Methodology to represent functions in logic $BL_{\supset}$

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Abstract. In 1996 Avron and Arieli present a four-valued logic with
its axiomatization, this logic is called $BL_{\supset}$, and is defined in terms of
four connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$. In this paper, we look over $BL_{\supset}$ and show
that it is a genuine paraconsistent logic, besides being paracomplete.
Furthermore, we show a methodology for representing functions through
a many-valued logic. In particular for expressing the Klein group, through
the logic $BL_{\supset}$.

Keywords: many-valued logics · genuine paraconsistent logic · Klein
group.

Introduction

Several authors ([4, 7, 8, 11] and [12]) agree in the definition of *paraconsistency*,
which accepts that a single contradiction should not imply every formula. For-
mally, this may be represented by $p, \neg p \nvdash q$ (where p and q are propositional
variables) [4]. In [6] the concept of genuine logic is presented as a logic that does
not obey either $p, \neg p \nvdash q$ or $\nvdash \neg(p \wedge \neg p)$.

Motivation

– In [10], da Costa and Krause point out some applications of paraconsistency
  logic into physics, particularly they talk about quantum theory:

> The possibility of using non-standard systems does not necessarily
> entail that classical logic is wrong, or that (in particular) quantum
> theory needs at the moment another logic. Physicists probably will
> continue to use classical (informal) logic in the near future. But
> we should realize that other forms of logic may help us in the
> better understanding of certain features of the quantum world as
> well, not easily treated by classical devices, as the concepts of
> complementarity and of non-individuality show [...]   [10, p.17].

- Graham Priest in a recent paper claims that using paraconsistent logic can be useful for treating the Gödel's First Incompleteness Theorem.

  > Much has been written about Gödel's First Incompleteness Theorem. Nearly always, this is on the assumption that the logic of the theory of arithmetic in question is classical –or at least intuitionistic– logic. Yet the use of a paraconsistent logic puts a distinctively new spin on matters [14].

- In 2015 Graham Priest shows a very interesting use of a 4-valued logic in Indian logic.

  - The catuṣkoṭi (*Greek*: tetralemma; *English*: four corners) is a venerable principle of Indian logic, which has been central to important aspects of reasoning in the Buddhist tradition [...]   [13, p.1].
  - [...] The four koṭis (corners) of the catuṣkoṭi are four options that one might take on a question. Given any question, there are four possibilities, *yes*, *no*, *both*, and *neither* [...]   [13, p.2].

- Arieli and Avron in [3] make some remarks in the advantage of using tetra-valued logics for artificial intelligence, for instance, they mention that:

  - In his well-known paper "How computer should think" Belnap (1977) argues that four-valued semantics is very suitable setting for computarized reasoning [...]   [3, p.97].
  - The main advantage of using $FOUR$ rather than three-valued systems is, of course, that it allows us to deal with both *types* of abnormal propositions in one system [...]   [3, p.125].
  - [...] This freedom to use more truth values does not add much; each one of the multi-valued logics considered here can actually be characterized by one of our four-valued logics.   [3, p.128].

- In modern algebra Klein group is studied, it is a group of cardinality 4, such that any element is its inverse. It is worth mentioning that there is not a group of three elements with such property (see section 1.3). It has applications in biology, specifically in molecular systems [1].

In this work, we deal with $BL_\supset$, one paraconsistent logic defined in [2]; it has four primitive connectives $\neg, \wedge, \vee, \rightarrow$. Our contributions are the following:

1. We provide a weak form of the substitution property for $BL_\supset$, we show that the regular substitution property does not hold in $BL_\supset$. See section 2, Theorem 2.
2. At the end of Section 2.1, we give a list of steps that allows us to express the operation of a group in terms of connectives of a logic, particularly in that section we show how to construct the operation of Klein group in terms of connectives of $BL_\supset$.
3. In Theorem 5, we show that every function $f : D^n \longrightarrow D$, where D is the set of values of $BL_\supset$ can be expressed in terms of a special set of functions.

