Approximation of parametrically given polyhedral sets

Oleg V. Khamisov
Melentiev Energy Systems Institute
Irkutsk, Russia
khamisov@isem.irk.ru

Abstract

In our paper we consider systems of linear inequalities which linearly depend on multidimensional parameters. A technique for approximating set of parameters for which the considered system is consistent is described. Approximations are constructed by means of convex and concave piece-wise linear functions. An illustrative example is given.

1 Intriduction

Parametric polyhedral set $D(p)$ is defined in the following way

$$D(p) = \{x \in X : g_i(x, p) \leq 0,\ i = 1, \dots, m\}, \tag{1}$$

where $X \subset R^n$ is a polytope ($X \neq \emptyset$), $g_i : R^n \times R^q,\ i = 1, \dots, m$ are bilinear functions, i.e. each function is linear in variable x when variable p is fixed and vise versa. Vector p is called a parameter and may vary within a given polytope $P \subset R^q,\ p \in P$. Define set

$$P^* = \{p \in P : D(p) \neq \emptyset\}.$$

In general, P^* is a nonconvex, disconnected and implicitly given set . We consider the following problem: find an outer P^*_{out} and an inner P^*_{in} explicit approximations of P^*:

$$P^*_{in} \subset P^* \subset P^*_{out}.$$

Similar problem were considered in [BorrelliEtAl03] and [JonesEtAl08] under some additional conditions which allow one to effectively apply the well elaborated linear programming parametric technique. Interval linear optimization [FiedlerEtAl06] is also tightly connected to the topic of our paper. However, here we suggest to use approximations generated by support function-majorants and function minorants [Khamisov99].

2 Approximation technique

Consider the following function

$$w(p) = \min_{x \in X} \max_{1 \leq i \leq m} g_i(x, p).$$

Then set P^* has the equivalent description

$$P^* = \{p \in P : w(p) \leq 0\}.$$

In: Sergei S. Goncharov, Yuri G. Evtushenko (eds.): Proceedings of the Workshop on Applied Mathematics and Fundamental Computer Science 2020 , Omsk, Russia, 30-04-2020, published at http://ceur-ws.org

Let $\tilde{p} \in P$ be given, find

$$\tilde{x} \in \operatorname*{Argmin}_{x \in X} \max_{1 \leq i \leq m} g_i(x, \tilde{p}) \tag{2}$$

and define function

$$\varphi(p, \tilde{p}) = \max_{1 \leq i \leq m} g_i(\tilde{x}, p). \tag{3}$$

Since functions g_i are liner in p function φ is a convex piece-wise linear function which satisfies two conditions

$$w(\tilde{p}) = \varphi(\tilde{p}, \tilde{p}), \;\; w(p) \leq \varphi(p, \tilde{p}) \;\; \forall p \in P. \tag{4}$$

Due to these conditions function φ is called a support function-majorant of function w. Rewrite now function w in the following way

$$w(p) = \min_{x \in X} \max_{1 \leq i \leq m} g_i(x, p) = \min_{x \in X} \max_{u \in S_u} \sum_{i=1}^{m} u_i g_i(x, p) = \max_{u \in S_u} \min_{x \in X} \sum_{i=1}^{m} u_i g_i(x, p), \tag{5}$$

where S_u is the standard simplex, $S_u = \left\{ u \in R^m : \sum_{i=1}^{m} u_i = 1, \; u_i \geq 0, \; i = 1, \ldots, m \right\}$. Let again $\tilde{p} \in P$ be given. Solve the corresponding max-min problem in (5) and find $\tilde{x}$ and $\tilde{u}$ such that

$$w(\tilde{p}) = \sum_{i=1}^{m} \tilde{u}_i g_i(\tilde{x}, \tilde{p}). \tag{6}$$

Define function

$$\psi(p, \tilde{p}) = \min_{x \in X} \sum_{i=1}^{m} \tilde{u}_i g_i(x, p). \tag{7}$$

Then, from (5) and (6) we have

$$w(\tilde{p}) = \psi(\tilde{p}, \tilde{p}), \;\; w(p) \geq \psi(p, \tilde{p}) \;\; \forall p \in P. \tag{8}$$

By construction function $\psi(\cdot, \tilde{p})$ is concave and due to the properties (8) is called a support function-minorant of function w.

