The Method of Reconstructing Discontinuous Functions Using Projections Data and Finite Fourier Sums

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Abstract. Product quality control is one of the important areas of the production management process. In particular, when designing and manufacturing some particularly important objects, there is a need to check their internal structure for defects in them.

The methods of computed tomography are by far the leading among the methods of flaw detection. Therefore, developing and researching methods for verifying the internal structure of multilayered objects using tomographic methods is an urgent task. This work is dedicated to this task.

This paper investigates the method of approximation of discontinuous functions of two variables by discontinuous splines. These functions describe the internal structure of the 2D body. The unknown parameters are found in them using projections coming from a computer tomograph. It is proposed to use discontinuous splines for the automatic representation of these functions, with known lines of discontinuity of a special form, in the form of a single analytical expression. It is also proposed to use the O. M. Lytvyn method for calculating the Fourier coefficients of two variables using periodic discontinuous splines of one variable and projections. This allows you to submit discontinuous functions in the form of a discontinuous spline sum and a finite Fourier sum. Thus, the proposed method does not require the decomposition of a discontinuous component into a Fourier series. This allows the approximation to be obtained using tomography data without the Gibbs phenomenon.

Keywords: computer tomography, reconstruction, image, discontinuous function, discontinuous spline, sum Fourier.

1 Introduction

As is known [1, 2, 3], the approximation of discontinuous functions of one and many variables by finite Fourier sums leads to the Gibbs phenomenon [4]. This phenomenon also occurs in computer tomography [5-7]. For this case, in the works of Gottlieb and Gustafsson [8, 9], various methods of constructing finite Fourier sums are given; Fourier coefficients are multiplied by the factors determined in a proper way in order to reduce the influence of the Gibbs phenomenon on the final result.

In [10], the method of an approximate representation of the function of two variables by finite Fourier sums was investigated, in which the Fourier coefficients are found helping projections using the direct and inverse Radon transformation This method is known as the Direct Fourier Method (DFM).

To obtain experimental data in the DFM method, a parallel scan scheme is used. The main idea of the DFM method is to find Fourier transforms of projections p and use them to find the Fourier transform of a function f that describes the investigated image.

Numerical implementation includes a discrete Fourier transform p and inverse discrete function transformation f using a fast Fourier transform.

Therefore, to restore the image $N \times N$ the method DFM falls into a number of very fast methods, with the number of arithmetic operations is proportional $N^2 \log N$. But there are two problems that arise with its numerical implementation. The first problem is the need to perform an interpolation procedure in Fourier's space, which is a complicated procedure.

The authors of the DFM method assert that all polynomial interpolation methods are not suitable for this purpose. Therefore, they propose another method of interpolation, constructed using Fourier series and the central section theorem (Sampling Theorem).

The second problem is that in practice contours of images have sharpening, angular points, which leads to an increase in the breaks in the function f. It is well known that in the approximation of this function by finite Fourier sums, the Gibbs phenomenon arises. It generates nonphysical oscillations in the form, that is oscillations, which the original does not have.

The authors of the DFM method claim that they have investigated several methods of combating the Gibbs phenomenon and as a result proposed two different ways to eliminate most of the oscillations.

The first one is a simple application of an exponential filter for Fourier coefficients.

The second way is using the fact that the amplitude of the oscillation is proportional to the jump of the function f. Jumps have different values for typical applications.

For example, a person's skull has a greater density than the brain inside, which leads to the largest jump of function near the surface of the skull.

Also the finite sum of Fejer [1, 2] are used, which has a smoothing effect, but not an increase in the accuracy of the result.

In this paper, it is proposed to generalize the method introduced in [11, 12] to the case of approximating discontinuous functions of two variables using projections and finite Fourier sums for one important class of discontinuous functions. This corresponds to a new approach to the problem of research, which does not lead to the Gibbs phenomenon.