1 Background

In this section, we present two of the more common ways of defining a logic and provide examples. In Section 1.1, we define a logic from the semantical point of view, particularly via many-valued systems. On the other hand, in Section 1.2, we give one axiomatic formal system for logic $BL_\supset$, [5, Section 4.2], which corresponds to the syntactical approach.

1.1 Many-valued logics

A way to define a logic is by truth values and interpretations. Many-valued systems generalize the idea of using the truth tables that are used to determine the validity of formulas in classical logic. Pioneers such as Łukasiewicz considered such many-valued systems as alternatives to the classical framework. As other authors do, we prefer to give to many-valued systems the benefit of the doubt about their status as logics.

The core of a many-valued system is its *domain* of values D, specifically D is a nonempty set of truth-values, where some of such values are special and identified as *designated*. Connectives (e.g. $\wedge$, $\vee$, $\rightarrow$, $\neg$) are then introduced as operators over D according to the particular definition of the logic. An *interpretation* is a function $I \colon \mathcal{L} \to D$ that maps atoms to elements in the domain, where $\mathcal{L}$ is the set formed by the atoms of language. The application of I is then extended to arbitrary formulas by mapping first the atoms to values in D, and then evaluating the resulting expression in terms of the connectives of the logic. A formula is said to be a *tautology* if, for every possible interpretation, the formula evaluates to a designated value. The most simple example of a many-valued logic is classical propositional logic where: $D = \{0, 1\}$, 1 is the unique designated value, and connectives are defined through the usual basic truth tables.

Not all many-valued logics must have the four connectives mentioned before, in fact, classical logic can be defined in terms of two of those connectives $\neg, \wedge$ (primitive connectives), and the other two (non-primitive) can be defined in terms of $\neg, \wedge$. In case of a logic having the implication connective, it should preserve tautologies, in the sense that if $x, x \to y$ are tautologies, then y is also a tautology. This restriction enforces the validity of Modus Ponens in the logic.

Since we will be working with several logics, we will use subindexes next to the connectives to specify to which logic they correspond, for example, $\neg_K$ corresponds to the connective $\neg$ of Kleene's logic. In those cases where the given logic is understood from the context, we drop such subindexes. Numbers 0, 1, 2 and 3 are part of the semantics of logic studied in this paper and were chosen only for convenience.

Kleene's 3-valued logic. The Kleene's 3-valued logic, denote here by K, is defined in [5]. Kleene's logic is a 3-valued logic with truth values in the domain $D = \{0, 1, 3\}$, where 3 is the only designated value[1]. Conjunction and disjunction are defined as the *min* and *max* functions respectively, namely $\alpha \wedge \beta = min(\alpha, \beta)$, and $\alpha \vee \beta = max(\alpha, \beta)$. The connectives $\rightarrow_K$ and $\neg_K$ are defined according to the tables given in Table 1. It is important to mention that in this paper we use the implication of Kleene as defined by Avron in [5, p.11].

$\rightarrow_K$	0	1	3
0	3	3	3
1	3	3	3
3	0	1	3

x	$\neg_K x$
0	3
1	1
3	0

Table 1. Truth tables of connectives $\rightarrow$ and $\neg$ in Kleene's logic.

The basic 3-valued paraconsistent logic PAC. We consider the domain of the logic PAC as $D = \{0, 2, 3\}$, this logic is a 3-valued paraconsistent logic with 2 and 3 as designated values[2]. The connectives $\neg$, $\wedge$ and $\vee$ have the same properties as those of the logic K. Table 2 shows the truth tables of connectives $\neg_{PAC}$ and $\rightarrow_{PAC}$ [5].

$\rightarrow_{PAC}$	0	2	3
0	3	3	3
2	0	2	3
3	0	2	3

x	$\neg_{PAC} x$
0	3
2	2
3	0

Table 2. Truth tables of connectives $\rightarrow$, $\neg$ in PAC.

Logic $BL_\supset$. This logic is a four-valued logic with truth values in the domain $D = \{0, 1, 2, 3\}$ where 2 and 3 are the designated values. The connectives $\wedge$ and $\vee$, as usually, correspond to the *greatest lower bound* (Glb) and the *least upper bound* (Lub), respectively. The connectives $\neg$ and $\rightarrow$ are defined according to the truth tables given in Table 3.

