics is given by redefining the modeling relation for $\mathbf{CG'_3}$ considering that the Kripke models for $\mathbf{CG'_3}$ are those for $\mathbf{G'_3}$ but changing the definition by the following one.

**Definition 10.** *A formula $\varphi$ is said to be e-valid on a model $\mathcal{M}$ for logic $\mathbf{CG'_3}$ if exists a point $x$ in $\mathcal{M}$ such that $(\mathcal{M}, x) \models_{\mathbf{G'_3}} \varphi$.*

**Lemma 1.** *Let $\varphi$ be a formula in the language of $\mathbf{CG'_3}$, then: $\models_{\mathbf{CG'_3}} \varphi$ if and only if for any Kripke model $\mathcal{M}$ for $\mathbf{CG'_3}$ it holds that $\varphi$ is e-valid.*

[1] We use the symbol $\models^*$ to define the modeling relation and avoid confusion with the symbol $\models$ that is used for tautologies.

## 4 Axiomatization of $\mathbf{CG'_3}$

In this section, we present an axiomatization for the $\mathbf{CG'_3}$ logic. We begin by defining a formal axiomatic theory whose language has four connective, bottom formula, conjunction, disjunction, and implication. Note that the connective disjunction is determined by the primitive connective.

Let $\mathbf{CG'_{3h}}$ be a formal axiomatic theory for $\mathbf{CG'_3}$ logic defined over the signature $\Sigma = \{\bot, \wedge, \rightarrow, \neg\}$, we define some other connectives, that can be considered as abbreviations as follows:

$$
\begin{array}{rll}
{\sim}\varphi := \varphi \rightarrow \bot & & \text{(Strong negation)} \\
\bullet\varphi := {\sim}{\sim}\varphi \wedge \neg\varphi & & \text{(Inconsistency operator)} \\
\varphi \vee \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) & & \text{(Disjunction logic)} \\
\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) & & \text{(Equivalence logic)}
\end{array}
$$

The set of well-formed formulas is constructed as usual, it is denoted as $\mathcal{L}_\Sigma$.

**Definition 11 ($\mathbf{CG'_{3h}}$).** *The logic $\mathbf{CG'_{3h}}$ is defined over the language $\mathcal{L}_\Sigma$ by the Hilbert calculus:*
***Axiom schemas:***

$$
\begin{array}{cr}
\alpha \rightarrow (\beta \rightarrow \alpha) & Ax1 \\
(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)) & Ax2 \\
(\alpha \wedge \beta) \rightarrow \alpha & Ax3 \\
(\alpha \wedge \beta) \rightarrow \beta & Ax4 \\
\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta)) & Ax5 \\
\alpha \rightarrow (\alpha \vee \beta) & Ax6 \\
\beta \rightarrow (\alpha \vee \beta) & Ax7 \\
(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta) \rightarrow \gamma) & Ax8 \\
(\alpha \rightarrow \beta) \vee \alpha & Ax9 \\
\alpha \vee \neg\alpha & Ax10 \\
\neg\varphi \rightarrow (\neg\neg\varphi \rightarrow \psi) & Ax11 \\
\bullet\alpha \rightarrow \alpha & Ax12
\end{array}
$$

***Inference rule:***

$$\frac{\alpha \quad \alpha \rightarrow \beta}{\beta}\,\boldsymbol{MP}$$

**Definition 12 (Derivation).** *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$ be a set of formulas. A derivation of $\varphi$ from $\Gamma$ in $\mathbf{CG'_{3h}}$ is a finite sequence $\varphi_1, \ldots, \varphi_n$ of formulas in $\mathcal{L}_\Sigma$ such that $\varphi_n$ is $\varphi$ and, for every $1 \leq i \leq n$, the following holds:*

1. *$\varphi_i$ is a instance of an axiom schema of $\mathbf{CG'_{3h}}$, or*
2. *$\varphi_i \in \Gamma$, or*
3. *there exist $j, k$ such that $\varphi_k = \varphi_j \rightarrow \varphi_i$ (and so $\varphi_i$ follows from $\varphi_j$ and $\varphi_k$ by **MP**).*

*We say that $\varphi$ is derivable from $\Gamma$ in $\mathbf{CG'_{3h}}$, denoted by $\Gamma \vdash_{\mathbf{CG'_{3h}}} \varphi$, if there exists a derivation of $\varphi$ from $\Gamma$ in $\mathbf{CG'_{3h}}$.*

The following meta-theorems of $\mathbf{CG'_{3h}}$ will prove to be quite useful, their demonstrations are straightforward.

