The Computational Modeling: Dynamic Quantum Model Approach

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Abstract. After S. Smale's works it became clear that in smooth dynamics the system of a general form is not structurally stable and therefore there is no strict mathematical basis for modeling and computational analysis of systems. The contradiction appeared in science: according to physicists dynamics is simple and universal. The solution to this problem was proposed based on the construction of dynamic quantum models (DQM). From the assumption that quantum effects are caused by unrecoverable "white noise", a certain mathematical model of quantum mechanics already follows and is essentially unambiguous. On the other hand, in this model spectral problems are reduced tothe usual perturbation theory of smooth dynamical systems. Thus, the construction of such models can be considered as an asymptotic method for solving spectral problems. But the definition of DQM is not formally related to Hamiltonian systems. DQM is defined and constructed universally for both Hamiltonian systems and systems with the truth function. As a result, for example, quantization with the Bohr-Sommerfeld condition also extends to systems with a truth function. Hopefully DQM opens for new applications. The most important is to seek assistance and cooperation in future research.

Keywords: modeling, computer simulation, structural stability, dynamical system, dynamic quantum model, Markov cascade.

1 Introduction

Increasingly, processes and systems are researched or developed through computer simulations and this trend is likely to continue [1]. Computational modeling has been used in physics, chemistry and related engineering for many decades because this is the only way the equations can be solved at all [2]. It consists from two steps: (i) modeling, i.e. finding a model description of a real system, and (ii) solving the resulting model equations using computational methods [3].

But if an arbitrarily small perturbation of the model leads to a qualitatively different picture of the dynamics, then such a model is not applicable to the real process: strictly speaking, perturbations are included in the definition of a model [4]. Also computational methods inevitably lead to errors of discretization and rounding in calculations [5]. Therefore traditionally the stability of a model with respect to rela-

tively small changes is a necessary condition for its correctness [6]. The qualitative invariance of a mathematical model under small perturbations is usually called structural stability [7].

However, in S. Smale's works [8] was shown, that there exist smooth dynamic systems whose neighborhoods do not contain any structurally stable system. This meant that there was no rigorous mathematical basis for modeling and computational analysis. The contradiction has appeared in science, because physicists believe that the dynamics is simple and universal [9].

The solution to this problem was proposed in [10] based on the construction of dynamic quantum models (DQM). It turned out that taking into account random fluctuations, necessary for the transition to the quantum model of reality, allows us to return in fact to the simple picture of A. Poincare's dynamics: a dense set of structurally stable systems.

DQM is so named because for Hamiltonian systems it is simply related to the corresponding Schrödinger equation. From the assumption that quantum effects are caused by unrecoverable "white noise", a certain mathematical model of quantum mechanics already follows and is essentially unambiguous [11]. Dynamics in it is described by Markov cascades (time is discrete). This model is simply connected with the traditional one: there is a simple correspondence between Markov cascades and quasisolutions of the corresponding Schrödinger equation. Thus, in a sense, DQM is a bridge between the formal calculus of quantum mechanics and the intuitive vision of physicists. On the other hand, in this model spectral problems are reduced to the usual perturbation theory of smooth dynamical systems. Thus, the construction of such models can be considered as an asymptotic method for solving spectral problems. This paper gives an example of such approach to the one-dimensional system with the quasiperiodic potential (Proposition 2).

But the definition of DQM is not formally related to Hamiltonian systems; it is defined for any ordinary differential equation or any diffeomorphism on any smooth Riemannian manifold. Hopefully this opens the way for absolutely new applications. For example, for applications to dynamic systems that using logical operations: algorithms, theorems, software applications. The use of fuzzy logic in DQM is in principle completely natural and even almost inevitable. Let $J = J(z)$ be a given smooth function on phase space ($0 \le J(z) \le 1$), equal to 1 on the true trajectory and 0 outside some neighborhood of it; we can interpret it as the function of truth. In this paper, DQM is defined and constructed universally for both Hamiltonian systems and systems with the truth function J. As a result, for example, the point of the DQM spectrum is interpreted exactly as the average value of truth for approximate logical conclusions. Quantization with the Bohr-Sommerfeld condition also extends to systems with a truth function (Proposition 1).

But the reverse is also true. If we construct an approximate model of given theorem or software application using the training of a neural network, then we get the DQM of these objects. With further training, these DQMs will approach the original object (in other words, they will converge to it according semiclassical limit). Perhaps this will allow a new approach to the problems of solvability in logic, there is some analogy with the theorem of the equivalence of structural stability and hyperbolicity proved

in [10]. DQM of systems with logical operations are always uniformly limited by the number of operations (see Section 2.2) and then for them solvability is not in doubt. Then everything depends on the semiclassical limit, more precisely, on the uniformity of the structural stability of DQM.