Assume, that $w(\tilde{p}) \leq 0$ and define set

$$P_{in}(\tilde{p}) = \{p \in P : \varphi(p, \tilde{p}) \leq 0\}. \tag{9}$$

From (4) we have $w(p) \leq 0 \;\; \forall p \in P_{in}(\tilde{p})$. By construction $P_{in}(\tilde{p})$ is a convex polyhedral set. Therefore, $P_{in}(\tilde{p})$ is a convex polyhedral *inner* approximation of P^*_{in}. In the case of $w(\tilde{p}) > 0$ we define set

$$P_{\emptyset}(\tilde{p}) = \{p \in P : \psi(p, \tilde{p}) > 0\}. \tag{10}$$

From (8) we have $w(p) > 0 \;\; \forall p \in P_{\emptyset}(\tilde{p})$. It follows from the concavity of $\psi(\cdot, \tilde{p})$ that $P_{\emptyset}(\tilde{p})$ is a convex set. Then the set $P_{out}(\tilde{p}) = P \setminus P_{\emptyset}(\tilde{p})$ is an *outer* approximation of P^*_{out}.

It follows from (3) and (9) that set $P_{in}(\tilde{p})$ has explicit description as the polyhedron

$$P_{in}(\tilde{p}) = \{p \in P : g_i(\tilde{x}, p) \leq 0, \; i = 1, \ldots, m\}. \tag{11}$$

The set $P_{out}(\tilde{p})$ is the complement of convex set $P_{\emptyset}(\tilde{p})$. Therefore, $P_{out}(\tilde{p})$ is nonconvex, can be disconnected and has a disjunctive structure [Balas18]. Assume, that vertices $v^1, \ldots, v^N$ of X are known, $X = conv\{v^1, \ldots, v^N\}$. Then, from (7) we have

$$\psi(p, \tilde{p}) = \min_{1 \leq j \leq N} \sum_{i=1}^{m} \tilde{u}_i g_i(v^j, p) \tag{12}$$

and

$$P_{\emptyset}(\tilde{p}) = \{p \in P : \sum_{i=1}^{m} \tilde{u}_i g_i(v^j, p) > 0, \; j = 1, \ldots, N\}. \tag{13}$$

Define sets

$$P_j(\tilde{p}) = \{p \in P : \sum_{i=1}^{m} \tilde{u}_i g_i(v^j, p) \le 0\},\ j = 1, \dots, N.$$

Then

$$P_{out}(\tilde{p}) = \bigcup_{j=1}^{N} P_j(\tilde{p}), \tag{14}$$

i.e. $P_{out}(\tilde{p})$ is a union of polyhedrons. Find scalars $\tilde{\gamma}_j$:

$$\sum_{i=1}^{m} \tilde{u}_i g_i(v^j, p) \le \tilde{\gamma}_j \ \ \forall p \in P,\ j = 1, \dots, N.$$

It is well-known, that by introducing 0-1 variables $z_j,\ j = 1, \dots, N$ the disjunctive structure of $P_{out}(\tilde{p})$ in (14) can be described as follows

$$P_{out}(\tilde{p}) = \left\{ p \in P : \exists z : \sum_{i=1}^{m} \tilde{u}_i g_i(v^j, p) \le \tilde{\gamma}_j z_j,\ z_j = 0 \bigvee 1,\ \ j = 1, \dots, N,\ \sum_{j=1}^{N} z_j = (N-1) \right\}.$$

We always can assume that X is a simplex. Since X is bounded it has an outer approximation by a simplex and since X is defined by a system of linear inequalities we can move all inequalities into the list of new functions g_i in (1). In this case new functions do not have parameter p, however the suggested approach is still correct.