**Proposition 1.** *The calculus $\mathbf{CG'_{3h}}$ satisfies the following properties:*

*(i)* $\Gamma, \alpha \vdash_{\mathbf{CG'_{3h}}} \beta$ *iff* $\Gamma \vdash_{\mathbf{CG'_{3h}}} \alpha \to \beta$ *(Deduction meta-theorem, DMT).*
*(ii) If* $\Gamma, \alpha \vdash_{\mathbf{CG'_{3h}}} \varphi$ *and* $\Gamma, \beta \vdash_{\mathbf{CG'_{3h}}} \varphi$ *then* $\Gamma, \alpha \vee \beta \vdash_{\mathbf{CG'_{3h}}} \varphi$.
*(iii) If* $\Gamma, \alpha \vdash_{\mathbf{CG'_{3h}}} \varphi$ *and* $\Gamma, \neg\alpha \vdash_{\mathbf{CG'_{3h}}} \varphi$ *then* $\Gamma \vdash_{\mathbf{CG'_{3h}}} \varphi$ *(Proof-by-cases).*

*Proof.*

(i) To prove that a Hilbert calculus satisfies DMT, it suffices to derive axioms Ax1 and Ax2, while MP must be the unique inference rule.
(ii) The demonstration is straightforward by applying axiom Ax8 and **MP** twice.
(iii) It is a direct consequence of item (ii) and axiom Ax10.

**Definition 13 (Valuations for $\mathbf{CG'_3}$).** *A function $v : \mathcal{L}_\Sigma \to \{0,1,2\}$ is a valuation for $\mathbf{CG'_3}$, or a $\mathbf{CG'_3}$-valuation, if it satisfies the following clauses:*

$$
\begin{array}{lll}
-v(\neg\alpha) = 0 & \textit{when} & v(\alpha) = 2 \\
-v(\alpha \wedge \beta) \in \{1,2\} & \textit{iff} & v(\alpha) \in \{1,2\} \textit{ and } v(\beta) \in \{1,2\} \\
-v(\alpha \to \beta) \in \{1,2\} & \textit{iff} & v(\alpha) = 0 \textit{ or } v(\beta) = 2 \textit{ or} \\
 & & v(\alpha) = v(\beta) = 1 \textit{ or } v(\alpha) = 2, v(\beta) = 1
\end{array}
$$

The set of all such valuations will be designated by $V^{\mathbf{CG'_3}}$.

For every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$, the following semantical consequences relation w.r.t. the set $V^{\mathbf{CG'_3}}$ of $\mathbf{CG'_3}$-valuations can be defined:

$$\Gamma \models_{\mathbf{CG'_3}} \varphi \textit{ if and only if, for every } v \in V^{\mathbf{CG'_3}},$$

$$\textit{if } v(\gamma) \in \{1,2\} \textit{ for every } \gamma \in \Gamma \textit{ then } v(\varphi) \in \{1,2\}.$$

**Theorem 2 (Soundness).** *For every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$:*

$$\textit{If } \Gamma \vdash_{\mathbf{CG'_{3h}}} \varphi \textit{ then } \Gamma \models_{\mathbf{CG'_3}} \varphi.$$

*Proof.* It suffices to verify that each axiom schema is a tautology in $\mathbf{CG'_3}$ and if $\alpha$ and $\beta$ are formulas such that $v(\alpha)$, $v(\alpha \to \beta) \in \{1,2\}$ then $v(\beta) \in \{1,2\}$ i.e. **MP** preserves tautologies.

The proof of completeness needs some definitions and results related to Tarskian Logic, see definition 3.

**Definition 14 (Maximal set).** *For a given Tarskian logic $\mathscr{L}$ over the language $\mathcal{L}$, let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$. The set $\Gamma$ is maximal non-trivial with respect to $\varphi$ in $\mathscr{L}$ if $\Gamma \nvdash_{\mathscr{L}} \varphi$ but $\Gamma, \psi \vdash \varphi$ for any $\psi \notin \Gamma$.*

**Definition 15 (Closed theory).** *Let $\mathscr{L}$ be a Tarskian logic, $A$ be a set of formulas $\Gamma$ is closed in $\mathscr{L}$, or a closed theory of $\mathscr{L}$, if the following holds for every formula $\psi$; $\Gamma \vdash_{\mathscr{L}} \psi$ if and only if $\psi \in \Gamma$.*

**Lemma 2.** *Any set of formulas maximal non-trivial with respect to $\varphi$ in $\mathscr{L}$ is closed, provided that $\mathscr{L}$ is Tarskian.*

*Proof.* Straightforward from Definition 3 and Definition 14.

