Modelling of the derivatives pricing with multifactor volatility

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Abstract. The pricing of options generated by diffusion processes, where diffusion depends on two groups of variables, was carried out. An algorithm for calculating the approximate price of derivatives and the accuracy of valuations has been developed, which allows to perform the analysis and to make precautionary to minimize the risk of derivatives pricing arising on the stock market. The method of finding the indicative price for a wide class of derivatives has been expanded. Using the spectral theory of self-adjoint operators in Hilbert space and the wave theory of singular and regular perturbations, an analytical formula of the approximate asset price was set, which was described by models with stochastic volatility dependent on l-fast variable and n-slow variable factors, $l \geq 1, n \geq 1,\ \ l \in N, n \in N$ and on local variable.

Keywords: derivative pricing, diffusion processes, Ornstein-Uhlenbeck process, spectral theory, singular and regular perturbation theory, stochastic volatility, Sturm-Liouville theory, Vasicek model.

I Introduction

At the moment, financiers have been drawn to the problem of the relationship between the price of an asset and its volatility. The asset price was found to be volatile. This has led to a number of works to refine this model. Empirical studies have shown that volatility is a time-dependent random variable [5]. Analytical models having stochastic volatility are proposed in [6]. In particular, they provide an opportunity to examine the price of assets that change continuously over time [7-10].

A spectral image of the density of one-dimensional diffusion was obtained in [12]. Spectral theory is an important tool for the analysis of financial models of diffusion in the study of the decomposition of the eigenfunctions of linear operators. Spectral theory has been used by many scientists, namely to forecast options prices [9], to find interest rate on securities [14], to simulate volatility of financial assets [13]. Both spectral theory and stochastic volatility models have become an indispensable tool in financial mathematics, due to the fact that derivative prices are subject to Brownian motion and correlate with volatility [1]. Study of stochastic volatility, in particular the volatility of an asset controlled by non-local diffusion [2].

Short-term interest rate dynamics models were considered in Vasicek's work [11] for derivatives pricing. Significant contribution to the theory of interest rates was made in [8-10], namely: finding the credit spread of credit market instruments, determining the price of interest rate options, determining the risk and return on derivatives of the stock market. The models developed by these scientists have their advantages and disadvantages, but each is used to increase the liquidity of the financial markets. The use of more complex models, despite their theoretical validity, causes complex multi-parameter functions of the profitability curve to be obtained, and this causes significant errors in the calculations.

Using spectral analysis, Linetsky [5] applied the spectral theory of self-adjoint operators to different models, and in particular to the Vasicek model. Lorig [11] considered short-term interest rates described by Vasicek's model with stochastic volatility dependent on two factors, one of which is fast and the other is slowly changing. The spectral theory and the theory of singular and regular perturbations is applied to self-adjoint operators in Hilbert spaces, which describe processes with multidimensional stochastic volatility having l-fast variable, n-slow variable factors, $l \geq 1, n \geq 1, \quad l \in N, n \in N$. In particular, this theory applies to the short-term interest rates described by Vasicek's model. The approximate price of the bonds and their profitabilty are calculated. Applying the Sturm-Liouville theory, Fredholm alternatives, and analyzing singular and regular perturbations at different time scales, we obtained explicit formulas for the approximation of bond prices and profitability.

The goal of the article is to develop an algorithm for finding the approximate price of derivatives and to find explicit formulas for finding their value based on the development of eigen functions and eigenvalues of self-adjoint operators using boundary tasks for singular and regular perturbations. To set the theorem of estimating the accuracy of option prices approximation.

The main advantage over other developed methods is that finding the price of derivatives is reduced to solving the problem of finding the eigenvalues and eigenfunctions of a particular equation that fits this model.

II Methodology and Data

Let X represent short interest rates. One of the most widely known models of short interest rates is the Vasicek model, in which X is modeled as an Ornstein-Uhlenbeck process with multidimensional stochastic volatility.

The Ornstein – Uhlenbeck process is described by a second-order differential equation ща parabolic type

$$\partial_t w(t,x) = \frac{\sigma^2}{2}\partial_x^2 w(t,x) + k(\theta - x)\partial_x w(t,x) \qquad (1)$$

Let's calculate the density of distribution of this process. To do this, consider the Cauchy problem for (1). With the initial condition

$$w(0,x) = w_0(x) - \delta \qquad (2)$$

where $w_0(x)$ – smooth finite function.

Let's apply the Fourier transform. In particular,

$$w(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ix\xi}\, v(t,\xi)d\xi,$$

i.e.

$$v(t,\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ix\xi}\, w(t,x)dx.$$

Then equation (1) reduces to equation

$$\partial_t v = \frac{\sigma^2}{2}(-i\xi)^2 v + k\xi\partial_\xi v + k(1 - i\theta\xi)v = 0 \qquad (3)$$

We will take into account that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ix\xi}\, w_x'(t,x)dx = \frac{1}{\sqrt{2\pi}}\Big[e^{ix\xi}xw|_{-\infty}^{+\infty}$$

$$-\left(\int_{-\infty}^{+\infty} i\xi e^{ix\xi}x\, w(t,x)dx + \int_{-\infty}^{+\infty} e^{ix\xi}\, u(t,x)dx\right)\Bigg] = -\xi\partial_\xi v - v.$$

The initial condition has the form

$$v(0,\xi) = v_0(\xi) \qquad (4)$$

The Cauchy problem (3), (4) for a linear non-uniform differential equation in partial first-order derivatives is solved by the method of characteristics

$$\frac{\partial v}{\partial t} - k\xi\frac{\partial v}{\partial \xi} = \left(-\frac{\sigma^2\xi^2}{2} + (1 - i\theta\xi)\right)v,$$

$$dt = \frac{d\xi}{k\xi}\ , \qquad (5)$$

$$dt = \frac{dv}{\left(-\frac{\sigma^2\xi^2}{2} + (1 - i\theta\xi)\right)v} \quad , \tag{6}$$

(5) and (6) are equations of characteristics.

From (5) we have the equation

$\ln|\xi| = -kt + \ln C\,,\ \ C > 0,\ \ \xi = Ce^{-kt}$

Substitute in (6)

$$dt = \frac{dv}{\left(-\frac{\sigma^2 C^2 e^{-2kt}}{2} + (1 - i\theta C e^{-kt})\right)v};$$

$$\int \frac{dv}{v} = \int_0^t \left(-\frac{\sigma^2 C e^{-k\beta}}{2} + k\big(1 - i\theta C e^{-k\beta}\big)\right) d\beta + \ln C_1, \quad C_1 > 0.$$

$$v = C_1 \exp\left\{\int_0^t \left(-\frac{\sigma^2 C e^{-k\beta}}{2} + k\big(1 - i\theta C e^{-k\beta}\big)\right) d\beta\right\}.$$

With $t = 0$, $v_0\big(\hat{\mathrm{C}}\big) = C_1$, $\hat{\mathrm{C}} = \hat{\xi}$.