Let us consider how $\tilde{x}$ and $\tilde{u}$ corresponding to a given $\tilde{p}$ can be obtained. Rewrite problem in (2) in the following way

$$\min_{x,\xi} \{\xi : g_i(x, \tilde{p}) \le \xi,\ i = 1, \dots, m,\ x \in X\}, \tag{15}$$

where ξ is an auxiliary unbounded scalar variable. Problem (15) is a linear programming problem which always has a finite solution. Let $(\tilde{x}, \tilde{\xi})$ be a solution. Then, obviously, $\tilde{x}$ satisfies inclusion (2) and $w(\tilde{p}) = \tilde{\xi}$. Write down the Lagrange function

$$L(\xi, x, u) = \xi + \sum_{i=1}^{m} u_i (g_i(x, \tilde{p}) - \xi).$$

The Lagrange dual problem is

$$\max_{u \ge 0} \min_{x \in X, \xi \in R} L(\xi, x, u) = \max_{u \ge 0} \min_{x \in X, \xi \in R} \left\{ \xi \left(1 - \sum_{i=1}^{m} u_i \right) + \sum_{i=1}^{m} u_i g_i(x, \tilde{p}) \right\} = \max_{u \in S_u} \min_{x \in X} \sum_{i=1}^{m} u_i g_i(x, \tilde{p}),$$

i.e. $\tilde{u}$ introduced in (6) is a dual solution of (15). Note also, that (15) is a linear programming problem and hence can be easily solved for any given $\tilde{p} \in P$.

Let us make some intermediate conclusions. For any arbitrary given $\tilde{p} \in P$ we solve linear programming problem (15) obtaining primal solution $(\tilde{x}, \tilde{\xi})$ and dual solution $\tilde{u}$. If $\tilde{\xi} \le 0$ then we know that system (1) is consistent not only for $p = \tilde{p}$ but also $\forall p \in P_{in}(\tilde{p})$, where $P_{in}(\tilde{p})$ is given in (11). If $\tilde{\xi} > 0$ then system (1) is inconsistent not only for $p = \tilde{p}$ but also $\forall p \in P_{\emptyset}(\tilde{p})$ in (10). The main result here consists in the following: checking a given parameter for feasibility we obtain a set with the same property.

Example. Sets $P = [-0.2, 1.3]$, $X = \{(x_1, x_2) : -5 \le x_j \le 5, j = 1, 2\}$. System (1) is defined by the following inequalities

$$g_1(x_1, x_2, p) = 5px_1 + 10x_2 + 2p - 10 \le 0,$$

$$g_2(x_1, x_2, p) = -2x_1 - 3px + 2 - 5p + 10.5 \le 0.$$

Set P^* is union of three intervals, $P^* = [-0.2, -0.05] \bigcup [0.075.0.94] \bigcup [1.235, 1.3]$. Function w and set P^* are shown in Fig. 1. In this example we check three values of parameter for feasibility and construct the corresponding sets.

First parameter $p^1 = 0.01$. Solving problem (15) with $\tilde{p} = p^1$ we obtain the corresponding primal solution $x^1 = (5, 1.015), \xi^1 = 0.4196$, and dual solution $u^1 = (0.003, 0.997)$. Since $w(p^1) = \xi^1 > 0$ we have $D(p^1) = \emptyset$. Set

X has four vertices $v^1 = (-5,-5),\ v^2 = (-5,5),\ v^3 = (5,-5),\ v^4 = (5,5)$. Support function-minorant of w has the following form (see (12))

$$\psi(p,p^1) = \min\{9.901p + 20.2585, -20.009p + 20.5585, 10.051p + 0.3185, -19.859p + 0.6185\} =$$

$$= \min\{10.051p + 0.3185, -19.859p + 0.6185\} \ \ \forall p \in [-0.2, 1.3].$$

Set $P_{\emptyset}(p^1) = \{p : \psi(p,p^1) > 0\}$ is open interval (-0.0317,0.0311). Therefore, $D(p) = \emptyset \ \forall p \in (-0.0317, 0.0311)$. Hence $P_{out}(p^1) = P \setminus P_{\emptyset}(p^1) = [-0.2, -0.0317] \bigcup [0.0311, 1.3]$. See Fig 1 for geometrical interpretation of function $\psi(\cdot, p^1)$ and set $P_{\emptyset}(p^1)$.