**Theorem 3 (Lindenbaum-Łos).** *Let $\mathscr{L}$ be a Tarskian and finitary logic over the language $\mathcal{L}$. Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$ be such that $\Gamma \nvdash_{\mathcal{L}} \varphi$. There exists then a set $\Delta$ such that $\Gamma \subseteq \Delta \subseteq \mathcal{L}$ with $\Delta$ maximal non-trivial with respect to $\varphi$ in $\mathscr{L}$.*

*Proof.* The demonstration can be found in [10, Theorem 22.2] and [1, Theorem 2.2.6].

$\mathbf{CG'_{3h}}$ is a Tarskian and finitary because $\mathbf{CG'_{3h}}$ is defined by a Hilbert calculus, so Theorem 3 holds. On the other hand, the following properties it holds:

**Lemma 3.** *If $\Delta$ is a maximal non-trivial set with respect to $\varphi \in \mathbf{CG'_{3h}}$, then for every formulas $\psi$ and $\gamma$, $\Delta$ satisfies the following properties:*

*(i)* $\Delta \vdash_{\mathbf{CG'_{3h}}} \psi$ *if and only if* $\psi \in \Delta$.
*(ii)* $(\psi \vee \gamma) \in \Delta$ *if and only if* $\psi \in \Delta$ *or* $\gamma \in \Delta$.
*(iii)* $(\psi \wedge \gamma) \in \Delta$ *if and only if* $\psi \in \Delta$ *and* $\gamma \in \Delta$.
*(iv)* $\psi \in \Delta$ *if and only if* $\neg\psi \notin \Delta$ *if and only if* $\neg\neg\psi \in \Delta$.

*Proof.* The proof of each item can be seen in [1].

**Proposition 2.** *The following formulas are theorems in $\mathbf{CG'_{3h}}$.*

*(i)* $\varphi \wedge \neg\varphi \leftrightarrow \bot$
*(ii)* $\neg \bullet \varphi \to \neg \bullet \neg\varphi$
*(iii)* $(\neg\neg\varphi \wedge \neg\gamma) \to \neg(\varphi \to \gamma)$
*(iv)* $(\bullet\varphi \wedge \neg \bullet \psi) \to \neg \bullet (\varphi \to \psi)$
*(v)* $(\neg \bullet \varphi \wedge \neg \bullet \psi) \to \neg \bullet (\varphi \to \psi)$
*(vi)* $(\varphi \wedge \psi) \to (\varphi \to \psi)$
*(vii)* $(\varphi \wedge \bullet\psi) \to \bullet(\varphi \to \psi)$
*(viii)* $(\neg\varphi \wedge \neg \bullet \varphi) \to (\varphi \to \psi)$
*(ix)* $\neg\varphi \to \neg \bullet (\varphi \to \psi)$
*(x)* $\neg \bullet \psi \to \neg \bullet (\varphi \to \psi)$
*(xi)* $(\bullet\varphi \wedge \bullet\psi) \to \neg \bullet (\varphi \to \psi)$
*(xii)* $\neg\varphi \to \neg(\varphi \wedge \psi)$
*(xiii)* $(\neg\varphi \wedge \neg \bullet \varphi) \to \neg \bullet (\varphi \wedge \psi)$
*(xiv)* $\neg\psi \to \neg(\varphi \wedge \psi)$
*(xv)* $(\neg\psi \wedge \neg \bullet \psi) \to \neg \bullet (\varphi \wedge \psi)$
*(xvi)* $(\bullet\varphi \wedge \bullet\psi) \to \bullet(\varphi \wedge \psi)$
*(xvii)* $(\bullet\varphi \wedge \psi \wedge \neg \bullet \psi) \to \bullet(\varphi \wedge \psi)$
*(xviii)* $(\varphi \wedge \neg \bullet \varphi \wedge \bullet\psi) \to \bullet(\varphi \wedge \psi)$

*(xix)* $(\neg \bullet \varphi \wedge \neg \bullet \psi) \to \neg \bullet (\varphi \wedge \psi)$

The proofs of each item in the Proposition 2 can be demonstrated using the axiom schemas and Modus Ponens.