The **paper goal** is 1) to build the foundations of the theory of dynamic quantum models (DQM); 2) to demonstrate the application of this theory for spectral problems of quantum mechanics.

The paper is organized as follows: in part 2 we synthesize the dynamic quantum model (DQM); in part 3 we demonstrate the application of DQM for spectral problems of quantum mechanics; part 4 concludes.

We had to omit proofs of some propositions in order to fit the paper format.

2 The Dynamic Quantum Model: Basic Definitions

2.1 DQM Definition

Let $p(x)$ be an n-dimensional smooth vector field on an n-dimensional smooth Riemannian manifold M, where $x(x_1, x_2, ..., x_n)$ are local Euclidean coordinates on M, $p_i(x) \in C^\infty(R^n)$ ($i = 1,...,n$). On each phase curve $x(t) \in M$ of the dynamical system generated by this vector field

$$\frac{dx_i}{dt} = p_i(x), \quad (i = 1,...,n) \qquad (1)$$

consider the integral of the "shortened action" $s(t) = \int\limits_{x(t)} p(x)dx = \int\limits_0^t \|p(\tau)\|^2 d\tau$,

where $\|p(\tau)\|^2 = \sum\limits_{i=1}^{n} p_i^2(\tau)$. The value of $s(t)$ on each curve $x(t)$, which is different from a fixed point, is diffeomorphically expressed in t and is called "optical time". Let ρ be a metric such that $s(t) = \int\limits_{x(t)} d\rho : \; d\rho = \|p(t)\|^2 dt$. The following is the heuristic derivation or explanation of the definition of dynamic quantum model (Definition 1).

So, the distance d traveled by a point along the path of (1) during the time Δt is equal to $d = \int\limits_0^{\Delta t} \|p(\tau)\| d\tau = \|p(t_c)\| \cdot \Delta t$, where $p_c = p(t_0)$ is the average value $(0 \le t_0 \le \Delta t)$. (Of course this is with a single bypass of trajectory during Δt: turning points are the special case). Further, we assume that the fluctuations generate "white noise" $\xi(t)$, acting on the configuration space with the dispersion $D\xi(t) = \sigma^2 t$, where the diffusion coefficient σ^2 is constant over the considered time interval. It

will take some time Δt, until the point moves to a distance d from the initial position, which exceeds the mean square error caused by $\xi(t)$ during the time Δt, i.e. $\|p_c\|\Delta t$ will exceed $\sqrt{\sigma^2 \Delta t}$. With such a minimal Δt $\|p_c\|\Delta t = \sigma\sqrt{\Delta t}$, whence $\sigma^2 = \|p_c\|^2 \Delta t$ and therefore

$$\Delta t = \frac{\sigma^2}{\|p_c\|^2}, \qquad d = \|p_c\|\Delta t = \sigma\sqrt{\Delta t} = \frac{\sigma^2}{\|p_c\|} \cdot \tag{2}$$

Here by assumption Δt is the minimal time interval after which it becomes possible to make a new measurement, the difference from which will exceed the error, i.e. get a significantly different measurement. Owing to (2) $\sigma^2 = \|p_c\|^2 \Delta t \approx \int_0^{\Delta t} \|p(\tau)\|^2 d\tau = s(\Delta t)$. Thus 1) the time interval between the nearest significant measurements is unchanged on the optical time scale and is equal to σ^2. (In other words, the distance between them in the metric ρ is equal to σ^2). 2) During this time "white noise" $\xi(t)$ generates an irremovable random error, the standard deviation of which is equal to the distance d between the nearest significant measurements along the trajectory.

Now suppose that the configuration space is one-dimensional, there is a turn point on the segment of the trajectory, the initial position is located near the turn point and move towards it. To pass a segment of the path, the ends of which are significantly different, the point must reach the turning point, and then (after turning) pass another segment of ρ – length σ^2. How much will the distance to the nearest significantly different measurement increase? The points on the segment ρ – lengths σ^2, including the turning point, are indistinguishable among themselves, only their average value is important. Therefore, the points in the interval between $\frac{3}{2}\sigma^2$ and $\frac{1}{2}\sigma^2$ to the turning point in time σ^2 will move to the segment centered just on the turning point; and after a while until the next significantly different position. So, only for points on a segment with length $\frac{1}{2}\sigma^2$ to a turning point, the distance to the nearest significantly different measurement will increase: on average by $\frac{1}{4}\sigma^2$.