2 Method of finite Fourier sums

The problem of image reconstruction is to restore the function $f(x,y)$ based on known projection data – the values of integrals γ_k along the lines L_k that cross the object of the study:

$$\int_{L_k} f(x,y)\,dl = \gamma_k,\ \ k = \overline{1,M}. \tag{1}$$

In the future we will assume, that the object of research belongs to the square $D = [0,1]^2$. This problem can be interpreted as a problem of studying the density $f(x,y)$ inside a body on the plane Oxy by methods of X-ray computerized tomography.

To solve the problem, the method proposed by O. M. Lytvyn in [12] was used. According to this method, the solution of the task was sought in the form of a Fourier sum.

$$f(x,y) \approx S_{N,N}(x,y) = \sum_{k=-N}^{N}\sum_{l=-N}^{N} F_{k,l} e^{i2\pi(kx+ly)}, \tag{2}$$

where the Fourier coefficients are calculated by the formula

$$F_{k,l} = \iint_D f(x,y) e^{-i2\pi(kx+ly)} dxdy.$$

The peculiarity and advantage of the developed method is that it proposes explicit formulas for the approximate calculation of the Fourier coefficients of the function of two variables by means the values of projections coming from a computer tomograph to a processor.

This led to the solution of the problem to the calculation of integrals. The choice of a system of straight lines by which projective data is given, and hence the form of integrals, and the form of formulas for their calculation, is determined by the values of the indices k and l in the Fourier sum.

To calculate the Fourier coefficients by means of projections, it is necessary to consider separately the cases concerning the signs k, l and their mutual position on the numerical axis ($k > l$, $k < l$, $k = l$). A detailed description of the method is given in [12].

3 Formulation of the problem and idea of the method

In this paper, we propose explicit formulas for the construction of discontinuous splines of two variables with first kind discontinuities on the boundary of a system of embedded one to two two-dimensional domains.

A method of their use is proposed for approximating the discontinuous functions of two variables by finite Fourier sums, in which Fourier coefficients are calculated only for that component of an approximate function, which is a continuous or continuously differentiable function.

The basic idea of the method is as follows: the discontinuous function $f(x, y)$ is replaced by the sum of the discontinuous spline $sp(x, y)$ and the continuous function $F(x, y)$.

$$f(x, y) = sp(x, y) + F(x, y). \tag{3}$$

In this paper, we assume that the function is periodic with period 1 by variable x and period 1 by variable y and has known breaks of the first kind at the boundaries $\Gamma_k : w_k(x, y) = 0$ domains $D_k : w_k(x, y) < 0,\ w_k(x, y) \in C^r(R^2),\quad k = \overline{0, M-1},$ $r = 0,1;\ D_0 \subset D_1 \subset \ldots \subset D_M$.

Theorem 1. If

$$f(x, y) = \begin{cases} f_0(x, y), w_0(x, y) < 0, \\ f_1(x, y), w_0(x, y) \geq 0, w_1(x, y) < 0, \\ \vdots \\ f_k(x, y), w_{k-1}(x, y) \geq 0, w_k(x, y) < 0, \\ \vdots \\ f_M(x, y), w_{M-1}(x, y) \geq 0, \end{cases} \tag{4}$$

then we can represent $f(x, y)$ in the following analytical form:

$$f(x, y) = \frac{1}{2}\left(f_0(x, y) + f_M(x, y) + \sum_{k=0}^{M-1} \left(f_{k+1}(x, y) - f_k(x, y) \right) \frac{\left| w_k(x, y) \right|}{w_k(x, y)} \right). \tag{5}$$

Proving. Let it $M = 1$. Then the function [13]:

$$f(x, y) = \frac{1}{2}\left(f_0(x, y) + f_1(x, y) + \left(f_1(x, y) - f_0(x, y) \right) \frac{\left| w_0(x, y) \right|}{w_0(x, y)} \right)$$

and has properties:

if $w_0(x, y) < 0$, then

$$f(x, y) = \frac{1}{2}\left(f_0(x, y) + f_1(x, y) - \left(f_1(x, y) - f_0(x, y) \right) \right) = f_0(x, y),$$

if $w_0(x, y) > 0$, then

$$f(x,y)=\frac{1}{2}\left(f_0(x,y)+f_1(x,y)+\left(f_1(x,y)-f_0(x,y)\right)\right)=f_1(x,y).$$

That is, the assertion of Theorem 1 for $M=1$ is fulfilled.