**Lemma 4 (The truth lemma).** *Let $h : Var \to D$ be a homomorphism from $Var$ in $D$ such that for every propositional variable $p$*

$$h(p) = \begin{cases} 0 \textit{ iff } p \notin \Delta, \bullet p \notin \Delta \\ 1 \textit{ iff } p \in \Delta, \bullet p \in \Delta \\ 2 \textit{ iff } p \in \Delta, \bullet p \notin \Delta \end{cases}$$

*where $\Delta$ is a maximal non-trivial set with respect to $\varphi \in \mathbf{CG'_{3h}}$. Then for all $\alpha \in \mathcal{L}_\Sigma$ is verified:*

$$h(\alpha) = \begin{cases} 0 \textit{ iff } \alpha \notin \Delta, \bullet \alpha \notin \Delta \\ 1 \textit{ iff } \alpha \in \Delta, \bullet \alpha \in \Delta \\ 2 \textit{ iff } \alpha \in \Delta, \bullet \alpha \notin \Delta \end{cases}$$

*Proof.* Let $\alpha$ be a formula and let $v$ be a valuation in $\mathbf{CG'_{3h}}$. The proof is done by induction on the complexity of $\alpha$.

**Base case.** If $\alpha = p$, where $p$ is a propositional variable, then affirmation holds by definition.

**Induction hypothesis.** Assume that the statement is verified for each formula of complexity less than $\alpha$; that is, if $\beta$ is a formula that is less complex than $\alpha$, then it is true that:

$$\begin{array}{lll} h(\beta) = 0 & \textit{if and only if} & \beta \notin \Delta, \bullet\beta \notin \Delta \\ h(\beta) = 1 & \textit{if and only if} & \beta \in \Delta, \bullet\beta \in \Delta \\ h(\beta) = 2 & \textit{if and only if} & \beta \in \Delta, \bullet\beta \notin \Delta \end{array}$$

Note that it is sufficient to prove the "**only if**" part of the statement, since the three conditions on the right side are incompatible in pairs, also $h(\beta)$ can only take one of the following values $0, 1, 2$. For example, if the first condition on the right side of the statement holds for a $\beta$ formula, then the other two conditions are false and therefore $h(\beta) \notin \{1, 2\}$. Thus, $h(\beta)$ must be 0.

**Case1 negation.** Let $\alpha = \neg\beta$, for some formula $\beta$. We analyze three cases.

I. Assume that $h(\alpha) = 0$. Given that $h(\alpha) = h(\neg\beta) = \neg h(\beta)$, we have that $\neg h(\beta) = 0$. By the table of negation, $h(\beta) = 2$. Note that $\beta$ has less complexity than $\alpha$, then by induction hypothesis $\beta \in \Delta$ and $\bullet\beta \notin \Delta$. Given that $\beta \in \Delta$ by Lemma 3, $\neg\beta \notin \Delta$. So $\alpha \notin \Delta$. On the other hand, we have that $\bullet\beta \notin \Delta$ by Lemma 3, $\neg \bullet \beta \in \Delta$. By Proposition 2(ii) and **MP**, we conclude $\neg \bullet \neg\beta \in \Delta$.

II. Assume that $h(\alpha) = 1$. Note that, this case is not verified, there is no formula whose negation takes the value of 1.

III. Assume that $h(\alpha) = 2$. Given that $h(\alpha) = h(\neg\beta) = \neg h(\beta)$, we have that $\neg h(\beta) = 2$. By the table of negation, $h(\beta) = 0$. Note that $\beta$ has less complexity than $\alpha$, then by induction hypothesis $\beta \notin \Delta$ and $\bullet\beta \notin \Delta$. Given that $\beta \notin \Delta$ by Lemma 3, $\neg\beta \in \Delta$. So $\alpha \in \Delta$. On the other hand, we have that $\bullet\beta \notin \Delta$ by Lemma 3, $\neg \bullet \beta \in \Delta$. By Proposition 2(ii) and **MP**, we conclude $\neg \bullet \neg\beta \in \Delta$.