Then v on the characteristics has the form

$$v = v_0(\xi \exp\{kt\}) \exp\left\{-\frac{\sigma^2}{2}\frac{(e^{-2kt} - 1)\xi^2 e^{2kt}}{-2k} + k\left(t - i\theta\,\frac{(e^{-kt} - 1)\xi e^{kt}}{-k}\right)\right\}$$

$$v(t,\xi) = v_0(\xi \exp\{kt\}) \exp\left\{-\frac{\sigma^2}{2}\frac{(e^{-2kt} - 1)\xi^2 e^{2kt}}{-2k} + k\left(t - i\theta\,\frac{(e^{-kt} - 1)\xi e^{kt}}{-k}\right)\right\}$$

$$w(t,x) = \frac{1}{\sqrt{2\pi}}\int (\exp\{-i\xi x\}) v(t,\xi) d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} (\exp\{-i\xi x\})$$

$$\exp\left\{-\frac{\sigma^2}{2}\frac{(e^{-2kt} - 1)\xi^2 e^{2kt}}{-2k} + k\left(t - i\theta\,\frac{(e^{-kt} - 1)\xi e^{kt}}{-k}\right)\right\} v(t,\xi) d\xi =$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp\left\{-i\gamma x \exp\left\{-kt - \frac{\sigma^2}{2}\frac{(e^{-2kt} - 1)\gamma^2}{-2k} + k\right.\right.$$

$$\left.\left.\left(t - i\theta\,\frac{(e^{-kt} - 1)\gamma}{-k}\right)\right\}\right\}\frac{v_0(\gamma)}{e^{kt}} d\gamma =$$

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}u_0(\gamma)\int\limits_{-\infty}^{+\infty}\exp\left\{-i\gamma x\exp\left\{(-kt-y)-\frac{\sigma^2}{2}\frac{(e^{-2kt}-1)\gamma^2}{-2k}\right.\right.$$

$$\left.\left.+k\left(t-i\theta\frac{(e^{-kt}-1)\gamma}{-k}\right)\right\}\right\}\frac{d\gamma dy}{e^{kt}}=\frac{1}{\sqrt{2\pi}e^{kt}}\int\limits_{-\infty}^{+\infty}e^{iy(xe^{-kt}-y)}$$

$$\exp\left\{-\frac{\sigma^2}{2}\frac{(e^{-2kt}-1)\gamma^2e^{2kt}}{-2k}+k\left(t-i\theta\frac{(e^{-kt}-1)\gamma e^{kt}}{-k}\right)\right\}d\gamma.$$

Let’s distinguish the complete square by γ

$$\frac{\sigma^2}{2}\frac{(1-e^{-2kt})\gamma^2}{2k}+2i\gamma\left(\frac{xe^{kt}-y}{2}-\frac{\theta(e^{-kt}-1)}{2}\right)\frac{\sigma(1-e^{-2kt})^{1/2}}{2\sqrt{k}}$$

$$\frac{2\sqrt{k}}{\sigma(1-e^{-2kt})^{1/2}}\pm i^2\left(\frac{xe^{kt}-y}{2}-\frac{\theta(e^{-kt}-1)}{2}\right)^2\frac{4k}{\sigma^2(1-e^{-2kt})}=$$

$$\left(\frac{\sigma(1-e^{-2kt})^{1/2}}{2\sqrt{k}}\gamma+i\left(\frac{xe^{kt}-y}{2}+\frac{\theta(e^{-kt}-1)}{2}\right)\frac{2\sqrt{k}}{\sigma(1-e^{-2kt})^{1/2}}\right)^2$$

$$+\left(\frac{xe^{kt}-y}{2}+\frac{\theta(e^{-kt}-1)}{2}\right)^2\frac{4k}{\sigma^2(1-e^{-2kt})};$$

Having replaced

$$\frac{\sigma(1-e^{-2kt})^{1/2}}{2\sqrt{k}}\gamma+i\left(\frac{xe^{kt}-y}{2}+\frac{\theta(e^{-kt}-1)}{2}\right)\frac{2\sqrt{k}}{\sigma(1-e^{-2kt})^{1/2}}=\frac{\alpha}{\sqrt{2}},$$

using the Cauchy integral theorem

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}e^{-\frac{\alpha^2}{2}}d\alpha=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty+i\left(xe^{-kt}+\theta\left(e^{-kt}-1\right)\right)\frac{\sqrt{k}}{\sigma\left(1-e^{-2kt}\right)^{1/2}}}^{+\infty+i\left(xe^{-kt}+\theta\left(e^{-kt}-1\right)\right)\frac{\sqrt{k}}{\sigma\left(1-e^{-2kt}\right)^{1/2}}}e^{-\frac{\alpha^2}{2}}d\alpha. \qquad (7)$$

Indeed $\oint_{(T)}e^{-\frac{\alpha^2}{2}}d\alpha=0$ – closed contour, $e^{-\frac{\alpha^2}{2}}$– analytical function, so, by Cauchy's integral theorem, the integral is zero.

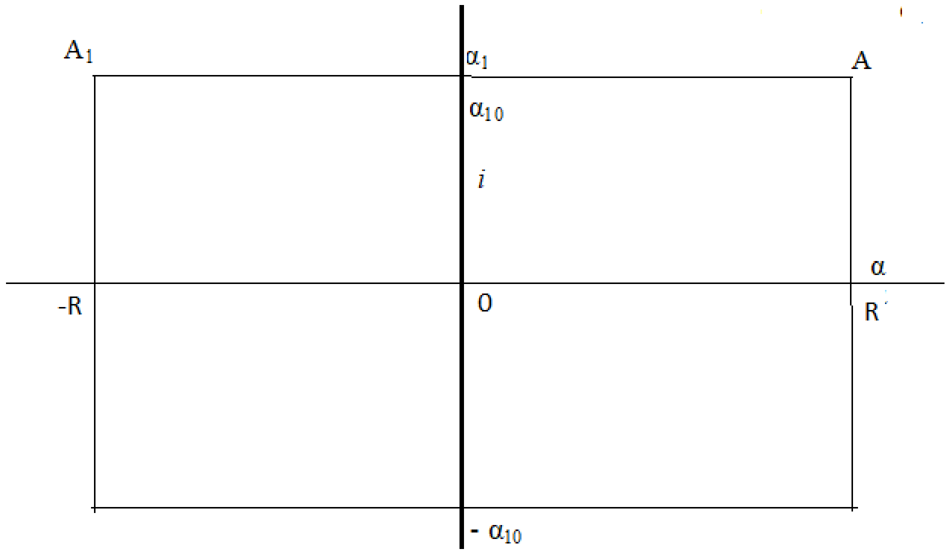

Fig. 1. Contour of integration

Let's take contour $(-RRAA_1(R))$ as (T), where

$$\alpha_{1\,0} = \frac{\left(xe^{-kt} + \theta(e^{-kt} - 1)\right)\sqrt{k}}{\sigma(1 - e^{-2kt})^{1/2}},$$

with $\left(xe^{-kt} + \theta(e^{-kt} - 1)\right) > 0$, if $\alpha_{1\,0} < \infty$, the contour is arranged symmetrically along the axis $O\alpha_1$ with $R \to +\infty$, $\int_{-R}^{R}$ will transform into $\int_{-\infty}^{+\infty}$, and $\int_{-A}^{A} \to \int_{-\infty}^{+\infty}$.

$$\int_{R}^{A} e^{-\frac{\alpha^2}{2}} d\alpha \to 0, \quad \int_{A_1}^{-R} e^{-\frac{\alpha^2}{2}} d\alpha \to 0, \tag{8}$$

with $R \to \infty$, therefore equality (7) holds.