Let it $M=2$. Then the function:

$$f(x,y)=\frac{1}{2}\left(f_0(x,y)+f_2(x,y)+\sum_{k=0}^{1}\left(f_{k+1}(x,y)-f_k(x,y)\right)\frac{|w_k(x,y)|}{w_k(x,y)}\right)$$

takes a look

$$f(x,y)=\frac{1}{2}\left(f_0(x,y)+f_1(x,y)+\left(f_1(x,y)-f_0(x,y)\right)\frac{|w_0(x,y)|}{w_0(x,y)}+\right.$$
$$\left.+\left(f_2(x,y)-f_1(x,y)\right)\frac{|w_1(x,y)|}{w_1(x,y)}\right).$$

This feature has the following properties:

if $w_0(x,y)<0$, $w_1(x,y)<0$, then

$$f(x,y)=\frac{1}{2}\left(f_0(x,y)+f_2(x,y)-\left(f_1(x,y)-f_0(x,y)\right)-\right.$$
$$\left.-\left(f_2(x,y)-f_1(x,y)\right)\right)=f_0(x,y),$$

if $w_0(x,y)>0$, $w_1(x,y)<0$, then

$$f(x,y)=\frac{1}{2}\left(f_0(x,y)+f_2(x,y)+\left(f_1(x,y)-f_0(x,y)\right)-\right.$$
$$\left.-\left(f_2(x,y)-f_1(x,y)\right)\right)=f_1(x,y),$$

if $w_0(x,y)>0$, $w_1(x,y)>0$, then

$$f(x,y)=\frac{1}{2}\left(f_0(x,y)+f_2(x,y)+\left(f_1(x,y)-f_0(x,y)\right)+\right.$$
$$\left.+\left(f_2(x,y)-f_1(x,y)\right)\right)=f_2(x,y).$$

That is, the assertion of Theorem 1 for $M=2$ is fulfilled.

Let $M\geq 2$, then formula (5) with $w_0(x,y)<0$, $w_1(x,y)<0$, ..., $w_{M-1}(x,y)<0$ gives equality

$$f(x,y)=\frac{1}{2}\left(f_0(x,y)+f_M(x,y)+\sum_{k=0}^{M-1}\left(f_{k+1}(x,y)-f_k(x,y)\right)(-1)\right)=f_0(x,y).$$

Let $1 \le p \le M-2$ and

$$w_0(x,y) > 0, ..., w_{p-1}(x,y) > 0, w_p(x,y) < 0, ..., w_{M-1}(x,y) < 0.$$

Then formula (5) gives equality

$$f(x,y) = \frac{1}{2}\Bigg(f_0(x,y) + f_M(x,y) + \sum_{k=0}^{p-1}\big(f_{k+1}(x,y) - f_k(x,y)\big)(1) +$$
$$+ \sum_{k=p}^{M-1}\big(f_{k+1}(x,y) - f_k(x,y)\big)(-1)\Bigg) = f_p(x,y).$$

The latter is obtained after the disclosure of the brackets and obvious transformations. Let $w_0(x,y) > 0,\ \ w_1(x,y) > 0, \ldots, w_{M-1}(x,y) > 0.$ Then formula (5) gives:

$$f(x,y) = \frac{1}{2}\left(f_0(x,y) + f_M(x,y) + \sum_{k=0}^{M-1}\big(f_{k+1}(x,y) - f_k(x,y)\big)(1)\right) = f_M(x,y).$$

Thus, all the assertions of Theorem 1 are fulfilled. Theorem 1 is proved.