**Case2 implication.** Let $\alpha = \beta \to \gamma$, for some formulas $\beta$, $\gamma$. We analyze three cases.

I. Assume that $h(\alpha) = 0$. Given that $h(\alpha) = h(\beta \to \gamma) = h(\beta) \to h(\gamma)$, we have that $h(\beta) \to h(\gamma) = 0$. By the table of implication, we analyze two cases.
   (a) $h(\beta) = 1, h(\gamma) = 0$. Note that $\beta$ and $\gamma$ has less complexity than $\alpha$, then by induction hypothesis $\beta \in \Delta$, $\bullet\beta \in \Delta$ and $\gamma \notin \Delta$, $\bullet\gamma \notin \Delta$. Given that $\beta \in \Delta$ and $\gamma \notin \Delta$ by Lemma 3, $\neg\neg\beta \wedge \neg\gamma \in \Delta$, applying the Proposition 2(iii) and **MP** we get $\neg(\beta \to \gamma) \in \Delta$. On the other hand, we have $\bullet\beta \in \Delta$ and $\bullet\gamma \notin \Delta$ by Lemma 3 we have $\bullet\beta \wedge \neg \bullet \gamma \in \Delta$ applying the Proposition 2(iv) and **MP** we get $\neg \bullet (\beta \to \gamma) \in \Delta$, i.e. $\bullet(\beta \to \gamma) \notin \Delta$.
   (b) $h(\beta) = 2, h(\gamma) = 0$. Note that $\beta$ and $\gamma$ has less complexity than $\alpha$, then by induction hypothesis $\beta \in \Delta$, $\bullet\beta \notin \Delta$ and $\gamma \notin \Delta$, $\bullet\gamma \notin \Delta$. This case is similar to the previous one applying the Proposition 2(v).

II. Assume that $h(\alpha) = 1$. Given that $h(\alpha) = h(\beta \to \gamma) = h(\beta) \to h(\gamma)$, we have that $h(\beta) \to h(\gamma) = 1$. By the table of implication, we analyze one case.
   (a) $h(\beta) = 2, h(\gamma) = 1$. Note that $\beta$ and $\gamma$ has less complexity than $\alpha$, then by induction hypothesis $\beta \in \Delta$, $\bullet\beta \notin \Delta$ and $\gamma \in \Delta$, $\bullet\gamma \in \Delta$. Given that $\beta \in \Delta$ and $\gamma \in \Delta$ then by Lemma 3, $\beta \wedge \gamma \in \Delta$, applying Proposition 2(vi) and **MP** we conclude that $\beta \to \gamma \in \Delta$. On the other hand, we have that $\beta \in \Delta$ and $\bullet\gamma \in \Delta$ then $\beta \wedge \bullet\gamma \in \Delta$ by Lemma 3. Then applying Proposition 2(vii) and **MP** we obtain, $\bullet(\beta \to \gamma) \in \Delta$.

III. Assume that $h(\alpha) = 2$. Given that $h(\alpha) = h(\beta \to \gamma) = h(\beta) \to h(\gamma)$, we have that $h(\beta) \to h(\gamma) = 2$. By the table of implication, we analyze three cases.
   (a) $h(\beta) = 0$. Note that $\beta$ has less complexity than $\alpha$, then by induction hypothesis $\beta \notin \Delta$ and $\bullet\beta \notin \Delta$. Then $\neg\beta \wedge \neg \bullet \beta \in \Delta$. Then applying Proposition 2(viii) and **MP** we obtain, $(\beta \to \gamma) \in \Delta$, On the other hand, we have that $\neg\beta \in \Delta$, then applying Proposition 2(ix) and **MP** we obtain, $\neg \bullet (\beta \to \gamma) \in \Delta$ i.e. $\bullet(\beta \to \gamma) \notin \Delta$.
   (b) $h(\gamma) = 2$. Note that $\gamma$ has less complexity than $\alpha$, then by induction hypothesis $\gamma \in \Delta$, $\bullet\gamma \notin \Delta$. Note that $\gamma \in \Delta$, applying Ax1 and **MP** we obtain, $\neg(\beta \to \gamma) \in \Delta$. On the other hand, $\neg \bullet \gamma \in \Delta$, applying Proposition 2(x) and **MP** we obtain, $\neg\bullet(\beta \to \gamma) \in \Delta$ i.e. $\bullet(\beta \to \gamma) \notin \Delta$.