Let's check that goes to zero at $R \to +\infty$.

$$\left|\int_{R}^{A} e^{-\frac{\alpha^2}{2}} d\alpha\right| = \left|\int_{0}^{|\alpha_{1\,0}|} e^{-\frac{(R+i\alpha_1)^2}{2}} d\alpha_1\right| \leq e^{-R^2} e^{|\alpha_{1\,0}|} |\alpha_{1\,0}|.$$

With fixed $x, \theta, k, t, \sigma,\quad e^{|\alpha_{1\,0}|} = const,$ then because

$$e^{-R^2} \to 0, \lim_{R\to\infty} \int_{R}^{A} e^{-\frac{\alpha^2}{2}} d\alpha = 0.$$

Similarly, $\int_{A_1}^{-R} e^{-\frac{\alpha^2}{2}} d\alpha \to 0,\ \mathrm{R} \to +\infty$. Will get

$$w(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0(\gamma) \exp\left\{-\left(\frac{xe^{kt}-y}{2} + \frac{\theta(e^{-kt}-1)}{2}\right)^2 \frac{4k}{\sigma^2(1-e^{-2kt})}\right\} dy,$$

so fundamental solution or the Green's function has the form

$$p(t,x,y) = \frac{1}{\sqrt{\pi \frac{\sigma^2}{k}(1-e^{-2kt})}} \exp\left\{-\frac{\left(xe^{kt}-y+\theta(e^{-kt}-1)\right)^2}{\frac{\sigma^2}{k}(1-e^{-2kt})}\right\}$$

$$= \frac{1}{\sqrt{\pi \frac{\sigma^2}{k}(1-e^{-2kt})}} \exp\left\{-\frac{\left(y-x+(x-\theta)(1-e^{-kt})\right)^2}{\frac{\sigma^2}{k}(1-e^{-2kt})}\right\}.$$

On the other hand, on the probabilistic side, the Green's function is the density of distribution.

Using the methods of spectral theory and the theory of singular and regular perturbations, we can find the approximate price of Ornstein-Uhlenbeck two-barrier options with multivariate volatility, as a self-function decomposition using infinitesimal generators of (l + n + 1)-dimensioned diffusions, $l \geq 1, n \geq 1,\ \ l \in N, n \in N$, that is, diffusion depends on one local variable, the l-dimensional fast-variable factor and the n-dimensional slow-variable factor. This work is an extension of [6, 11, 13], in [11] l = 1 and n = 1.

Process X can represent many economic phenomena and processes. For example, inventory value, index price, reliable short interest, etc. More broadly, X is an external factor that characterizes the cost of any of the above processes. By physical measure $\mathbb{P}$ of process X, we understand process X, which has an instant drift $v(X_t)$ and stochastic volatility $a(X_t) f(\mathrm{Y}_{1t}, \dots, \mathrm{Y}_{\mathrm{lt}}, \mathrm{Z}_{1\mathrm{t}}, \dots, \mathrm{Z}_{\mathrm{nt}}) > 0$, which contains both components: local $a(X_t)$and non-local $f(\mathrm{Y}_{1t}, \dots, \mathrm{Y}_{\mathrm{lt}}, \mathrm{Z}_{1\mathrm{t}}, \dots, \mathrm{Z}_{\mathrm{nt}})$. It should be noted, that infinitesimal generators for Y_j and Z_i have a form $\forall\ i, j$

$$\mathfrak{L}_{Y_j}^{\epsilon_j} = \frac{1}{\epsilon_j}\left(\frac{1}{2}{\beta_j}^2(y_j)\partial_{y_j y_j}^2 + \alpha_j(y_j)\partial_{y_j}\right), \mathfrak{L}_{Z_i}^{\delta_i} = \delta_i\left(\frac{1}{2}{g_i}^2(z_i)\partial_{z_i z_i}^2 + c_i(z_i)\partial_{z_i}\right),$$

are characterized by the values $\frac{1}{\epsilon_j}$ and $\delta_i$respectively. Thus, $\mathrm{Y}_1, \dots, \mathrm{Y}_\mathrm{l}$ and $\mathrm{Z}_1, \dots, \mathrm{Z}_\mathrm{n}$ have an internal timeline $\epsilon_j > 0$ and $\frac{1}{\delta_i} > 0$. Let's consider $\epsilon_j << 1$ and $\delta_i << 1$, to make the inner time scale Y_j small and the inner time scale Z_i – large. Therefore, Y_j , $j = \overline{1, l}$, are fast variables, and Z_i , $i = \overline{1, n}$ are slowly variable factors. Note that $\mathfrak{L}_{Y_j}^{\epsilon_j}$ and $\mathfrak{L}_{Z_i}^{\delta_i}$ have the form of the Ornstein-Uhlenbeck process [20]

$$\partial_t w = \frac{1}{2}\bar{\sigma}^2 x^2 \partial_{xx}^2 w + rx\partial_x w - rw,$$

the right part of which has a form

$$\tfrac{1}{2}\bar{\sigma}^2 x^2 y'' + rxy' - ry = 0, \tag{9}$$

Let's reduce (9) to the equation $y'' + \lambda y = 0$, where $\lambda = const$, so that there is no first derivative in the obtained equation, that is, we look for a solution in the form

$$y = s(x)v(x),$$

where $v(x) -$ new unknown function, and let's choose $s(x)$ in such way to have

$$v'' + \lambda(x)v = 0. \tag{10}$$

Having reduced into (10), we will have $\bar{\sigma}^2 x^2 s' + rxs = 0$, herefrom

$$s(x) = \exp\left\{-\int \frac{rx}{\bar{\sigma}^2 x^2}dx\right\} = \exp\left\{-\frac{r}{\bar{\sigma}^2}\ln x\right\}.$$

$$\lambda(x) = -\frac{d}{dx}\left(\frac{rx}{\bar{\sigma}^2 x^2}\right) - \left(\frac{rx}{\bar{\sigma}^2 x^2}\right)^2 - \frac{2r}{\bar{\sigma}^2 x^2} = -\frac{r}{\bar{\sigma}^2 x^2} - \frac{r^2}{\bar{\sigma}^4 x^2},$$

will have

$$v'' - \frac{r}{\bar{\sigma}^2 x^2}\left(1 + \frac{1}{\bar{\sigma}^2}\right)v = 0.$$

Thus,

$$y = s(x)v(x), \text{ де } s(x) = \exp\left\{-\tfrac{r}{\bar{\sigma}^2}\ln x\right\}.$$

In our case

$$m(x) = \frac{2}{\bar{\sigma}^2 x^2}\exp\left\{\frac{2r}{\bar{\sigma}^2}\ln x\right\}, \quad \mathfrak{s}(x) = s^2(x).$$

In order that in equation (10) was not a function of $\lambda(x)$, but $\lambda = const$, then we replace the variables $x = \sigma(\theta)$ in (9).