4 Construction of a split spline

Let's introduce the functions:

$$f_k^-(x,y) = \lim_{\substack{(u,v)\to(x,y)\\ w_k(u,v)<0,\\ w_k(x,y)=0}} f(u,v),\ \ f_k^+(x,y) = \lim_{\substack{(u,v)\to(x,y)\\ w_k(u,v)>0,\\ w_k(x,y)=0}} f(u,v),\ \ k = \overline{0, M-1}.$$

Construct auxiliary functions:

$$h_0(x,y) = f_0^-(x,y),\ \ h_k(x,y) = \frac{-f_{k-1}^+(x,y)w_k(x,y) + f_k^-(x,y)w_{k-1}(x,y)}{w_{k-1}(x,y) - w_k(x,y)},$$

$$k = \overline{1, M-1},\ \ h_M = f_{M-1}^+(x,y).$$

Denote:

$$sp(x,y) = \begin{cases} h_0(x,y),\ \ w_0(x,y) < 0, \\ h_k(x,y),\ \ w_{k-1}(x,y) \ge 0,\ \ w_k(x,y) < 0, \\ k = \overline{1, M-1} \\ h_M(x,y),\ \ w_{M-1}(x,y) \ge 0. \end{cases} \tag{6}$$

Theorem 2. Function $sp(x,y)$ is a discontinued spline with properties:

$$\lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)<0,\\ w_p(x,y)=0}} sp(u,v) = \lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)<0,\\ w_p(x,y)=0}} f(u,v) = f_p^-(x,y),$$

$$\lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)>0,\\ w_p(x,y)=0}} sp(u,v) = \lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)>0,\\ w_p(x,y)=0}} f(u,v) = f_p^+(x,y), \quad p=\overline{0,M-1}.$$

Proving. Let $1 \le p \le M-2$, then

$$\lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)<0,\\ w_p(x,y)=0}} sp(u,v) = \lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)<0,\\ w_p(x,y)=0}} h_p(u,v) = \lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)<0,\\ w_p(x,y)=0}} \frac{-f_{p-1}^+(u,v)w_p(u,v)+f_p^-(u,v)w_{p-1}(u,v)}{w_{p-1}(u,v)-w_p(u,v)} =$$

$$= \lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)<0,\\ w_p(x,y)=0}} \frac{-f_{p-1}^+(u,v)w_p(u,v)}{w_{p-1}(u,v)-w_p(u,v)} + \lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)<0,\\ w_p(x,y)=0}} \frac{f_p^-(u,v)w_{p-1}(u,v)}{w_{p-1}(u,v)-w_p(u,v)} =$$

$$= 0 + f_p^-(x,y) = f_p^-(x,y).$$

Similarly

$$\lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)>0,\\ w_p(x,y)=0}} sp(u,v) = \lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)>0,\\ w_p(x,y)=0}} h_{p+1}(u,v) = \lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)>0,\\ w_p(x,y)=0}} \frac{-f_p^+(u,v)w_{p+1}(u,v)+f_{p+1}^-(u,v)w_p(u,v)}{w_p(u,v)-w_{p+1}(u,v)} =$$

$$= \lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)>0,\\ w_p(x,y)=0}} \frac{-f_p^+(u,v)w_{p+1}(u,v)}{w_p(u,v)-w_{p+1}(u,v)} + \lim_{\substack{(u,v)\to(x,y)\\ w_p(u,v)>0,\\ w_p(x,y)=0}} \frac{f_{p+1}^-(u,v)w_p(u,v)}{w_p(u,v)-w_{p+1}(u,v)} =$$

$$= f_p^+(x,y) + 0 = f_p^+(x,y).$$

Thus, the function $sp(x,y)$ has the same unilateral boundaries as the function $f(x,y)$ on the lines $\Gamma_p : w_p(x,y)=0,\ p=\overline{1,M-2}$. Let $p=0$. Then

$$\lim_{\substack{(u,v)\to(x,y)\\ w_0(u,v)<0,\\ w_0(x,y)=0}} sp(u,v) = \lim_{\substack{(u,v)\to(x,y)\\ w_0(u,v)<0,\\ w_0(x,y)=0}} h_0(u,v) = \lim_{\substack{(u,v)\to(x,y)\\ w_0(u,v)<0,\\ w_0(x,y)=0}} f_0^-(u,v) = f_0^-(x,y),$$