$\rightarrow_{BL_\supset}$	0	1	2	3
0	3	3	3	3
1	3	3	3	3
2	0	1	2	3
3	0	1	2	3

x	$\neg_{BL_\supset} x$
0	3
1	1
2	2
3	0

Table 3. Truth tables of connectives $\rightarrow$ and $\neg$ in logic $BL_\supset$.

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[1] The reason for considering this domain is that these values and the behavior of its connectives coincide with part of the logic $BL_\supset$.

[2] We take this domain with the purpose of PAC be a fragment of $BL_\supset$ logic.

The logic $BL_\supset$ is represented in Fig. 1, note that if we consider only the values 0, 1 and 3 (right part of figure 1) we obtain the Kleene's logic while if we take the values 0, 2 and 3 (left part of figure 1) we have the PAC logic.
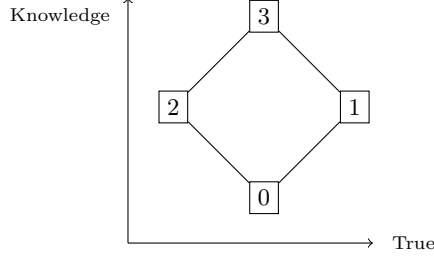

Fig. 1. Lattice logic $BL_\supset$.

As A. Avron mentions in [2], $BL_\supset$ is *interlaced* [3] and hence satisfies that $1 \wedge 2 = 0$ and $1 \vee 2 = 3$. As a consequence of this result, we can take $D = \{1, 2\}$. However, for simplicity we use $D = \{0, 1, 2, 3\}$ and from now on whenever we put D we will be referring to this set.

For Theorem 2 in Section 2 we need the following recursive substitution property definition (on α).

Definition 1. *[15] Let α, β be formulas we define the proposition obtained by replacing all occurrences of an atom p in α by β as follows:*

$$\alpha[\beta/p] = \begin{cases} \alpha & \text{if } \alpha \text{ atomic and } \alpha \neq p \\ \beta & \text{if } \alpha = p \end{cases}$$

$(\alpha_1 \square \alpha_2)[\beta/p] = \alpha_1[\beta/p] \square \alpha_2[\beta/p]$, where $\square$ is any of the binary connectives and α_1, α_2 any formulas.

$(\neg \alpha)[\beta/p] = \neg \alpha[\beta/p]$.

1.2 The system HBL

Let us consider HBL, a formal axiomatic theory for $BL_\supset$ [2] formed by the primitive logical connectives: $\neg, \rightarrow, \wedge$ and $\vee$, and the logical constants 0, 1, 2 and 3. We also consider one logical connective defined in terms of the primitive ones:

$$\alpha \leftrightarrow \beta := (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$$

the well-formed formulas are constructed, as usual, the axiom schemas are:

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[3] This means that each one of $\wedge, \vee, \otimes$ and $\oplus$ is monotonic with respect to both $\leq_t$ and $\leq_k$ [2, Definition 2.2, p.3]

I1	$\alpha \to (\beta \to \alpha)$
I2	$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
I3	$((\alpha \to \beta) \to \alpha) \to \alpha$
C1	$(\alpha \wedge \beta) \to \alpha$
C2	$(\alpha \wedge \beta) \to \beta$
C3	$\alpha \to (\beta \to (\alpha \wedge \beta))$
D1	$\alpha \to (\alpha \vee \beta)$
D2	$\beta \to (\alpha \vee \beta)$
D3	$(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$
N1	$\neg(\alpha \vee \beta) \leftrightarrow \neg\alpha \wedge \neg\beta$
N2	$\neg(\alpha \wedge \beta) \leftrightarrow \neg\alpha \vee \neg\beta$
N3	$\neg\neg\alpha \leftrightarrow \alpha$
N4	$\neg(\alpha \to \beta) \leftrightarrow \alpha \wedge \neg\beta$

and as the only inference rule: Modus Ponens

$$\frac{\alpha \quad \alpha \to \beta}{\beta}$$

Logic $BL_\supset$ is soundness and completeness with respect to this axiomatization.