In the general case, one should take into account those points that move towards the caustic K – the set of singular points at which the direction of motion changes and are located at ρ – distance $\frac{1}{2}\sigma^2$ from K. To pass a segment of the path, the ends of which are significantly different, such points must reach K, and then (turning) to pass another segment of ρ – length σ^2 in a new direction. Therefore, the travel time

for these points will increase on average by $\frac{1}{4}\sigma^2$. The jump in the time interval at turning points on the optical time scale is a quantum-mechanical phenomenon traditionally taken into account by means of the Morse index (when establishing a connection with the Schrödinger equation, it turns out that $\frac{1}{4}\sigma^2 = \frac{\pi h}{2}$, i.e. $\sigma^2 = 2\pi h$). Generally speaking, there may be features on the caustic other than turn points, however, such a singular point splits into several turn points with an arbitrarily small stir [6]. At such points, the number $\frac{1}{4}\sigma^2 \cdot \mu = \mu\frac{\pi h}{2}$ is added to the ρ - length, where μ is the Morse index of the singular point. Here μ is an integer equal to the number of turning points that arose during small stir and passed in the positive direction to the caustic minus the number of turning points traveled in the negative direction.

So, a dynamic quantum model first shifts each point along the phase curve of a given dynamic system over the optical time σ^2 (or ρ – length σ^2) and in a neighborhood of the caustic this shift increases abruptly by $\frac{1}{4}\mu\sigma^2 = \mu\frac{\pi h}{2}$. And then randomly shifts on a distance not exceeding the length of the trajectory from the original to the new point. The following rigorous definition summarizes this description. The definition of a dynamic quantum model is given for an arbitrary dynamic system (1) on an arbitrary compact Riemannian manifold M.

Definition 1. By a dynamic quantum model (DQM) for dynamical system (1) we mean the Markov cascade with the transition function) $P(x, A)$, which associates with each point x of the trajectory of (1) and an open subset A of the configuration space probability of getting from x to A in one iteration:

$$P(x, A) = \frac{1}{\sqrt{2\pi t}\sigma} \int_A e^{-(y-Gx)^2/2\sigma^2 t} dy ,$$

where t is the shift time from x to Gx along the path of the ρ-length $2\pi h$ or $2\pi h + \mu\frac{\pi h}{2}$ in the neighborhood of the caustic, $\sigma^2 = 2\pi h$. Given the initial distribution, we obtain a Markov process P with this initial distribution and the transition function $P(y, A)$: if μ_t is the distribution at time t, Δt is the lag between the two nearest measurements, then the DQM sets new distribution $P(\mu_t) = \mu_{t+\Delta t}$ at time $t + \Delta t$.

2.2 DQM Eigenvalues and Markov Deviations

Our goal is to determine pure states and eigenvalues of DQM. And now, along with the discreteness of the measurement process, its limited time will be essential. Of course, the measurement process cannot continue indefinitely, but here its duration is dictated by the very definition of DQM. Namely, the duration of the measurement, in

principle, cannot exceed on order $\frac{1}{h}$ since further the measurement errors with dispersion $\sigma^2 t$ (where the diffusion coefficient σ^2 is small of order h) are no longer small and the notion of trajectory loses its meaning. (And you can only talk about the average values for the ensemble, as in statistical physics). Therefore, we limit the time to a certain limiting value T of order $\frac{1}{h}$ ($T \sim \frac{1}{h}$): $T \le \frac{B}{h}$, where B > 0 is a constant. (In general, we say that the quantity u = u (h) in a DQM is of order h^k ($u \sim h^k$ or $u = O(h^k)$), if $|u| \le Ch^k$. And u = u (h) is exactly of the order h^k ($u \sim = h^k$), if $ch^k \le |u| \le Ch^k$ for some constants C, c > 0).

Let us now consider the problem: it is required to experimentally determine the location of the point at which a given point of the phase space (x; p) passes under the action of dynamical system (1) with the greatest possible accuracy. Note that the time interval between the two nearest significant measurements is $\sigma^2 = 2\pi h$ on the optical time scale or of order $\frac{h}{\|p\|^2}$ on the usual scale in the neighborhood of the point (x; p) (see (2)). In addition, the duration of the measurements is limited by the value of $T \sim \frac{1}{h}$. Therefore, the position of the point in the next significant measurement can be obtained, in principle, only a finite number of times, namely $N \sim T \Big/ \frac{h}{\|p\|^2} \sim \frac{\|p\|^2}{h^2}$ numbers. But this position is determined each time with the unrecoverable error, the standard deviation of which is equal to $d \approx \frac{\sigma^2}{\|p\|}$ (see (2)) , i.e. $d \sim \frac{h}{\|p\|}$. Therefore, the averaging of all such measurements, i.e. then the best approximation to the unperturbed value, which in principle can be achieved, differs from it by an order of value $\frac{d}{\sqrt{N}} \sim \frac{h}{\|p\|} \Big/ \sqrt{\frac{\|p\|^2}{h^2}} \sim \frac{h^2}{\|p\|^2}$. Such a deviation is given at each point z of the phase space, i.e. defines a vector field $Z(z)$. So,