Take now the second value of the parameter, $p^2 = 0.6$. The corresponding problem (15) ($\tilde{p} = p^2$) has solutions $x^2 = (5, -0.737),\ \xi^2 = -1.1729,\ u^2 = (0.153, 0.847)$. Since $\xi^2 < 0$ set $D(p^2) \neq \emptyset$. From (3) we obtain

$$\varphi(p,p^2) = \max\{27p - 17.37, -2.789p + 0.5\},$$

$P_{in}(p^2) = \{p : \varphi(p,p2) \leq 0\} = [0.179, 0.643]$ and $D(p) \neq \emptyset \ \forall p \in [0.179, 0.643]$.

Third parameter $p^3 = 1.1$. The corresponding primal and dual solutions $x^3 = (5, -1.857),\ \xi^3 = 1.1286,\ u^3 = (0.248, 0.752)$. In this case $\xi^3 > 0$ and $D(p^3) = \emptyset$. Support function-minorant

$$\psi(p,p^3) = \min\{14.216p - 14.504, -8.344p + 10.296, -20.744p + 25.336\}.$$

Set $P_{\emptyset}(p^3) = \{p : \psi(p,p^3) > 0\}$ is interval $(1.02, 1.22)$.

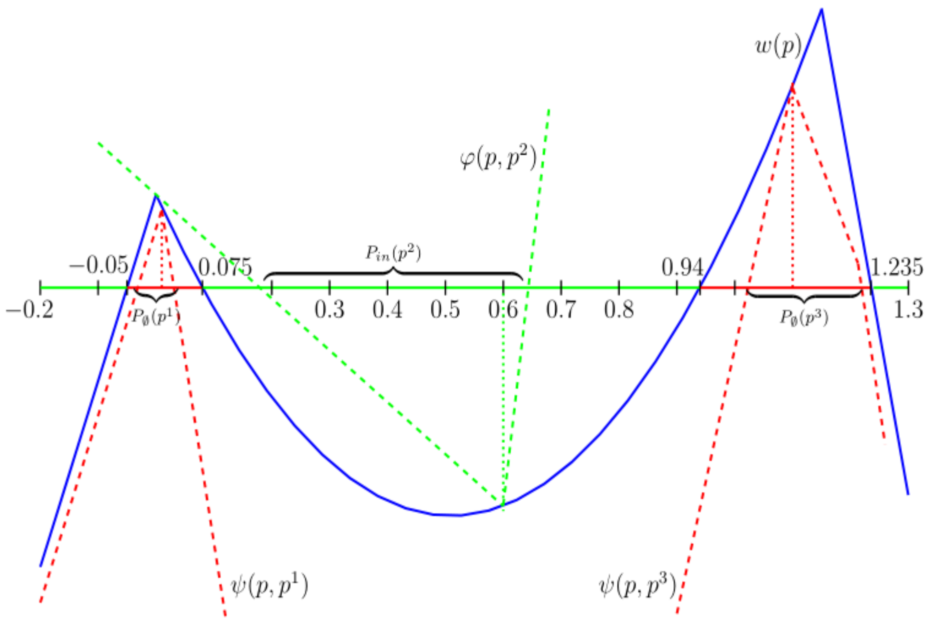

Figure 1: Geometrical interpretation of the Example.

3 An approximation procedure

Parameter $p \in P$ such that $D(p) \neq \emptyset$ is called feasible parameter and infeasible otherwise. Below we describe a procedure which is based on generating finite number of random parameters in P and checking feasibility property of them. Since every parameter generates a set containing either only other feasible parameters or only infeasible parameters such procedure is a covering type procedure. In general, we finish with a collection of sets with feasible parameters and collection of sets with infeasible parameters.

It is assumed that vertices $v^1, \ldots, v^N$ of set X are known. We also fix maximum number of iterations $\overline{k} > 1$ in advance. The procedure has the following description.