$$y'_\theta \theta'_x = \frac{1}{\sigma'_\theta \theta} y'_\theta; \;\; y''_{x^2} = y''_{\theta^2}\frac{1}{\sigma'^2_\theta} - y'_\theta \frac{\sigma''_{\theta^2}}{\sigma'^3_\theta},$$

substituting in (10) we obtain

$$\frac{1}{2}\bar{\sigma}^2 \sigma' x^2 y'' + \left(rx\sigma'^2 - \frac{1}{2}\bar{\sigma}^2\sigma'' x^2\right)y' - r\sigma'^3 y = 0, \tag{11}$$

Let's apply to (11) the same considerations as in (8) $y = s^* v$ then

$$s^* = \exp\left\{-\int \frac{rx\sigma'^2 - \frac{1}{2}\bar{\sigma}^2\sigma'' x^2}{\bar{\sigma}^2\sigma' x^2}d\theta\right\} = \sqrt{\sigma'}\, s.$$

Let's write the value of new $\lambda^*(\theta)$.

$$\lambda^*(\theta) = -\frac{d}{d\theta}\left(\frac{rx\sigma'^2 - \frac{1}{2}\bar{\sigma}^2\sigma'' x^2}{\bar{\sigma}^2\sigma' x^2}\right) - \left(\frac{rx\sigma'^2 - \frac{1}{2}\bar{\sigma}^2\sigma'' x^2}{\bar{\sigma}^2\sigma' x^2}\right)^2 - \frac{2r}{\bar{\sigma}^2 x^2},$$

For reasons of solution, let's make a substitution $x = \exp\left\{\frac{\bar{\sigma}^2}{\sqrt{2}}\theta\right\}$ to find λ^*

$$\lambda^* = -\frac{d}{d\theta}\left(\frac{r}{\sqrt{2}\bar{\sigma}} - \frac{\bar{\sigma}}{2\sqrt{2}}\right) - \left(\frac{r}{\sqrt{2}\bar{\sigma}} - \frac{\bar{\sigma}}{2\sqrt{2}}\right)^2 - r = -\left(\frac{r}{\sqrt{2}\bar{\sigma}} - \frac{\bar{\sigma}}{2\sqrt{2}}\right)^2 - r$$
$$= -\frac{1}{2}\left(\frac{r}{\bar{\sigma}} - \frac{\bar{\sigma}}{2}\right)^2 - r,$$

we will have

$$v'' - \left(\frac{1}{2}\left(\frac{r}{\bar{\sigma}} - \frac{\bar{\sigma}}{2}\right)^2 + r\right)v = 0,$$

$$k^2 - \frac{1}{2}\left(\frac{r}{\bar{\sigma}} - \frac{\bar{\sigma}}{2}\right)^2 - r = 0, \quad k_{1\,2} = \pm\sqrt{\frac{1}{2}\left(\frac{r}{\bar{\sigma}} - \frac{\bar{\sigma}}{2}\right)^2 + r},$$

need solution to be equal 0 at points L and R , and x^{k_1} and x^{k_2} do not give such a picture, so we will replace the variables $x = Le^{\theta}$, we'll have

$$\frac{\bar{\sigma}^2}{2}v'' - \left(\frac{1}{2}\left(\frac{r}{\bar{\sigma}} - \frac{\bar{\sigma}}{2}\right)^2 + r\right)v = 0,$$

or

$$\frac{\bar{\sigma}^2}{2}v'' - \left(\frac{\nu^2}{2} + r\right)v = 0,$$

on eigen values and eigen functions we will explore

$$\frac{\bar{\sigma}^2}{2}v'' - \left(\frac{\nu^2}{2} + r\right)v = \lambda v, \qquad \frac{\bar{\sigma}^2}{2}v'' = -\lambda_n^2 v, \;\; L \le x \le R, \qquad 0 \le \theta \le \ln\frac{R}{L},$$

then

$$\frac{\bar{\sigma}^2}{2}k^2 = -\lambda_n^2, \;\; \lambda_n > 0, \qquad k^2 = -\frac{2}{\bar{\sigma}^2}\lambda_n^2, \;\; k_{1\,2} = \pm\frac{i\sqrt{2}}{\bar{\sigma}}\lambda_n.$$

The general solution has the form

$$v = C_1\cos\left(\frac{\sqrt{2}}{\bar{\sigma}}\lambda_n\theta\right) + C_2\sin\left(\frac{\sqrt{2}}{\bar{\sigma}}\lambda_n\theta\right),$$

Let's check the fulfilling of boundary conditions, if $\theta = 0$, then $C_1 = 0$, with $\theta = \ln\frac{R}{L}$,

$$\sin\left(\frac{\sqrt{2}}{\bar{\sigma}}\lambda_n\ln\frac{R}{L}\right) = 0, \quad \frac{\sqrt{2}}{\bar{\sigma}}\lambda_n\ln\frac{R}{L} = n\pi,$$

therefrom

$$\lambda_n = \frac{n\pi\bar{\sigma}}{\sqrt{2}\ln\frac{R}{L}}, \quad v_n(\theta) = \sin\frac{n\pi\bar{\sigma}}{\sqrt{2}\ln\frac{R}{L}}\theta.$$

From the point that

$$\int_0^{\ln\frac{R}{L}} \sin^2\frac{n\pi\bar{\sigma}}{\sqrt{2}\ln\frac{R}{L}}\theta\,d\theta = \frac{1}{2}\ln\frac{R}{L}.$$

We have

$$v_n = \frac{\sin\frac{n\pi\bar{\sigma}}{\sqrt{2}\ln\frac{R}{L}}\theta}{\sqrt{\frac{1}{2}\ln\frac{R}{L}}} = \frac{\sin\frac{n\pi\bar{\sigma}\ln\frac{x}{L}}{\sqrt{2}\ln\frac{R}{L}}}{\sqrt{\frac{1}{2}\ln\frac{R}{L}}}.$$

$$\psi_n(x) = \left(\frac{n\pi\bar{\sigma}}{2\ln\frac{R}{L}}\right)^2 \left(\lambda_n^2 + \frac{1}{2}\left(\frac{r}{\bar{\sigma}} - \frac{\bar{\sigma}}{2}\right)^2 + r\right) \sqrt{\frac{2x}{\ln\frac{R}{L}}} \exp\left\{-\frac{r}{\bar{\sigma}^2}\ln x\right\} \sin\frac{n\pi\bar{\sigma}\ln\frac{x}{L}}{\sqrt{2}\ln\frac{R}{L}},$$

Let’s find the scalar product

$$\left(\psi_n(x), \psi_m(x)\right) = \int_L^R \frac{x\bar{\sigma}^2}{\ln\frac{R}{L}} \exp\left\{-\frac{2r}{\bar{\sigma}^2}\ln x\right\} \sin\left(\frac{n\pi\ln\frac{x}{L}}{\sqrt{2}\ln\frac{R}{L}}\right) \sin\left(\frac{m\pi\ln\frac{x}{L}}{\sqrt{2}\ln\frac{R}{L}}\right)$$

$$\frac{2}{x^2\bar{\sigma}^2}\exp\left\{\frac{2r}{\bar{\sigma}^2}\ln x\right\}dx = \frac{1}{\ln\frac{R}{L}}\int_L^R \frac{2}{x}\sin\left(\frac{n\pi\ln\frac{x}{L}}{\ln\frac{R}{L}}\right)\sin\left(\frac{m\pi\ln\frac{x}{L}}{\ln\frac{R}{L}}\right)dx =$$

$$2\int_0^1 \sin n\pi t \sin m\pi t\, dt = 2\int_0^1 [\cos(n-m)\pi t + \cos(n+m)\pi t]dt =$$

$$-\frac{\sin(n-m)\pi t}{(n-m)\pi}\Big|_0^1 - \frac{\sin(n+m)\pi t}{\pi}\Big|_0^1 = 0, \quad n \neq m.$$