1. h^2 is the least in order error, with which the coordinates of the point in he phase space can be known, and thus the values observed at the point. Values whose difference in order of value is less than h^2 are not experimentally distinguishable.
2. As a result of averaging the maximum number of maximally accurate measurements, we arrive to a dynamical system generated not by the diffeomorphism G, but by its perturbation $\overline{G} = G + Z$.

Definition 2. The Markov deviation $Z(z)$ is a smooth vector field on phase space such that 1)

$$\|Z(z)\| \le B\frac{h^2}{\|p\|^2}$$

($B > 0$) is a constant of dynamical system (1) (i.e., the length $Z(z)$ does not exceed in order $\frac{h^2}{\|p\|^2}$ for all points z of the phase space);

2) for any initial point $z_0 = z(t_0)$ on the phase curve $z(t)$ of the dynamical system (1) and the time instant $\bar{t}$ in optical time ($|\bar{t}| < T \sim \frac{1}{h}$) we have

$$|\int_{t_0}^{\bar{t}}(\int_{t_0}^{t}(Z(z(s),e(s))ds)dt| \le Bh^2, \quad (3)$$

where $e(t)$ is the unit normal vector of the closed phase curve at the point $z(t)$, $B > 0$ is the constant of dynamical system (1).

Property (3) of the Markov deviation is due to the fact that, by construction, the vector $Z(z(t))$ has a random orientation, therefore, the pluses and minuses of the accumulations of its projections on the unit vectors are compensated. Therefore, the integral of the accumulation of projections along the phase curve is experimentally indistinguishable from zero.

If instead of a given time limit $T \sim \frac{1}{h}$ we take $T_1 \sim \frac{1}{h}$ ($T_1 \neq T$), then we obtain another Markov deviation; similarly, when replacing the zero point in time. So the Markov deviation is a smooth vector field that depends on the parameters; further, it can be assumed to be a general view field.

2.3 DQM Pure States and Eigenvalues. Quantization of Spectrum in DQM.

The physical meaning of the eigenvalues is that these are all values of energy that can be the result of reliable, i.e. the most accurate measurement (ideally of the order of h^2). But as a result of the most accurate experiments, as we have seen, in reality the dynamics is studied not of the diffeomorphism G, but of its perturbation $\overline{G} = G + Z$. Let $J = J(z)$ be a given smooth function on phase space. We can interpret it as the Hamiltonian (energy in the phase space) or as a function of truth ($0 \le J(z) \le 1$), equal to 1 on the true trajectory and 0 outside some neighborhood of it. Given the irremovable errors of the Markov deviation, the discreteness of the measurement process and its limited time, we arrive at the maximum number of the most accurate measurements $\frac{1}{N_t}\sum_{i=0}^{N_t} J(\overline{G}^i z)$, where z is the point of phase space, Z is the general view Markov deviation, $\overline{G} = G + Z$ is a diffeomorphism, N_t is the maximum number of significantly different measurements over time $t \le T$.

In terms of meaning the eigenvalue of the spectrum is associated with some pure stationary state of the dynamical system: any reliable measurement in this state leads to an acceptable error (ideally with a maximum accuracy of the order of h^2) and only

to this value. But in a DQM any point in the phase space is always known with the irremovable error of order h (see (2)). As a result, we average generally speaking over trajectories with a starting point not z, but some $\tilde{z}$, that is distant from z by a distance of the order h. Therefore, the carrier of the state associated with some eigenvalue of α must contain a ball with a diameter of exactly the order of h: otherwise, any reliable experiment with a significant probability will lead to values significantly different from α. Hence

Definition 3. Let $\overline{G} = G + Z$, where Z is a general view Markov deviation; N_t is the number of all iterations of the diffeomorphism in time t; α is a real number. Let D = $D_{\alpha h}$ be the set of points z of the phase space such that for all sufficiently large t < T ~ = $\frac{1}{h}$

$$\left| \frac{1}{N_t} \sum_{i=0}^{N_t} J(\overline{G}^i z) - \alpha \right| < Bh^2 ,$$

where B is a constant. Then, if for any Z of a general view and sufficiently small h, the set $D_{\alpha h}$ contains a ball with a diameter of exactly the order of h, then α will be called the eigenvalue of the DQM for dynamical system (1), and $D_{\alpha h}$ will be called the carrier of the pure state corresponding to this eigenvalue.