Theorem 1. *[2, Section 3][Sound and Complete]*

$$\Gamma \vdash_{BL_\supset} \alpha \ \text{if and only if} \ \Gamma \models_{BL_\supset} \alpha.$$

1.3 Klein group

In group theory, the Klein group $\langle G, \cdot \rangle$ has several representations (in [9] we can see some of its representations), this group is formed by four elements $G = \{0, 1, 2, 3\}$, where each element is its own inverse and 0 is the neutral element. Table 4 shows the product of the elements of G.

$\cdot$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 4. Product of G.

One more motivation for studying $BL_\supset$ logic is that it is very close to Klein group since as we can see in Theorem 4, the operation given in Table 4 can be expressed in terms of functions in logic $BL_\supset$. On the other hand, the applications of G are varied, as we mention in the Introduction, due to their characteristic property that each element is its own inverse. It is worth mentioning that it is not possible to build a group of three elements with such property, to see this fact, suppose that there is a group of 3 elements where each element is its own inverse; under this assumption it is possible to prove that the group has only two elements, this put in evidence the importance of extending the 3-valued logics to the 4-valued logics.

2 Some results about the representation of functions in $BL_\supset$ logic

In this section, we present some results on $BL_\supset$ logic. But, above all, we focus our study on the presentation of a methodology for representing functions, from D^n to D.

Remark 1.

a) Logic $BL_\supset$ satisfies *Modus Ponens*. Indeed, if $I(\alpha) \in \{2,3\}$ and $\models \alpha \to \beta$, then $I(\beta) \in \{2,3\}$.

b) Any logic that has axioms $I1$, $I2$ and Modus Ponens as the only inference rule satisfies the hypothetical syllogism, i.e. $\alpha \to \beta$, $\beta \to \theta \vdash \alpha \to \theta$. Particularly in $BL_\supset$, the hypothetical syllogism is satisfied.

c) The deduction theorem is satisfied in $BL_\supset$, this means that if $\Gamma, \alpha \models_{BL_\supset} \beta$, then $\Gamma \models_{BL_\supset} \alpha \to \beta$.

d) The connective $\leftrightarrow_{BL_\supset}$ introduced at the beginning of Section 1.2 defines an equivalence relation among formulas. It is reflexive, symmetric and transitive.

e) $BL_\supset$ is a genuine paraconsistent logic, namely, there are formulas $\alpha, \beta \in BL_\supset$ such that $\alpha, \neg\alpha \not\models_{BL_\supset} \beta$ and $\not\models_{BL_\supset} \neg(\alpha \wedge \neg\alpha)$. For verifying the first one $\alpha, \neg\alpha \not\models_{BL_\supset} \beta$, it is enough to consider the value of 2 for α and 1 for β. For $\not\models_{BL_\supset} \neg(\alpha \wedge \neg\alpha)$ consider the value 1.

f) $BL_\supset$ is a paracomplete logic, this means that there exist some formula α in $BL_\supset$ such that $\not\models_{BL_\supset} \alpha \vee \neg\alpha$, just consider the value 1.

Definition 2. *We say that the binary connective* **OP** *has the substitution property if by substituting equivalent parts we obtain equivalent propositions, this is if* $\models \alpha_1$ **OP** α_2, *then* $\models \beta[\alpha_1/p]$ **OP** $\beta[\alpha_2/p]$, *where p is an atom.*

The connective $\leftrightarrow$ of logic $BL_\supset$ does not satisfy the substitution property. Consider for instance $\alpha_1 = \gamma \to \gamma$ and $\alpha_2 = \theta \to \theta$. Note that in this case $\models \alpha_1 \leftrightarrow \alpha_2$ ($\models (\gamma \to \gamma) \leftrightarrow (\theta \to \theta)$). To verify this, let I be any interpretation, then we have two cases:

1. $I(\theta) \neq 2$ y $I(\gamma) \neq 2$, in this situation, $I(\alpha_1 \leftrightarrow \alpha_2) = 3$.
2. $I(\theta) = 2$, o $I(\gamma) = 2$ then $I(\alpha_1 \leftrightarrow \alpha_2) = 2$.