With $n = m$

$$\int_0^1 \sin^2 n\pi t\, dt = \int_0^1 (1 - cos2n\pi t)dt = 1 - \frac{\sin 2n\pi t}{2n\pi}\Big|_0^1 = 1.$$

Thus,

$$\left(\psi_n(x), \psi_m(x)\right) = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

III Results and analysis

Let X be the securities without paying dividends on an asset (for example, stock, index, etc.). Often, X is modelled as a geometric Brownian motion with constant volatility (e.g. Black-Scholes formula) [7]. Consider X as a model of geometric Brownian motion with multidimensional stochastic volatility. In particular, $\widetilde{\mathbb{P}}$ dynamics in X are given by:

$$dX_t = rX_t dt + f(\mathrm{Y}_1, \dots, \mathrm{Y}_\mathrm{l}, \mathrm{Z}_1, \dots, \mathrm{Z}_\mathrm{n}) X_t d\widetilde{W}_t^x\ , h(X_t) = 0,$$

Let’s calculate the approximate price of the double barrier of the option defined on X.

Let us write down the operator $\langle \mathfrak{L}_2 \rangle$ and its associated densities at speed $m(x)$

$$\langle \mathfrak{L}_2 \rangle = \frac{1}{2}\bar{\sigma}^2 x^2 \partial_{xx}^2 + rx\partial_x - r, \qquad m(x) = \frac{2}{\bar{\sigma}^2 x^2} exp\left(\frac{2r}{\bar{\sigma}^2}\ln x\right). \qquad (12)$$

For a double barrier option with L and R barrier values, the payout is:

$$H(X_t)\mathbb{I}_{\{\tau>t\}} = (X_t - K)^+\mathbb{I}_{\{\tau>t\}},\ I = (L,R),\ 0 < L < K < R,$$

To calculate the value of this parameter, at first, it is needed to find the eigenvalues of the operator $\langle\mathfrak{L}_2\rangle$ with boundary conditions

$$\lim_{x\to L}\psi_n(x) = 0,\ \lim_{x\to R}\psi_n(x) = 0.$$

It should be noted that a regular killing of boundary conditions at the ends L and R are entered. Equation

$$-\langle\mathfrak{L}_2\rangle\psi_n = \lambda_n\psi_n, \qquad \psi_n \in dom(\langle\mathfrak{L}_2\rangle)$$

with the above boundary conditions can be found in [4]

$$\psi_n(x) = \frac{\bar\sigma\sqrt{x}}{\sqrt{\ln\left(\frac{R}{L}\right)}}\exp\left(-\frac{r}{\bar\sigma^2}\ln x\right) sin\left(\frac{n\pi\ln\left(\frac{x}{L}\right)}{\ln\left(\frac{R}{L}\right)}\right), \qquad n = 1,2,3,\dots,$$

$$\lambda_n = \frac{1}{2}\left(\frac{n\pi\bar\sigma}{\ln(\frac{R}{L})}\right)^2 + \left(\frac{\nu^2}{2} + r\right), \qquad \nu = \frac{r}{\bar\sigma} - \frac{\bar\sigma}{2}.$$

Let's write down expressions for operators $\mathcal{A}_j$ and $\mathcal{B}_i$

$$\mathcal{A}_j = -\vartheta_{3j}x\partial_x x^2\partial_{xx}^2 - \vartheta_{2j}x^2\partial_{xx}^2,\ \mathcal{B}_i = -\vartheta_{1i}x\partial_x - \vartheta_{0i}. \qquad (13)$$

On the basis [12] let's calculate $\mathcal{A}_{jk,n}$, $\mathcal{B}_{ik,n}$ and $\widetilde{\mathcal{B}}_{ik,n}$. For $k \neq n$ let's find

$$\mathcal{A}j_{k,n} = -\vartheta_{2j}\left(\frac{(-1+(-1)^{k+n})knr}{(k^2-n^2)\bar\sigma^2\ln\left(\frac{R}{L}\right)}\right)$$

$$-\vartheta_{3j}\left(\frac{(-1+(-1)^{k+n})kn\left(4n^2\pi^2\bar\sigma^4 + (-12r^2 + 4r\bar\sigma^2 + \bar\sigma^4)\ln^2\left(\frac{R}{L}\right)\right)}{2(k^2-n^2)\bar\sigma^4\ln^3\left(\frac{R}{L}\right)}\right),$$

$$\mathcal{B}_{ik,n} = \vartheta_{i1}\frac{2(-1+(-1)^{k+n})kn}{(k-n)(k+n)\ln\left(\frac{R}{L}\right)},$$

$$\widetilde{\mathcal{B}}_{ik,n} = -\vartheta_{1i}\bar\sigma'(\Upsilon_{k,n}) - \vartheta_{i0}\bar\sigma'\left(\frac{8(-1+(-1)^{k+n})knr\ln\left(\frac{R}{L}\right)}{(k^2-n^2)^2\pi^2\bar\sigma^3}\right),$$

$$v_{k,n} \coloneqq \frac{4nkr(\ln(L) - (-1)^{k+n}\ln(R))}{(k^2-n^2)\bar\sigma^3\ln\left(\frac{R}{L}\right)}$$

$$-\frac{2(-1+(-1)^{k+n})kn\left((k-n)(k+n)\pi^2\bar\sigma^4 - 2r(-2r+\bar\sigma^2)\ln^2\left(\frac{R}{L}\right)\right)}{(k^2-n^2)^2\pi^2\bar\sigma^5\ln\left(\frac{R}{L}\right)},$$

And for $k = n$ let's find

$$\mathcal{A}_{jn,n} = -\vartheta_{3j}\left(\frac{1}{\bar{\sigma}^3}\left(\frac{3n^2\pi^2\nu}{\ln^2\left(\frac{R}{L}\right)} - \nu^3\right) - \frac{1}{\bar{\sigma}^2}\left(\nu^2 - \frac{n^2\pi^2}{\ln^2\left(\frac{R}{L}\right)}\right)\right.$$

$$\left. -\vartheta_{j2}\left(\frac{1}{\bar{\sigma}^2}\left(\nu^2 - \frac{n^2\pi^2}{\ln^2\left(\frac{R}{L}\right)}\right) + \frac{\nu}{\bar{\sigma}}\right)\right),$$

$$\mathcal{B}_{in,n} = \vartheta_{i1}\left(\frac{2r - \bar{\sigma}^2}{2\bar{\sigma}^2}\right) - \vartheta_{i0},$$

$$\widetilde{\mathcal{B}}_{in,n} = -\vartheta_{i1}\bar{\sigma}'\left(\frac{1}{\bar{\sigma}} - \frac{r\nu\left(\ln^2(R) - \ln^2(L)\right)}{\bar{\sigma}^4\ln\left(\frac{R}{L}\right)}\right) - \vartheta_{i0}\bar{\sigma}'\left(\frac{1}{\bar{\sigma}} - \frac{r\left(\ln^2(R) - \ln^2(L)\right)}{\bar{\sigma}^3\ln\left(\frac{R}{L}\right)}\right).$$