Thus, all points of a DQM spectrum are formally determined only with an accuracy of the order of h^2, but this corresponds precisely to their meaning. By definition, the domain $D_{\alpha h}$ is an open $\overline{G}$ - invariant subset of the phase space.

So, to define DQM means to set: 1) the Markov process in accordance with Definition 1; 2) the Markov deviation Z of general view or, what is the same, diffeomorphism $\overline{G} = G + Z$ in accordance with Definition 2.

Consider the two–dimensional dynamical system (1), the compact phase space Λ of which is filled with closed phase curves. After the smooth change of variables, in canonical coordinates, this is the dynamics of uniform rotation along concentric circles. If we interpret J as a function of truth, then its values on each circle, concentric to the true path (true circle), are constants (i.e., they do not depend on a point on this circle). At the semantic level, with such interpretation, we are talking about transitions to equivalent propositions.

Proposition 1. The DQM eigenvalues of the given dynamical system, with accuracy of the order h^2, are equal to the values of J (z) on the phase circles in Λ, the ρ - length of which satisfies the Bohr - Sommerfeld condition

$$\int p(x)\,dx = \pi I = \pi h(n + \frac{1}{2})$$

and only they.

3 The Spectrum of Schrödinger Equation with Quasiperiodic Potential and DQM

Consider the Schrödinger equation

$$ih\frac{\partial\psi}{\partial t} = H(\psi) = -h^2\frac{\partial^2\psi}{\partial x^2} + U(x)\psi \qquad (4)$$

with the quasiperiodic potential $U(x) = \cos x + \varepsilon \cdot \cos \lambda x$ $(\varepsilon, \lambda > 0)$. For this Schrödinger equation we construct its dynamic quantum model (DQM), i.e. in accordance with Definition 1, the perturbation of the corresponding classical system by the Markov process. There is a simple connection between these Markov processes and the quasisolutions of the Schrödinger equation. The connection between the Schrödinger equation and the corresponding DQM is based on a following modification of the traditional asymptotic expansion of solution (4).

Lemma 1. Let $\psi(x,t) = \varphi(x,t)e^{\frac{i}{h}S(x,t)}$ is some quasisolution (4), i.e.

$$ih\frac{\partial\psi}{\partial t} = H(\psi) = -h^2\frac{\partial^2\psi}{\partial x^2} + U(x,t)\psi + O(h^2),$$

where $\varphi(x,t) = \varphi_h(x,t)$ and $S(x,t) = S_h(x,t)$ are real-valued. Then

$$\left(\frac{\partial S(x,t)}{\partial t} + \left(\frac{\partial S(x,t)}{\partial x}\right)^2 + U(x) - h\frac{\partial^2 S(x,t)}{\partial x^2}\right)\varphi(x,t) -$$

$$-h\left(\frac{\partial\varphi(x,t)}{\partial t} + 2\frac{\partial S(x,t)}{\partial x}\frac{\partial\varphi(x,t)}{\partial x} - h\frac{\partial^2\varphi(x,t)}{\partial x^2}\right) = O(h^2). \qquad (5)$$

The converse is also true: if $S(x,t) = S_h(x,t)$ and $\varphi(x,t) = \varphi_h(x,t)$ is some quasisolution (5), then $\psi(x,t) = \varphi(x,t)e^{\frac{i}{h}S(x,t)}$ is some quasisolution (4).

Now we show how the DQM of the Schrödinger equation is constructed from its two-dimensional classical analogue

$$\frac{dx}{dt} = p \qquad \frac{dp}{dt} = -\frac{dU}{dx}. \qquad (6)$$

Let us consider separately the terms of equation (5): Hamilton-Jacobi perturbed equation

$$\frac{\partial S(x,t)}{\partial t} + \left(\frac{\partial S(x,t)}{\partial x}\right)^2 + U(x) - h\frac{\partial^2 S(x,t)}{\partial x^2} = 0 \qquad (7)$$

and the Einstein-Fokker-Planck diffusion equation

$$\frac{\partial\varphi(x,t)}{\partial t} + 2\frac{\partial S(x,t)}{\partial x}\frac{\partial\varphi(x,t)}{\partial x} - h\frac{\partial^2\varphi(x,t)}{\partial x^2} = 0, \qquad (8)$$

in which $S(x,t)$ is determined from (7). If for $S(x,t)$ (7) holds with an accuracy of order h^2 and for $\varphi(x,t)$ (8) holds with an accuracy of order h, then the function $\psi(x,t) = \varphi(x,t)e^{\frac{i}{h}S(x,t)}$ satisfies (4) with an accuracy of the order h^2, i.e. it is a quasisolution of the Schrödinger equation.