Step 0. *Set* $\mathcal{P}_{\mathbf{in}} = \emptyset,\ \mathcal{P}_{\emptyset} = \emptyset,\ k = 1,\ k_{\mathbf{in}} = 0,\ k_{\emptyset} = 0$*;*

Step 1. *Choose randomly parameter* $p^k \in P$*;*

Step 2. If $p^k \in \bigcup\limits_{P_{in} \in \mathcal{P}_{\mathbf{in}}} P_{in}$ then goto step 7;

Step 3. *If* $p^k \in \bigcup\limits_{P_{\emptyset} \in \mathcal{P}_{\emptyset}} P_{\emptyset}$ *then goto step 7;*

Step 4. *Solve problem (15) with* $\tilde{p} = p^k$*. Let* (x^k, ξ^k) *and* u^k *be primal and dual solutions.*

Step 5. *If* $\xi^k > 0$ *then define* $P^k_{\emptyset} = P_{\emptyset}(\tilde{p})$ *in (13) for* $\tilde{p} = p^k$ *and* $\tilde{u} = u^k$*, and set* $\mathcal{P}_{\emptyset} = \mathcal{P}_{\emptyset} \bigcup P^k_{\emptyset},\ k_{\emptyset} = k_{\emptyset} + 1$*;*

Step 6. *If* $\xi^k \leq 0$ *then define* $P^k_{\mathbf{in}} = P_{\mathbf{in}}(\tilde{p})$ *in (11) for* $\tilde{p} = p^k$ *and* $\tilde{x} = x^k$*, and set* $\mathcal{P}_{\mathbf{in}} = \mathcal{P}_{\mathbf{in}} \bigcup P^k_{in},\ k_{\mathbf{in}} = k_{\mathbf{in}} + 1$*;*

Step 7. *Set k=k+1;*

Step 8. *If* $k > \overline{k}$ *then stop, otherwise goto step 1.*

When the procedure stops we have a collection $\mathcal{P}_{\mathbf{in}} = \{P^1_{\mathbf{in}}, \ldots, P^{k_{\mathbf{in}}}_{\mathbf{in}}\}$ of feasible parameters sets and a collection $\mathcal{P}_{\emptyset} = \{P^1_{\emptyset}, \ldots, P^{k_{\emptyset}}_{\emptyset}\}$ of infeasible parameters sets. Therefore, we can define P^*_{in} and P^*_{out} in the following way

$$P^*_{in} = \bigcup_{i=1}^{k_{\mathbf{in}}} P^i_{\mathbf{in}},\ P^*_{out} = P \setminus \left(\bigcup_{i=1}^{k_{\emptyset}} P^{k_{\emptyset}}_{\emptyset} \right).$$

In the example considered above we have $\overline{k} = 3,\ k_{\mathbf{in}} = 1,\ k_{\emptyset} = 2$ and

$$\mathcal{P}_{\mathbf{in}} = \{[0.179, 0.643]\},\ \mathcal{P}_{\emptyset} = \{(-0.0317, 0.0311), (1.02, 1.22)\}.$$

Therefore,

$$P^*_{in} = [0.179, 0.643],\ P^*_{out} = [-0.2, -0.0317] \bigcup [0.0311, 1.02] \bigcup [1.22, 1.3].$$

Theoretically the suggested procedure is finite since P is a compact set, hence can be covered by a finite number of convex sets. However, in order to increase the efficiency other covering techniques, e.g. borrowed from global optimization technology [EvtushenkoEtAl17] can be used.

Acknowledgements

The paper was supported by RFBR grant No. 18-07-01432.

References

[Balas18] E. Balas. *Disjunctive programming.* Springer Nature Switzerland AG, 2018.

[BorrelliEtAl03] F. Borrelli, A. Bemporad, and M. Morari. Geometric algorithm for multiparametric linear programming. *Journal of Optimization Theory and Applications*, 108(3):515–540, 2003.

[EvtushenkoEtAl17] Yu.G.. Evtushenko, M.A. Posypkin, L.A. Rybak, and A.V. Turkin. Finding sets of solutions to systems of nonlinear inequalities. *Computational Mathematics and Mathematical Physics*, 57(8):1241–1247, 2017.

[FiedlerEtAl06] M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, and K. Zimmermann. *Linear optimization problems with inexact data.* Springer, 2006

[JonesEtAl08] C.N. Jones, E.C.. Kerrigan, and J.M. Maciejowski. On polyhedral projection and parametric programming. *Journal of Optimization Theory and Applications*, 138(2):207–220, 2008.

[Khamisov99] O. Khamisov. On optimization properties of functions with a concave minorant. *Journal Global Optimization*, 14:79–101, January 1999.