Calculation c_n can be found in [1-2]

$$c_n = (\psi_\tau(\cdot), (\cdot - K)^+) = \frac{L^{\frac{\nu}{\bar{\sigma}}}}{\log\left(\frac{R}{L}\right)}\left(L\Phi_n(\nu + \bar{\sigma}) - K\Phi_n(\nu)\right),$$

$$\Phi_n(\gamma) \coloneqq \frac{2}{\omega_n^2 + z^2}$$

$$(\exp(\mathfrak{K}\gamma)\,(\omega_n\cos(\omega_n\mathfrak{K}) - \gamma sin(\omega_n\mathfrak{K}) - \exp(\mathfrak{U}\gamma)\,(-1)^n\omega_n),$$

$$\omega_n \coloneqq \frac{n\pi}{\mathfrak{U}},\quad \mathfrak{K} \coloneqq \frac{1}{\bar{\sigma}}\ln\left(\frac{K}{L}\right),\quad \mathfrak{U} \coloneqq \frac{1}{\bar{\sigma}}\ln\left(\frac{R}{L}\right).$$

The approximate price of the options can always be calculated and plotted, individually in each respective timeline, in the same way as for two components [3].

Let’s calculate the approximate price of a zero coupon bond.

Let’s write the operator $\langle\mathfrak{L}_2\rangle$ and its associated densities at speed $m(x)$

$$\langle\mathfrak{L}_2\rangle = \frac{1}{2}\bar{\sigma}^2\partial_{xx}^2 + \kappa(\bar{\theta} - x)\partial_x - x, \tag{14}$$

$$m(x) = \frac{2}{\bar{\sigma}^2}exp\left(\frac{-k}{\bar{\sigma}^2}(\bar{\theta} - x)^2\right), \qquad \bar{\theta} = \theta - \frac{1}{\kappa}\overline{f\Omega}.$$

For a zero coupon bond, the full payout is:

$$H(X_t) = \mathbb{I}_{\{\tau > t\}} = 1. \tag{15}$$

The differential equation corresponds to operator $\langle\mathfrak{L}_2\rangle$.

$$\frac{1}{2}\bar{\sigma}^2 y'' + \kappa(\theta - x)y' - xy = 0,$$

Let's make a substitution to avoid the first derivative in the obtained equation:

$$y = s(x)u(x),\ y'(x) = s'^{(x)}u(x) + s(x)u'(x),$$

$$y''(x) = s''(x)u(x) + 2s'(x)u'(x) + s(x)u''(x),$$

where $u(x) -$ unknown function, $s(x) -$ will be picked from the obtained equation

$$\frac{1}{2}\bar{\sigma}^2 s(x)u''(x) + \frac{1}{2}\bar{\sigma}^2 2s'(x)u'(x) + \frac{1}{2}\bar{\sigma}^2 s''(x)u(x)$$

$$+ \kappa(\theta - x)\left(s(x)u'(x) + s'(x)u(x)\right) - xs(x)u(x) = 0,$$

therefrom

$$\kappa(\theta - x)s(x) + \bar{\sigma}^2 s'(x) = 0,$$

i.e.

$$\frac{s'(x)}{s(x)} = -\frac{\kappa(\theta - x)}{\bar{\sigma}^2}, \qquad s(x) = \exp\left\{\frac{\kappa(\theta - x)^2}{2\bar{\sigma}^2}\right\}.$$

We will get the equation

$$\frac{1}{2}\bar{\sigma}^2 s(x)u''(x) + \left(\frac{1}{2}\bar{\sigma}^2 s''(x) + \kappa(\theta - x)s'(x) - xs(x)\right)u(x) = 0.$$

Because

$$s(x) = \exp\left\{\frac{\kappa(\theta - x)^2}{2\bar{\sigma}^2}\right\},$$

then

$$s'(x) = -\frac{\kappa(\theta - x)}{\bar{\sigma}^2}\exp\left\{\frac{\kappa(\theta - x)^2}{2\bar{\sigma}^2}\right\},$$

$$s''(x) = \left(\frac{\kappa}{2} - x + \frac{\kappa^2(\theta - x)^2}{\bar{\sigma}^2}\right)\exp\left\{-\frac{\kappa(\theta - x)^2}{2\bar{\sigma}^2}\right\},$$

By substituting we will get

$$c_n = (\psi_n, 1) = \frac{2}{\bar{\sigma}}\sqrt{\frac{\pi}{\kappa}}\mathcal{N}_n A^n e^{-A^2/4}, \qquad \psi_n = \mathcal{N}_n \exp\left(-A\xi - \frac{1}{2}A^2\right)H_n(\xi + A),$$

$$\mathcal{N}_n = \left(\sqrt{\frac{\kappa}{\pi}}\frac{\bar{\sigma}}{2^{n+1}n!}\right)^{1/2}, A = \frac{\bar{\sigma}}{\kappa^{3/2}},$$

$$\xi = \frac{\sqrt{\kappa}}{\bar{\sigma}}(x - \bar{\theta}), \qquad \lambda_n = \bar{\theta} - \frac{\bar{\sigma}^2}{2\kappa^2} + \kappa n, \qquad n = 0,1,2,\ldots$$

Let’s substitute all values of variables into ψ_n

$$\psi_n = \left(\sqrt{\frac{\kappa}{\pi}}\frac{\bar{\sigma}}{2^{n+1}n!}\right)^{1/2}\exp\left(-\frac{\bar{\sigma}}{\kappa^{3/2}}\frac{\sqrt{\kappa}}{\bar{\sigma}}(x - \bar{\theta}) - \frac{1}{2}\frac{\bar{\sigma}^2}{\kappa^3}\right)$$
$$H_n\left(\frac{\sqrt{\kappa}}{\bar{\sigma}}(x - \bar{\theta}) + \frac{\bar{\sigma}}{\kappa^{3/2}}\right),$$

where $\bar{\theta} = \theta - \frac{1}{\kappa}\overline{f\Omega}$, and H_n- the Hermite polynomial.