According to section 2.3 the DQM of the Schrödinger equation is determined by 1) the smooth dynamics from (7) and 2) its stochastic perturbation from (8).

1) Consider the first two terms of the asymptotic expansion of the solution $S(x, t)$ of equation (7) with respect to the small parameter h:

$$S(x, t) = S_0(x,t) + hS_1(x,t) + O(h^2). \quad (9)$$

Substituting this expansion into (7), in a first approximation, we obtain the Hamilton - Jacobie equation

$$\frac{\partial S_0(x,t)}{\partial t} + \left(\frac{\partial S_0(x,t)}{\partial x}\right)^2 + U(x) = 0$$

On the trajectory (6) γ with the energy level $\alpha = -\frac{\partial S_0(x,t)}{\partial t}$, this equation takes the form

$$\left(\frac{\partial S_0(x,t)}{\partial x}\right)^2 + U(x) = \alpha. \quad (10)$$

The velocity on γ is $\frac{\partial S_0(x,t)}{\partial x} = \pm\sqrt{\alpha - U(x)} = \pm p_0(x)$ regardless of t. Assuming that $S_1(x)$ is also independent of t, we find from (7) outside the turning points $\bar{x}$ ($\frac{\partial S_0(\bar{x},t)}{\partial x} = 0$)

$$\frac{\partial S_1}{\partial x} = \frac{1}{2}\frac{\partial^2 S_0}{\partial x^2} \Big/ \frac{\partial S_0}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}\ln\left|\frac{\partial S_0}{\partial x}\right|. \quad (11)$$

Now the smooth DQM dynamics corresponding to the perturbed Hamilton - Jacobi equation (7) is defined by the system:

$$\frac{dx}{dt} = \frac{\partial S}{\partial x} = \frac{\partial S_0}{\partial x} + h\frac{\partial S_1}{\partial x} = \pm p_0(x) + h\frac{\partial S_1}{\partial x}; \qquad \frac{dp}{dt} = -\frac{dU}{dx},$$

where in h - neighborhoods of turning points we smooth $S_1(x)$, preserving it with an accuracy of the order of h.

2) DQM also includes stochastic disturbance, "white noise" in the configuration space. If each point x_0, in accordance with the smooth dynamics of the DQM, moves along the trajectory (6) during time Δt to the point $x = x_0 + \int_{t_0}^{t_0+\Delta t} \frac{\partial S}{\partial x}(x(\tau),\tau)d\tau$; and at this time scattering occurs, defined by the normal distribution with the dispersion Δt, then [12] the distribution density $\varphi(x, t)$ at the time $t = t_0 + \Delta t$ is a solution of the diffusion equation

$$\frac{\partial}{\partial t}\varphi(x,t) = -\frac{\partial S(x,t)}{\partial x}\frac{\partial \varphi(x,t)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 \varphi(x,t)}{\partial x^2}. \quad (12)$$

This equation is equivalent to (8): if $\varphi(x, t)$ is a solution of (8), then this is a solution of (12) for $\sigma^2 = h$.

Let H (γ) be the energy value on the trajectory (6) γ. Consider the integral of "shortened action" on γ (t) = (x(t), p (t)) : s(t) = $\int_{\gamma(t)} p(x)dx = \int_0^t \|p(\tau)\|^2 d\tau$. On each trajectory γ (t) other than a fixed point, the quantity s(t) is diffeomorphically expressed through t and is called the optical time. Let ρ be a metric such that s(t) = $\int_{r(t)} d\rho$: $d\rho = \|p(t)\|^2 dt$. For a closed trajectory γ (γ(0) = γ($\bar{\tau}$)) $2\pi I(\gamma)$ = s($\bar{\tau}$) = $\int_\gamma p(x)dx$ is ρ - length of this trajectory, i.e. optical time of its bypass.