In both cases $\alpha_1 \leftrightarrow \alpha_2$ evaluates designated, therefore $\models \alpha_1 \leftrightarrow \alpha_2$. On the other hand $\not\models \beta[\alpha_1/p] \leftrightarrow \beta[\alpha_2/p]$. To see this, consider $\beta = \neg p$, $I(\gamma) = 2$ and $I(\theta) = 3$, we have that $I(\neg(\gamma \to \gamma) \leftrightarrow \neg(\theta \to \theta)) = 0$. Therefore, $\not\models \neg(\gamma \to \gamma) \leftrightarrow \neg(\theta \to \theta)$.

Definition 3. *Let α, β be formulas of $BL_\supset$, we define the connective $\Leftrightarrow$ as follows:* $\alpha \Leftrightarrow \beta := (\alpha \leftrightarrow \beta) \wedge (\neg\alpha \leftrightarrow \neg\beta)$

Remark 2. It is not difficult to see that $\alpha \Leftrightarrow \beta$ is a tautology if and only if α and β have the same truth value.

Theorem 2 (Substitution). *If $\models \alpha_1 \Leftrightarrow \alpha_2$, then $\models \theta[\alpha_1/p] \Leftrightarrow \theta[\alpha_2/p]$, where p is an atom.*

Sketch of proof. The proof is done by induction on the length of θ.
1. The atomic case is identical to the version in classical logic.
2. $\theta = \neg\theta_1$ immediately by the definition of $\leftrightarrow$.
3. $\theta = \theta_1 \wedge \theta_2$ it is derived from positive logic and De Morgan's laws.
4. $\theta = \theta_1 \rightarrow \theta_2$ is obtained thanks to the axiom $\neg(\alpha \rightarrow \beta) \leftrightarrow \alpha \wedge \neg\beta$.
5. $\theta = \theta_1 \vee \theta_2$ it is derived from case 2, 3 and De Morgan's laws.

The following result shows that if α is a semantical consequence of Γ in $BL_\supset$, then it is also a semantical consequence in Kleene and PAC.

Theorem 3. *[15] If $\Gamma \models_{BL_\supset} \alpha$, then $\Gamma \models_K \alpha$ and $\Gamma \models_{PAC} \alpha$.*

Proof. We proceed by contrapositive. Assume that $\Gamma \nvDash_K \alpha$, then exist a model M of Γ such that $M(\alpha) = 0$ or $M(\alpha) = 1$. Since 0 and 1 are not designated in $BL_\supset$ we have to $\Gamma \nvDash_{BL_\supset} \alpha$. Similarly, if we assume that $\Gamma \nvDash_{PAC} \alpha$, then exist a model M of Γ such that $M(\alpha) = 0$ which is not designated in $BL_\supset$. Therefore $\Gamma \nvDash_{BL_\supset} \alpha$.

The reciprocal of the previous theorem is not fulfilled. Consider the following counter-example. Let $\Gamma = \{p, \neg p\}$ and $\alpha = q \vee \neg q$. Note that there is no truth value assignment in Kleene that makes Γ true. Then $\Gamma \models_K \alpha$ by vacuity. On the other hand, the only truth value that models Γ in PAC is 2 and since $q \vee \neg q$ is a tautology in PAC we have to $\Gamma \models_{PAC} \alpha$. Finally, in $BL_\supset$ 2 is the only values that models Γ but α is not a tautology, just consider the truth value 1 for q, so also α worth 1 that is not a designated value. Therefore, $\Gamma \nvDash_{BL_\supset} \alpha$.

2.1 Expressing the operation of Klein group by means of function in logic $BL_\supset$

To express the operation of Klein group in $BL_\supset$, we introduce some special functions as follows.

Definition 4. *For all $x \in \{0, 1, 2, 3\}$ we define the connectives:*

$$t(x) := x \wedge (\neg x \rightarrow 0)$$
$$f(x) := \neg x \wedge (x \rightarrow 0)$$
$$\bot(x) := (\neg x \rightarrow 0) \wedge (x \rightarrow 0)$$

The reader can easily verify the behavior of these connectives, we use them to define functions that identify precisely each one of the values of $BL_\supset$, it means that, with the help of the connectives t, f and $\bot$ of Definition 4, we define four functions that give 3 for a specific value of x and 0 in another case.