Let’s find the scalar product $\left(\psi_n(x), \psi_m(x)\right)$:

$$\left(\psi_n(x), \psi_m(x)\right) = \int_{-\infty}^{+\infty}\sqrt{\frac{\kappa}{\pi}}\frac{\bar{\sigma}}{\sqrt{2^{n+1}2^{m+1}n!\,m!}}\exp\left(-\frac{2\bar{\sigma}}{\kappa}(x - \bar{\theta}) - \frac{\bar{\sigma}^2}{\kappa^3}\right)$$
$$H_n\left(\frac{\sqrt{\kappa}}{\bar{\sigma}}(x - \bar{\theta}) + \frac{\bar{\sigma}}{\kappa^{3/2}}\right)H_m\left(\frac{\sqrt{\kappa}}{\bar{\sigma}}(x - \bar{\theta}) + \frac{\bar{\sigma}}{\kappa^{3/2}}\right)\frac{2}{\bar{\sigma}^2}\exp\left(\frac{\kappa}{\bar{\sigma}^2}(x - \bar{\theta})^2\right)dx$$
$$= \sqrt{\frac{\kappa}{\pi}}\frac{1}{\sqrt{2^n 2^m n!\,m!}}\int_{-\infty}^{+\infty}\exp\left\{-\left(\frac{\kappa}{\bar{\sigma}^2}(x - \bar{\theta}) + \frac{\bar{\sigma}}{\kappa^{3/2}}\right)^2\right\}$$

$$H_n\left(\frac{\sqrt{\kappa}}{\bar{\sigma}}(x-\bar{\theta})+\frac{\bar{\sigma}}{\kappa^{3/2}}\right)H_m\left(\frac{\sqrt{\kappa}}{\bar{\sigma}}(x-\bar{\theta})+\frac{\bar{\sigma}}{\kappa^{3/2}}\right)dx.$$

After replacing the variable $\frac{\sqrt{\kappa}}{\bar{\sigma}}(x-\bar{\theta})+\frac{\bar{\sigma}}{\kappa^{3/2}}=\alpha$ we will have

$$\big(\psi_n(x),\psi_m(x)\big)=\frac{1}{\sqrt{\pi 2^n 2^m n!\,m!}}\int_{-\infty}^{+\infty}\exp\{-\alpha^2\}H_n(\alpha)H_m(\alpha)d\alpha=$$

$$=\frac{1}{\sqrt{\pi\, n!\,m!}}\int_{-\infty}^{+\infty}(-1)^{n+m}\frac{d^n}{dx^n}e^{-\frac{\alpha^2}{2}}\frac{d^m}{dx^m}e^{-\frac{\alpha^2}{2}}d\alpha.$$

Because

$$\int_{-\infty}^{+\infty}\left(e^{-\frac{\alpha^2}{2}}\right)^{(n)}\left(e^{-\frac{\alpha^2}{2}}\right)^{(m)}d\alpha=(-1)^n\int_{-\infty}^{+\infty}\left(e^{-\frac{\alpha^2}{2}}\right)^{(m+n)}e^{-\frac{\alpha^2}{2}}d\alpha=$$

$$(-1)^m\int_{-\infty}^{+\infty}\left(e^{-\frac{\alpha^2}{2}}\right)^{m+n}e^{-\frac{\alpha^2}{2}}d\alpha.$$

therefrom it follows that

$$\int_{-\infty}^{+\infty}\left(e^{-\frac{\alpha^2}{2}}\right)^{(n)}\left(e^{-\frac{\alpha^2}{2}}\right)^{(m)}d\alpha=\begin{cases}\neq 0, & m=n,\\ 0, & m=n.\end{cases}$$

$$\frac{1}{2^n n!\sqrt{\pi}}\int_{-\infty}^{+\infty}\left(\frac{d^n}{dx^n}e^{-\frac{\alpha^2}{2}}\right)^2 d\alpha=\frac{1}{2^n n!\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-\frac{\alpha^2}{2}}\left(e^{-\frac{\alpha^2}{2}}\right)^{(2n)}d\alpha.$$

By integrating by the parts, given that such an integral contains pair degrees α^k decreases and reduces to $\int_{-\infty}^{+\infty}e^{-\frac{\alpha^2}{2}}d\alpha$, and integral which contains α^k, $k-$ not pair equals zero, $2n$ times, taking by parts we will get $\big(\psi_n(x),\psi_m(x)\big)=1$.

To find the price of the bond with payments $H(X_t)=\mathbb{I}_{\{\tau>t\}}=1$, it's needed to solve the equation (15) on finding the eigenvalues at the segment $I=(-\infty,\infty)$ with $\langle\mathfrak{L}_2\rangle$, according to (15). As both ends $-\infty$ and ∞ are natural boundaries, then the solution has a form [3]

$$\psi_n=\left(\sqrt{\frac{\kappa}{\pi}}\frac{\bar{\sigma}}{2^{n+1}n!}\right)^{1/2}\exp\left(-\frac{\bar{\sigma}}{\kappa^{3/2}}\frac{\sqrt{\kappa}}{\bar{\sigma}}(x-\bar{\theta})-\frac{1}{2}\frac{\bar{\sigma}^2}{\kappa^3}\right)H_n\left(\frac{\sqrt{\kappa}}{\bar{\sigma}}(x-\bar{\theta})+\frac{\bar{\sigma}}{\kappa^{3/2}}\right),$$

H_n – the Hermite polynomials [18], which have a form

$$H_n=(-1)^n e^{-\frac{x^2}{2}}\frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right)=\sum_{j=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^j}{2^j}\frac{n!}{j!\,(n-2j)!}x^{n-2j}.$$

Let's write down expressions for operators $\mathcal{A}_j$ and $\mathcal{B}_j$:

$$\mathcal{A}_j=-\vartheta_{j3}\partial^3_{xxx}-\big(\vartheta_{j2}+\mathfrak{U}_{j2}\big)\partial^2_{xx}-\mathfrak{U}_{j1}\partial_x,\qquad \mathcal{B}_j=\vartheta_{j1}\partial_x-\vartheta_{j0}.$$

Operators $\mathcal{A}_{jk,n}, \mathcal{B}_{jk,n}, \ \widetilde{\mathcal{B}}_{jk,n}$ are written on the basis of recurrence ratios:

$$\partial_x H_n = 2nH_{n-1}, \qquad 2xH_n = H_{n+1} + \partial_x H_n,$$

$$\mathcal{A}_{jk,n} = -\vartheta_{j3}\left\{\sum_{m=0}^{3\wedge n}\binom{3}{m}\left(\frac{-1}{\kappa}\right)^{3-m}\left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)^{m}\frac{n!\,\mathcal{N}_n}{(n-m)!\,\mathcal{N}_{n-m}}\delta_{k,n-m}\right\}$$
$$-(\vartheta_2 + \mathfrak{U}_{j2})\left\{\sum_{m=0}^{3\wedge n}\binom{2}{m}\left(\frac{-1}{\kappa}\right)^{2-m}\left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)^{m}\frac{n!\,\mathcal{N}_n}{(n-m)!\,\mathcal{N}_{n-m}}\delta_{k,n-m}\right\}$$
$$-\mathfrak{U}_{j1}\left\{\left(\frac{-1}{\kappa}\right)\delta_{k,n} + \left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)\frac{n!\,\mathcal{N}_n}{(n-1)!\,\mathcal{N}_{n-1}}\delta_{k,n-1}\right\},$$

$$\mathcal{B}_{jk,n} = -\vartheta_1\left\{\left(\frac{-1}{\kappa}\right)\delta_{k,n} + \left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)\frac{n!\,\mathcal{N}_n}{(n-1)!\,\mathcal{N}_{n-1}}\delta_{k,n-1}\right\} - \vartheta_{j0}\delta_{k,n},$$