Lemma 2. For all sufficiently small h, the eigenvalue α of equation (4) with accuracy of the order of h^2 is equal to the value of the energy $H(\gamma)$ on some closed path (6) γ. These and only these trajectories γ are such, for which the Bohr - Sommerfeld condition holds with accuracy of order h^2:

$$I(\gamma) = h(n + \frac{1}{2}) \qquad (n = 0, 1, \ldots). \qquad (13)$$

Proof. Suppose that, on a closed trajectory (11) γ, (13) holds. Using the DQM construction, we show that $\alpha = H(\gamma)$ is the eigenvalue of equation (4) with an accuracy of the order of h^2. Let $S_0(x, t)$ be the action on γ, i.e. the solution of the Hamilton - Jacobi equation (10) on γ $\left(\frac{\partial S_0(x,t)}{\partial x}\right)^2 + U(x) = \alpha = H(\gamma)$. As the initial condition for $S_0(x, t)$ we take

$$S_0(x,0) = \int_{x_0}^{x} p_0(y)dy, \qquad (14)$$

where $p_0(y) = \sqrt{\alpha - U(y)}$ is velocity on γ, $p_0(x_0) = 0$, i.e. x_0 is the abscissa of the turning point on γ. Then

$$S_0(x,t) = S_0(x,0) - \alpha t + k_t \cdot \pi I, \qquad (15)$$

where k_t is the number of turns in time t, πI is the ρ - length by γ between turning points. And for $S(x, t) = S_0(x,t) + hS_1(x,t)$, where in accordance with the construction of DQM $S_1(x, t) = S_1(x)$ from (11), equality (12) is approximately satisfied:

$$\frac{\partial S(x,t)}{\partial t} + \left(\frac{\partial S(x,t)}{\partial x}\right)^2 + U(x) - h\frac{\partial^2 S(x,t)}{\partial x^2} = O(h^2). \qquad (16)$$

We assume that the velocity function $p_0(x)$ on the closed curve γ without loss of generality is analytic with an accuracy of the order of h^2, otherwise approximating $p_0(x)$ on γ analytic with this accuracy. Then outside the neighborhood of the turning points this is also true for $S_0(x,t)$, $S_1(x)$, and in the neighborhood of the turning

point $S_1(x)$ we can continue analytically. When turning, the sign of $\frac{\partial S_0(x,t)}{\partial x} = \pm p_0(x)$ changes, which in view of (11), implies the transition $S_1(x)$ to another branch of the logarithm. This means adding to the value of the logarithm, $\frac{1}{2}\pi i$ to S_1 and then $\frac{1}{2}\pi h i$ to S. Therefore, in view of (15)

$$S(x,t) = S_0(x,t) + hS_1(x,t) = S(x,0) - \alpha t + k_t(\pi I + \frac{\pi h i}{2}), \qquad (17)$$

where, as in (17), k_t is the number of turns in time t.

By definition, DQM also includes a stochastic perturbation defined on γ by equation (12) with a diffusion coefficient $\sigma^2 = h$. Let $\varphi_0(x)$ be the density of the stationary state for such a process. Then, for such dynamics at the initial density $\varphi_0(x) = \varphi(x, 0)$, the solution of (12) $\varphi(x, t)$ is different from $\varphi_0(x)$ by order h for any finite time t: $\varphi(x, t) - \varphi_0(x) \sim h$ (due to scattering by "white noise" outside the limits of γ). Thus

$$\frac{\partial S(x,t)}{\partial x}\frac{\partial \varphi_0(x)}{\partial x} - \frac{h}{2}\frac{\partial^2 \varphi_0(x)}{\partial x^2} = O(h),$$

which is equivalent to this option (8):

$$\frac{\partial \varphi_0(x)}{\partial t} + 2\frac{\partial S(x,t)}{\partial x}\frac{\partial \varphi_0(x)}{\partial x} - h\frac{\partial^2 \varphi_0(x)}{\partial x^2} = O(h). \qquad (18)$$

Now replace h by hi in (18), (17) and (16). Then

$$\frac{\partial \varphi_0(x)}{\partial t} + 2\frac{\partial S(x,t)}{\partial x}\frac{\partial \varphi_0(x)}{\partial x} - ih\frac{\partial^2 \varphi_0(x)}{\partial x^2} = O(h), \qquad (19)$$

$$\frac{\partial S(x,t)}{\partial t} + \left(\frac{\partial S(x,t)}{\partial x}\right)^2 + U(x) - hi\frac{\partial^2 S(x,t)}{\partial x^2} = O(h^2), \qquad (20)$$

$$S(x,t) = S(x,0) - \alpha t + k_t(\pi I - \frac{\pi h}{2}). \qquad (21)$$

Let on γ $\bar{S}(x) = S(x,0)$ to the first turning point and $\bar{S}(x) = S(x,0) + \pi I - \frac{\pi h}{2}$ then to the second. Then $S(x,t) = \bar{S}(x) - \alpha t + m_t(2\pi I - \pi h)$, where m_t is the number of complete walks γ in time t, $k_t = 2m_t$ or $k_t = 2m_t + 1$. Hence