Definition 5. *Let $g_i : D \longrightarrow D$, $i \in \{0, 1, 2, 3\}$ given by:*

- $g_0(x) = \neg\bot(x) \wedge f(x)$

- $g_1(x) = \bot(x)$
- $g_2(x) = \neg f(x) \wedge \neg(\neg x \to 0)$
- $g_3(x) = t(x) \wedge \neg(x \to 0)$

The truth table of each one of the functions g_i, $i \in \{0, 1, 2, 3\}$ is presented in Table 2.1. Note that for each $i \in \{0, 1, 2, 3\}$ and $x \in D$:

$$g_i(x) = \begin{cases} 3 & x = i \\ 0 & x \neq i \end{cases}$$

x	$g_0(x)$
0	3
1	0
2	0
3	0

x	$g_1(x)$
0	0
1	3
2	0
3	0

x	$g_2(x)$
0	0
1	0
2	3
3	0

x	$g_3(x)$
0	0
1	0
2	0
3	3

Table 5. Functions g_0, g_1, g_2 and g_3.

Definition 6. *Let* $f_{ij} : D \times D \longrightarrow D$, i, $j \in \{0, 1, 2, 3\}$ *be the function given by:*

$$f_{ij}(x, y) = g_i(x) \wedge g_j(y) = \begin{cases} 3 & x = i, \ y = j \\ 0 & in\ other\ case \end{cases}$$

The functions f_{ij} allow us to place 3 in any position in Table 4. If we want to have any other value $0, 1$ or 2, it is enough to take the conjunction of this function f_{ij} with a function that evaluates the desired value.

The final representation of G

The main objective is to build a two-variable function that models the product of G. Note that as we have mentioned, the functions f_{ij} of Definition 6 can be redefined to obtain any entry from Table 4 as shown below.

Definition 7. *Let* $f_{ij} : D \times D \longrightarrow D$, i, j, $k \in \{0, 1, 2, 3\}$ *be function given by:*

$$f_{ij}^k(x, y) = g_i(x) \wedge g_j(y) \wedge k = \begin{cases} k & x = i, \ y = j \\ 0 & in\ other\ case \end{cases}$$

To have better control, we introduce functions that allow us to represent each one of the lines of the product of G.

Definition 8. *Let* $h_i : D \times D \longrightarrow D$, $i \in \{1, 2, 3, 4\}$ *be function given by:*

$$h_1(x,y) = f_{00}^0(x,y) \vee f_{01}^1(x,y) \vee f_{02}^2(x,y) \vee f_{03}^3(x,y)$$
$$h_2(x,y) = f_{10}^1(x,y) \vee f_{11}^0(x,y) \vee f_{12}^3(x,y) \vee f_{13}^2(x,y)$$
$$h_3(x,y) = f_{20}^2(x,y) \vee f_{21}^3(x,y) \vee f_{22}^0(x,y) \vee f_{23}^1(x,y)$$
$$h_4(x,y) = f_{30}^3(x,y) \vee f_{31}^2(x,y) \vee f_{32}^1(x,y) \vee f_{33}^0(x,y)$$

Each h_i, $i \in \{1,2,3,4\}$ provides the i-th row of Table 4 and in the rest of entries gives 0 as shown in Table 6.

$h_1(x,y)$	0 1 2 3
0	0 1 2 3
1	0 0 0 0
2	0 0 0 0
3	0 0 0 0

$h_2(x,y)$	0 1 2 3
0	0 0 0 0
1	1 0 3 2
2	0 0 0 0
3	0 0 0 0

$h_3(x,y)$	0 1 2 3
0	0 0 0 0
1	0 0 0 0
2	2 3 0 1
3	0 0 0 0

$h_4(x,y)$	0 1 2 3
0	0 0 0 0
1	0 0 0 0
2	0 0 0 0
3	3 2 1 0

Table 6. Functions h_1, h_2, h_3, h_4.