$$\widetilde{\mathcal{B}}_{jk,n} = -\vartheta_{j1}\bar{\sigma}'\left\{\left[\left(\frac{-1}{\kappa}\right)\left(\frac{1}{2\bar{\sigma}} - \frac{\bar{\sigma}}{\kappa^3} - \frac{n}{\bar{\sigma}}\right)\right]\delta_{k,n}\right.$$
$$+\left[\left(\frac{-1}{\kappa}\right)\left(\frac{4}{\kappa^{\frac{3}{2}}}\right) + \left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)\left(\frac{1}{2\bar{\sigma}} - \frac{\bar{\sigma}}{\kappa^3} - \frac{n}{\bar{\sigma}}\right)\right]\frac{n!\,\mathcal{N}_n}{(n-1)!\,\mathcal{N}_{n-1}}\delta_{k,n-1}$$
$$+\left[\left(\frac{-1}{\kappa}\right)\left(\frac{-2}{\bar{\sigma}}\right) + \left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)\left(\frac{4}{\kappa^{\frac{3}{2}}}\right)\right]\frac{n!\,\mathcal{N}_n}{(n-2)!\,\mathcal{N}_{n-2}}\delta_{k,n-2}$$
$$\left.+\left[\left(\frac{2\sqrt{\kappa}}{\bar{\sigma}}\right)\left(\frac{-2}{\bar{\sigma}}\right)\right]\frac{n!\,\mathcal{N}_n}{(n-3)!\,\mathcal{N}_{n-3}}\delta_{k,n-3}\right\} - \vartheta_{i0}\bar{\sigma}'$$
$$\left\{\left(\frac{1}{2\bar{\sigma}} - \frac{\bar{\sigma}}{\kappa^3} - \frac{n}{\bar{\sigma}}\right)\delta_{k,n} + \left(\frac{4}{\kappa^{\frac{3}{2}}}\right)\frac{n!\,\mathcal{N}_n}{(n-1)!\,\mathcal{N}_{n-1}}\delta_{k,n-1} + \left(\frac{-2}{\bar{\sigma}}\right)\frac{n!\,\mathcal{N}_n}{(n-2)!\,\mathcal{N}_{n-2}}\delta_{k,n-2}\right\}$$
$$-\vartheta_{i1}\overline{f\Omega}'\left\{\left(\frac{1}{\kappa^3}\right)\delta_{k,n} + \left(\frac{-4}{\bar{\sigma}\kappa^{\frac{3}{2}}}\right)\frac{n!\,\mathcal{N}_n}{(n-1)!\,\mathcal{N}_{n-1}}\delta_{k,n-1} + \left(\frac{4}{\bar{\sigma}^2}\right)\frac{n!\,\mathcal{N}_n}{(n-2)!\,\mathcal{N}_{n-2}}\delta_{k,n-2}\right\}$$
$$-\vartheta_{i0}\overline{f\Omega}'\left\{\left(\frac{-1}{\kappa^2}\right)\delta_{k,n} + \left(\frac{2}{\bar{\sigma}\sqrt{\kappa}}\right)\frac{n!\,\mathcal{N}_n}{(n-1)!\,\mathcal{N}_{n-1}}\delta_{k,n-1}\right\}.$$

Calculation of c_n can be found in [1-3]

$$c_n = (\psi_n, 1) = \frac{2}{\bar{\sigma}}\sqrt{\frac{\pi}{\kappa}}\mathcal{N}_n A^n e^{-A^2/4}.$$

For zero coupon bonds, the profitability curve is considered more often rather than the price of the bond itself. Return $R^{\bar{\epsilon},\overline{\delta'}}$ in zero-coupon bonds, for which one dollar is paid at time t is determined by the ratio:

$$w^{\bar{\epsilon},\overline{\delta'}} = \exp\left(-R^{\bar{\epsilon},\overline{\delta'}}t\right),$$

Let's get an approximation for a zero coupon bond, sorting it out both bond prices $w^{\bar{\epsilon},\overline{\delta'}}$ and return $R^{\bar{\epsilon},\overline{\delta'}}$ by degrees $\sqrt{\epsilon_j}$ and $\sqrt{\delta_i}$:

$$w_{\overline{0},\overline{0'}}+\sum_{j=1}^{l}\sqrt{\epsilon_j}w_{\overline{1_j},\overline{0'}}+\sum_{i=1}^{r}\sqrt{\delta_i}w_{\overline{0},\overline{1'_i}}+\cdots=$$

$$\exp\left\{-\left(R_{\overline{0},\overline{0'}}+\sum_{j=1}^{l}\sqrt{\epsilon_j}R_{\overline{1_j},\overline{0'}}+\sum_{i=1}^{n}\sqrt{\delta_i}R_{\overline{0},\overline{1'_i}}\right)t\right\}$$

$$=e^{-R_{\overline{0},\overline{0'}}t}+\sum_{j=1}^{l}\sqrt{\epsilon_j}R_{\overline{1_j},\overline{0'}}\,e^{-R_{\overline{0},\overline{0'}}t}+\sum_{i=1}^{r}\sqrt{\delta_i}R_{\overline{0},\overline{1'_i}}\,e^{-R_{\overline{0},\overline{0'}}t}+\cdots.$$

Grouped by degrees $\sqrt{\epsilon_j}$ and $\sqrt{\delta_i}$ we will get:

$$R^{\overline{\epsilon},\overline{\delta'}}\approx R_{\overline{0},\overline{0'}}+\sum_{j=1}^{l}\sqrt{\epsilon_j}R_{\overline{1_j},\overline{0'}}+\sum_{i=1}^{r}\sqrt{\delta_i}R_{\overline{0},\overline{1'_i}},$$

$$R_{\overline{0},\overline{0'}}=-\frac{1}{t}\ln\left(w_{\overline{0},\overline{0'}}\right),\qquad R_{\overline{1_j},\overline{0'}}=\frac{-w_{\overline{1_j},\overline{0'}}}{tw_{\overline{0},\overline{0'}}},\qquad R_{\overline{0},\overline{1'_i}}=\frac{-w_{\overline{0},\overline{1'_i}}}{tw_{\overline{0},\overline{0'}}}.$$

Note: the drawings are built by component in each corresponding timeline, similarly for the two components as in [2].

IV Conclusions

Thus, the studies conducted in the work allow us to draw the following conclusions.

An algorithm for finding the approximate price of derivatives has been developed and explicit formulas have been found for finding their value based on the decomposition of eigen functions and eigenvalues of self-adjoint operators using boundary tasks for singular and regular perturbations. The theorem of estimating the accuracy of derivatives prices approximation is established, on the scales of systems of slow and fast variable factors on which volatility of derivative financial instruments depends.

The general method of finding the approximate price for a wide range of derivatives has been obtained. It has been established that derivative payments can be path-dependent, and the underlying process may exhibit a jump whose intensity depends on multidimensional volatility. The price of options depends on the stochastic multidimensional volatility, which is described by a path-dependent process. Finding the price of derivatives comes down to the task of finding the eigenvalues and eigenfunctions of a particular equation that fits this model.

The approximate price of bonds and their profitability are determined by the methods of spectral theory and wave perturbation theory. The spectral theory and the theory of singular and regular perturbations have been applied to short-term interest rates described by the Vasicek model. The approximate price of the bonds and their profitability are calculated.

The main advantage of the reviewed pricing methodology is that, by combining methods with spectral theory, regular perturbation theory and singular perturbation theory, it reduces to solving equations on finding eigenfunctions and eigenvalues.

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