$$\lim_{\substack{(u,v)\to(x,y)\\ w_0(u,v)>0,\\ w_0(x,y)=0}} sp(u,v) = \lim_{\substack{(u,v)\to(x,y)\\ w_0(u,v)>0,\\ w_0(x,y)=0}} h_1(u,v) = \lim_{\substack{(u,v)\to(x,y)\\ w_0(u,v)>0,\\ w_0(x,y)=0}} \frac{-f_0^+(u,v)w_1(u,v)+f_1^-(u,v)w_0(u,v)}{w_0(u,v)-w_1(u,v)} =$$

$$= \lim_{\substack{(u,v)\to(x,y)\\ w_0(u,v)>0,\\ w_0(x,y)=0}} \frac{-f_0^+(u,v)w_1(u,v)}{w_0(u,v)-w_1(u,v)} + \lim_{\substack{(u,v)\to(x,y)\\ w_0(u,v)>0,\\ w_0(x,y)=0}} \frac{f_1^-(u,v)w_0(u,v)}{w_0(u,v)-w_1(u,v)} =$$

$$= f_0^+(x,y)+0 = f_0^+(x,y).$$

Thus, the function $sp(x,y)$ has the same unilateral boundaries on the lines $\Gamma_p : w_p(x,y)=0,\ p=\overline{0, M-2}$ as the function $f(x,y)$ and the same breaks of the first kind. To prove the assertions of the theorem for $p = M-1$, we give the following chain of equalities.

$$\lim_{\substack{(u,v)\to(x,y)\\ w_{M-1}(u,v)<0,\\ w_{M-1}(x,y)=0}} sp(u,v) = \lim_{\substack{(u,v)\to(x,y)\\ w_{M-1}(u,v)<0,\\ w_{M-1}(x,y)=0}} h_{M-1}(u,v) =$$

$$= \lim_{\substack{(u,v)\to(x,y)\\ w_{M-1}(u,v)<0,\\ w_{M-1}(x,y)=0}} \frac{-f_{M-2}^+(u,v)w_{M-1}(u,v)+f_{M-1}^-(u,v)w_{M-2}(u,v)}{w_{M-2}(u,v)-w_{M-1}(u,v)} =$$

$$= \lim_{\substack{(u,v)\to(x,y)\\ w_{M-1}(u,v)<0,\\ w_{M-1}(x,y)=0}} \frac{-f_{M-2}^+(u,v)w_{M-1}(u,v)}{w_{M-2}(u,v)-w_{M-1}(u,v)} + \lim_{\substack{(u,v)\to(x,y)\\ w_{M-1}(u,v)<0,\\ w_{M-1}(x,y)=0}} \frac{f_{M-1}^-(u,v)w_{M-2}(u,v)}{w_{M-2}(u,v)-w_{M-1}(u,v)} =$$

$$= 0 + f_{M-1}^-(x,y) = f_{M-1}^-(x,y).$$

$$\lim_{\substack{(u,v)\to(x,y)\\ w_{M-1}(u,v)>0,\\ w_{M-1}(x,y)=0}} sp(u,v) = \lim_{\substack{(u,v)\to(x,y)\\ w_{M-1}(u,v)>0,\\ w_{M-1}(x,y)=0}} h_M(u,v) = \lim_{\substack{(u,v)\to(x,y)\\ w_{M-1}(u,v)>0,\\ w_{M-1}(x,y)=0}} f_{M-1}^+(u,v) = f_{M-1}^+(x,y).$$

Theorem 2 is proved.

In the case if the function $f_{k+1}^+(x,y) - f_k^-(x,y) \neq 0$ is in some parts of at least one line $w_k(x,y)=0$, then its decomposition in the finite Fourier sum is accompanied by the Gibbs phenomenon. For this case, in the writings of Gottlieb and Gustafsson [8, 9] various methods of constructing finite Fourier sums are given; Fourier coefficients are multiplied by the corresponding factors.

5 Implementation of the method

In this article we propose a solution to the problem of approximating discontinuous functions using of the finite Fourier sums as follows. We give $f(x,y)$ in the form of the sum of a discontinuous spline $sp(x,y),$ which has the same unilateral boundaries on the lines of the discontinuity as the approximate function $f(x,y),$ and the finite Fourier sum $T_N(x,y),$ which approximates the difference between the function $f(x,y)$ and the indicated spline.