Finally, for all x, $y \in D$ the product of group is expressed by the function

$$\phi(x,y) = h_1(x,y) \vee h_2(x,y) \vee h_3(x,y) \vee h_4(x,y). \tag{1}$$

As a summary, we describe in a general way the process of representing the operation of the Klein group.

1. Construction of unary functions g_i's which provide 3 for a specific value and 0 otherwise.
2. With the help of the functions g_i's, we build the functions of two variables f_{ij} to place 3 in a specific position of Table 4, and to place 0 in another case. Furthermore, we generalize these functions to obtain the exact value in each product entry of G, this with the functions f_{ij}^k.
3. Using the appropriate functions f_{ij}^k we express the operation of G.

Theorem 4. *There is a function of two variables given in the expression (1) such that it represents the Klein group.*

Proof. It follows immediately by the previous construction.

Remark 3. Functions f_{ij}^k can be generalized to functions of n-variables. This is:

$$f_{i_1,...,i_n}^k(x_1,...,x_n) = \begin{cases} k & x_1 = i_1,\ x_2 = i_2,...,\ x_n = i_n \\ 0 & \text{in other case.} \end{cases}$$

Theorem 4 can be generalized and its proof is a generalization of the previous construction. But, before to do it, consider the next example. Let $B = \{(0,0,0),(1,1,1),(0,2,0),(0,3,1)\} \subset D^3$, $f : B \longrightarrow D$ be the function:

$$f(x_1, x_2, x_3) = \begin{cases} 0 & \text{if } x_1 = x_2 = x_3 \\ 2 & \text{if } x_1 = 0 \text{ y } x_2 \neq 0 \end{cases}$$

Now let's express f in terms of f_{ij}^k. We need to calculate $f(0,0,0)$, $f(1,1,1)$, $f(0,2,0)$ and $f(0,3,1)$, due to space limitations, here we just calculate $f(1,1,1)$ and $f(0,2,0)$, the two remaining terms can be calculated analogously.

$f(1,1,1) = 0 = f_{1,1,1}^0(1,1,1) = f_{i_1^2,i_2^2,i_3^2}^{k_2}(1,1,1)$, where $i_1^2 = i_2^2 = i_3^2 = 1$ and $k_2 = 0$.

$f(0,2,0) = 2 = f_{0,2,0}^2(0,2,0) = f_{i_1^3,i_2^3,i_3^3}^{k_3}(0,2,0)$, where $i_1^3 = i_3^3 = 0$, $i_2^3 = 2$ and $k_3 = 2$.

Therefore,
$$f(x_1, x_2, x_3) = f_{i_1^1,i_2^1,i_3^1}^{k_1}(x_1,x_2,x_3) \vee f_{i_1^2,i_2^2,i_3^2}^{k_2}(x_1,x_2,x_3) \vee f_{i_1^3,i_2^3,i_3^3}^{k_3}(x_1,x_2,x_3)$$
$$\vee f_{i_1^4,i_2^4,i_3^4}^{k_4}(x_1,x_2,x_3)$$
$$= \bigvee^4 f_{i_1^l,i_2^l,i_3^l}^{k_l}(x_1,x_2,x_3)$$

Theorem 5. *Every function $f : D^n \longrightarrow D$, can be expressed in terms of the functions $f_{i_1,...,i_n}^k$. Furthermore,*

$$f(x_1,...,x_n) = \bigvee_{l=1}^{m} f_{i_1^l,...,i_n^l}^{k_l}(x_1,...,x_n);$$

for some $m \in \mathbb{N}$.

3 Conclusions and future work

We have provided a methodology to represent functions from D^n to D using the connectives of $BL_\supset$. In particular, we express the binary operation that defines the Klein group. As future work, we could consider at least to lines of research.

First, explore the possibility to express answer set programming (without strong negation) by using four valued logic instead of the actual three-valued Gödel logic. That goal seems to be direct. Then, by taking advantage of the four values, one should be able to express new useful operators.

Second, consider extending the logic Four to fifth-valued logic, in a way that the new valued could represent the notion of ineffability with potential applications in semi-automatic constructions of complex literature.

4 Acknowledgment

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