(c) $h(\beta) = 1, h(\gamma) = 1$. Note that $\beta$ and $\gamma$ has less complexity than $\alpha$, then by induction hypothesis $\beta \in \Delta$, $\bullet\beta \in \Delta$ and $\gamma \in \Delta$, $\bullet\gamma \in \Delta$. Given that $\beta \in \Delta$ and $\gamma \in \Delta$, then $\beta \wedge \gamma \in \Delta$. Applying Proposition 2(vi) and **MP** we obtain, $(\beta \to \gamma) \in \Delta$. On the other hand, $(\bullet\beta \wedge \bullet\gamma) \in \Delta$ and applying Proposition 2(xi) and **MP** we obtain, $\neg \bullet (\beta \to \gamma) \in \Delta$ i.e. $\bullet(\beta \to \gamma) \notin \Delta$.

**Case3 conjunction.** Let $\alpha = \beta \wedge \gamma$, for some formulas $\beta$, $\gamma$. We analyze three cases.

I. Assume that $h(\alpha) = 0$. Given that $h(\alpha) = h(\beta \wedge \gamma) = h(\beta) \wedge (\gamma)$, we have that $h(\beta) \wedge h(\gamma) = 0$. By the table of conjunction, we analyze two cases.
   (a) $h(\beta) = 0$. Note that $\beta$ has less complexity than $\alpha$, then by induction hypothesis $\beta \notin \Delta$ and $\bullet\beta \notin \Delta$. Given that $\beta \notin \Delta$, then $\neg\beta \in \Delta$, applying Proposition 2(xii) and **MP** we obtain, $\neg(\beta \wedge \gamma) \in \Delta$. Therefore, $(\beta \wedge \gamma) \notin \Delta$. On the other hand, given that $\beta \notin \Delta$ and $\bullet\beta \notin \Delta$, then $\neg\beta \wedge \neg \bullet \beta \in \Delta$, applying Proposition 2(xiii) and **MP** we obtain, $\neg \bullet (\beta \wedge \gamma) \in \Delta$
   (b) $h(\gamma) = 0$. Note that $\gamma$ has less complexity than $\alpha$, then by induction hypothesis $\gamma \notin \Delta$ and $\bullet\gamma \notin \Delta$. Given that $\gamma \notin \Delta$, then $\neg\gamma \in \Delta$. Applying Proposition 2(xiv) and **MP** we obtain, $\neg(\beta \wedge \gamma) \in \Delta$, so $(\beta \wedge \gamma) \in \Delta$. On the other hand, $\neg\gamma \in \Delta$ and $\bullet\gamma \notin \Delta$, then $\neg\gamma \wedge \neg \bullet \gamma \in \Delta$, applying Proposition 2(xv) and **MP** we obtain, $\neg \bullet (\beta \wedge \gamma) \in \Delta$. Therefore, $\bullet(\beta \wedge \gamma) \notin \Delta$.

II. Assume that $h(\alpha) = 1$. Given that $h(\alpha) = h(\beta \wedge \gamma) = h(\beta) \wedge h(\gamma)$, we have that $h(\beta) \wedge h(\gamma) = 1$. By the table of conjunction, we analyze three cases.
   (a) $h(\beta) = 1, h(\gamma) = 1$. Note that $\beta$ and $\gamma$ has less complexity than $\alpha$, then by induction hypothesis $\beta \in \Delta$, $\bullet\beta \in \Delta$ and $\gamma \in \Delta$, $\bullet\gamma \in \Delta$. Given that $\beta \in \Delta$ and $\gamma \in \Delta$ then $\beta \wedge \gamma \in \Delta$. On the other hand $\bullet\beta \in \Delta$ and $\bullet\gamma \in \Delta$, then $\bullet\beta \wedge \bullet\gamma \in \Delta$. Applying Proposition 2(xvi) and **MP** we obtain, $\bullet(\beta \wedge \gamma) \in \Delta$.
   (b) $h(\beta) = 1, h(\gamma) = 2$. Note that $\beta$ and $\gamma$ has less complexity than $\alpha$, then by induction hypothesis $\beta \in \Delta$, $\bullet\beta \in \Delta$ and $\gamma \in \Delta$, $\bullet\gamma \notin \Delta$. Given that $\beta \in \Delta$ and $\gamma \in \Delta$ then $\beta \wedge \gamma \in \Delta$. On the other hand, given that $\bullet\beta \in \Delta$, $\gamma \in \Delta$ and $\neg \bullet \gamma \in \Delta$ then $\bullet\beta \wedge \gamma \wedge \neg \bullet \gamma \in \Delta$. Applying Proposition 2(xvii) and **MP** we obtain, $\bullet(\beta \wedge \gamma) \in \Delta$.
   (c) $h(\beta) = 2, h(\gamma) = 1$. Note that $\beta$ and $\gamma$ has less complexity than $\alpha$, then by induction hypothesis $\beta \in \Delta$, $\bullet\beta \notin \Delta$ and $\gamma \in \Delta$, $\bullet\gamma \in \Delta$. This case is similar to the previous one applying the Proposition 2(xviii).