In this article we propose a solution to the problem of approximating discontinuous functions using of the finite Fourier sums as follows

$$\varphi(x,y) = f(x,y) - sp(x,y),$$

$$\varphi(x,y) \approx T_N(x,y).$$

For further use the following statement.

Theorem 3. The function $\varphi(x,y)$ belongs to the class of continuous periodic functions, i. e.

$$\varphi(x,y) \in \tilde{C}[0,1]^2 : \varphi(x,y) \in C[0,1]^2 : \varphi(0,y) = \varphi(1,y), \varphi(x,0) = \varphi(x,1).$$

The proof follows from the fact that, according to Theorem 2, the function $\varphi(x,y)$ on the lines of the discontinuity will have unilateral boundaries equal to zero. That is, they are continuous throughout the region $[0,1]^2$.

Theorem 4. If a function $f(x,y)$ is nonperiodic and has first-kind breaks only on lines inside the domain $[0,1]^2$, then function:

$$\tilde{f}(x,y) = f(x,y) - \psi(x,y),$$

$$\psi(x,y) = f(0,y)(1-x) + f(1,y)x + f(x,0)(1-y) +$$

$$+ f(x,1)y - \left[f(0,0)(1-y) + f(0,1)y\right](1-x) - \left[f(1,0)(1-y) + f(1,1)y\right]x$$

is periodic with period 1 for both variables, if it periodically extends to the entire plane Oxy.

Proving. Find traces [11] of the function $\psi(x,y)$ on all four sides of the square $[0,1]^2$. As a result we will receive:

$$f(0,y) = f(0,y) - \psi(0,y) = f(0,y) - f(0,y) = 0,$$

$$f(1,y) = f(1,y) - \psi(1,y) = f(1,y) - f(1,y) = 0,$$

$$\tilde{f}(x,0) = f(x,0) - \psi(x,0) = f(x,0) - f(x,0) = 0,$$

$$\tilde{f}(x,1) = f(x,1) - \psi(x,1) = f(x,1) - f(x,1) = 0.$$

Thus, the function $\tilde{f}(x,y)$, that we consider to be extended periodically to the entire plane Oxy, will satisfy the conditions:

$$\tilde{f}(x+T,y) = \tilde{f}(x,y),\ \tilde{f}(x,y+T) = \tilde{f}(x,y),\ T = 1.$$

This means that the function $\tilde{f}(x,y)$ is periodic.

Theorem 4 is proved.

Remark. In this way, we can represent each nonperiodic function $f(x,y)$ in the form:

$$f(x,y) = \psi(x,y) + \tilde{f}(x,y)$$

and approximate the Fourier sums only $\tilde{f}(x,y),$ which allows a periodic extension to the entire plane Oxy.

6 A general algorithm for the approximation of a discontinuous function with the help of discontinuous splines and projective data

We now formulate a general algorithm for approximation a discontinuous function $f(x,y)$ with the help of discontinuous splines and projections coming from a computer tomograph, considering the lines of discontinuity and one-sided boundaries known.

Step 1. We construct a discontinuous spline as a function $sp(x,y),$ considering also known as boundaries $f_k^+(x,y)$ and $f_k^-(x,y)$ in the form:

$$f_k^+(x,y) = f_{k+1}\left(x - w_k^*(x,y)\frac{\partial w_k^*(x,y)}{\partial x}, y - w_k^*(x,y)\frac{\partial w_k^*(x,y)}{\partial y} \right),$$

$$f_k^-(x,y) = f_k\left(x - w_k^*(x,y)\frac{\partial w_k^*(x,y)}{\partial x}, y - w_k^*(x,y)\frac{\partial w_k^*(x,y)}{\partial y}\right),$$

$$w_k^*(x,y) = \frac{w_k(x,y)}{-\sqrt{\left(\frac{\partial w_k(x,y)}{\partial x}\right)^2 + \left(\frac{\partial w_k(x,y)}{\partial y}\right)^2}}.$$