III. Assume that $h(\alpha) = 2$. Given that $h(\alpha) = h(\beta \wedge \gamma) = h(\beta) \wedge h(\gamma)$, we have that $h(\beta) \wedge h(\gamma) = 2$. By the table of conjunction, we analyze one case.
   (a) $h(\beta) = 2, h(\gamma) = 2$. Note that $\beta$ and $\gamma$ has less complexity than $\alpha$, then by induction hypothesis $\beta \in \Delta$, $\bullet\beta \notin \Delta$ and $\gamma \in \Delta$, $\bullet\gamma \notin \Delta$. Given that $\beta \in \Delta$ and $\gamma \in \Delta$ then $\beta \wedge \gamma \in \Delta$. On the other hand, $\neg \bullet \beta \in \Delta$ and $\neg \bullet \gamma \in \Delta$ then $\neg \bullet \beta \wedge \neg \bullet \gamma \in \Delta$. Applying Proposition 2(xix) and **MP** we obtain, $\neg \bullet (\beta \wedge \gamma) \in \Delta$. Hence $\bullet(\beta \wedge \gamma) \notin \Delta$.

**Theorem 4.** *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$, with $\Gamma$ maximal non-trivial with respect to $\varphi$ in $\mathbf{CG'_3}$. The mapping $v : \mathcal{L}_\Sigma \to \{0, 1, 2\}$ defined by:*

$$v(\psi) \in \{1, 2\} \textit{ if and only if } \psi \in \Gamma$$

*for all $\psi \in \mathcal{L}_\Sigma$, is a valuation for $\mathbf{CG'_3}$.*

*Proof.* The demonstration is straightforward. It suffices prove that $v$ satisfies all the clauses of Definition 13

The completeness of $\mathbf{CG'_{3h}}$ is then an immediate consequence of Theorem 4 and Theorem 3.

**Corollary 1 (Completeness of $\mathbf{CG'_{3h}}$).** *For every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$:*

$$\textit{If } \Gamma \models_{\mathbf{CG'_3}} \varphi \textit{ then } \Gamma \vdash_{\mathbf{CG'_{3h}}} \varphi.$$

*Proof.* Assume that $\Gamma \nvdash_{\mathbf{CG'_3}} \varphi$ by Theorem 3, let $\Delta$ be a maximal non-trivial set with respect to $\varphi$ in $\mathbf{CG'_3}$ extending $\Gamma$. By Theorem 4, there is an $\mathbf{CG'_3}$-valuation $v$, such that $v[\Gamma] \subseteq \{1, 2\}$ as $\Gamma \subseteq \Delta$, but $v(\varphi) = 0$ as $\varphi \notin \Delta$. Therefore, $\Gamma \not\models_{\mathbf{CG'_3}} \varphi$ and the theorem follows by contraposition.

## 5 Conclusions

The logic $\mathbf{CG'_3}$ was first developed by Osorio et al., in 2014 [5]. $\mathbf{CG'_3}$ is defined by its many-valued semantics the matrix of $\mathbf{CG'_3}$ logic is given by $M = \langle D, D^*, F\rangle$; where the domain is $D = \{0, 1, 2\}$ and the set of designated values is $D^* = \{1, 2\}$. This logic is a paraconsistent logic that can be viewed as an extension of the well-known logic $\mathbf{G'_3}$ also introduced by Osorio, in 2008 [6]. A Kripke-type semantics for $\mathbf{CG'_3}$ was later developed, by Borja et al., in 2016 [3]. Recently in 2019, Pérez-Gaspar et al. gave an axiomatization Hilbert type for $\mathbf{CG'_3}$ using the Kalmár technique [8]. In this paper, we extend studies on this logic by presenting some results relating deductive notions with its model-theoretic counterparts. We summarize results in this paper as follows.

- We developed a Hilbert-type axiomatization inspired by the Lindenbaum-Łos technique.
- Soundness is also proved.
- The main result of the paper is a completeness proof. Contrastingly with the proof using Kalmár's technique, this proof is based on maximal theories concerning a sentence.

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