Step 2. Find the difference:

$$\varphi(x,y) = f(x,y) - sp(x,y).$$

Step 3. For the function $\varphi(x,y)$ we find:

$$\tilde{\varphi}(x,y) = \varphi(x,y) - \psi(x,y),$$

where

$$\psi(x,y) = \varphi(0,y)(1-x) + \varphi(1,y)x + \varphi(x,0)(1-y) +$$

$$+\varphi(x,1)y - \left[\varphi(0,0)(1-y) + \varphi(0,1)y\right](1-x) - \left[\varphi(1,0)(1-y) + \varphi(1,1)y\right]x.$$

Step 4. Submit the function $f(x,y)$ as a sum:

$$f(x,y) = sp(x,y) + \psi(x,y) + \varphi(x,y). \tag{7}$$

Remark. In formula (7), the sum of first two terms is a no periodic discontinuous component of a function $f(x,y)$ on a given system of lines; the third term is a component of a function that allows a periodic extension to the entire plane Oxy.

Since $\tilde{\varphi}(x,y)$ is a continuous, periodic, and also if derivatives $\tilde{\varphi}^{(1,0)}(x,y)$, $\tilde{\varphi}^{(0,1)}(x,y)$ are continuous, it can be approximately represented as a finite Fourier sum, the Fourier coefficients of which are found by means of projections by the method of O. M. Lytvyn [12].

Note, that from a computer tomograph projections come from an unknown function $f(x,y)$, therefore projections of the function $\tilde{\varphi}(x,y)$ will be based on the formula:

$$\int\limits_{kx+ly=t_p} \tilde{\varphi}(x,y)ds = \int\limits_{kx+ly=t_p} f(x,y)ds - \int\limits_{kx+ly=t_p} \left[sp(x,y) + \psi(x,y)\right]ds. \tag{8}$$

7 Example

Denote:

$$w(x,y)=\sqrt{(x-0.5)^2+(y-0.5)^2}.$$

If the lines $w_k(x,y)=0, k=\overline{0,M-1}$ are circles $w_k(x,y)\equiv w(x,y)-r_k=0$, $k=\overline{0,M-1}$, that is:

$$D_0: w(x,y)<r_0,\ldots,\ D_k: w(x,y)<r_k,\ k=\overline{1,M-1},\ D_M=[0,1]^2$$

then the functions $f_k(x,y)$ will have the following boundary properties:

$$f_k^-(x,y)=f_k\left(r_k\cos\theta, r_k\sin\theta\right), k=\overline{0,M-1},$$

$$f_k^+(x,y)=f_{k+1}\left(r_k\cos\theta, r_k\sin\theta\right), k=\overline{0,M-1},$$

$$r\geq 0, 0\leq\theta\leq 2\pi.$$

Here r and θ coordinates of the point (x,y) in the polar coordinate system with the center at the point $(0.5, 0.5)$.

In this case, the formula (6) for $sp(x,y)$ the functions $h_k(x,y)$ can be written as:

$$h_0(x,y)=f_0\left(r_0\cos\theta, r_0\sin\theta\right),\ldots,h_k(x,y)=$$

$$=f_k\left(r_{k-1}\cos\theta, r_{k-1}\sin\theta\right)\frac{w(x,y)-r_k}{r_{k-1}-r_k}+f_k\left(r_k\cos\theta, r_k\sin\theta\right)\frac{w(x,y)-r_{k-1}}{r_k-r_{k-1}},\ k=\overline{1,M-1},$$

$$h_M(x,y)=f_M\left(r_{M-1}\cos\theta, r_{M-1}\sin\theta\right).$$

The results of the computational experiment and their analysis are planned to be presented in further developments.

8 Conclusions

1. In this paper, we propose a general method for the approximation of an unknown function $f(x,y)$ by means of projections, coming from a computer tomograph, for the case, when known line break function $f(x,y)$ and known unilateral boundaries functions for $f_k(x,y)$.

2. To introduce the proposed method to practice, the authors plan to develop and study the method of finding the lines of discontinuity and unilateral boundaries of an unknown function $f(x, y)$ on the specified lines.

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