$$e^{\frac{i}{h}S(x,t)} = e^{\frac{i}{h}(S(x,0) - \alpha t + k_t(\pi I - \frac{\pi h}{2}))} = e^{\frac{i}{h}(\bar{S}(x) - \alpha t + m_t(2\pi I - \pi h))} = e^{\frac{i}{h}(\bar{S}(x) - \alpha t)} \qquad (22)$$

since by condition (13) $2\pi I - \pi h = 2\pi h(n + \frac{1}{2}) - \pi h = 2\pi h n$ (n = 0, 1, …). Let

$$\psi(x,t) = \varphi_0(x)e^{\frac{i}{h}S(x,t)} = \varphi_0(x)e^{\frac{i}{h}(\bar{S}(x) - \alpha t)} \in L^2. \qquad (23)$$

In virtue of (19) and (20)

$$(\frac{\partial S(x,t)}{\partial t}+\left(\frac{\partial S(x,t)}{\partial x}\right)^2+U(x)-hi\frac{\partial^2 S(x,t)}{\partial x^2})\varphi(x,t)-$$

$$-hi\cdot(\frac{\partial \varphi_0(x)}{\partial t}+2\frac{\partial S(x,t)}{\partial x}\frac{\partial \varphi_0(x)}{\partial x}-ih\frac{\partial^2 \varphi_0(x)}{\partial x^2})=O(h^2). \quad (24)$$

Continuing this motion, we multiply by $e^{\frac{i}{h}S(x,t)}$ both sides of (24). But then from Lemma 1, with the same accuracy of order h^2, (4) also holds for the function (23):

$$ih\frac{\partial \psi}{\partial t}=-h^2\frac{\partial^2 \psi}{\partial x^2}+U(x)\psi+O(h^2).$$

Hence in view of (23)

$$\alpha\cdot\varphi_0(x)e^{\frac{i}{h}\bar{S}(x)}=\mathbf{H}(\varphi_0(x)e^{\frac{i}{h}\bar{S}(x)})+O(h^2), \quad (25)$$

where $\mathbf{H}=-h^2\frac{\partial^2}{\partial x^2}+U(x)$ is Schrödinger operator. Equality(25) means [7] that the pair $(\alpha,\ \varphi_0(x)e^{\frac{i}{h}\bar{S}(x)})$ is quasisolution (quasimode) of the stationary Schrödinger equation $\alpha\psi=\mathbf{H}\psi$ with a small parameter h^2. It follows that α is an eigenvalue of the operator **H** with order accuracy h^2.

Indeed, assuming that d is the distance from α to the spectrum of the operator **H**, $\|\ \|$ is the norm in L^2, and $R_\alpha=(\alpha E-H)^{-1}$ is resolvent **H**, we obtain for $\psi=\varphi_0(x)e^{\frac{i}{h}\bar{S}(x)}$ in virtue of (25)

$$\frac{1}{d}=\|R_\alpha\|\geq\frac{\|R_\alpha(\alpha E-H)\psi\|}{\|(\alpha E-H)\psi\|}=\frac{\|\psi\|}{\|(\alpha E-H)\psi\|}\geq\frac{1}{Bh^2},$$

where $B>0$, whence $d\leq B\,h^2$ and then the lemma follows from the Weyl criterion, QED.

Proposition 2. For Schrödinger equation (9) with a quasiperiodic potential $U(x)$ for all sufficiently small h in the scattering region, the spectrum is continuous for all ε, λ> 0. In the region of vibrational motions

1. the spectrum is discrete if λ is rational.
2. If λ is irrational, then the point spectrum with increasing ε monotonically expands to its closure – small segments around the original (at ε = 0) eigenvalues and for ε of order h occupies this entire region. With a further increase in ε to an order of $1/h$, such a picture of the spectrum is preserved, only the region of vibrational motions expands ultimately to the entire space. For ε of order $1/h$, with increasing ε, the point spectrum monotonously narrows to segments converging to discrete points.

Proof. At $\varepsilon=0$, the potential $U(x)=\cos x$ is periodic. In this case, in the region $|H(\gamma)|\leq 1$ of the phase space of the dynamical system (6) (in the region of vibrational

motions), the trajectories γ are closed. By virtue of Lemma 2, the spectrum in this region is discrete, and its points with an accuracy of the order of h^2 are equal to the energy values H (γ) on such trajectories γ for which condition (13) is satisfied. In the region |H(γ)| > 1 (the scattering region), the trajectories γ are unbounded and unclosed, so the spectrum here is continuous and, with accuracy of the order of h^2, is equal to the energy H(γ) on the trajectories from this region. For the scattering region, this will always be true with increasing ε.

As ε increases from zero, the periodicity of the potential $U(x)$ disappears, on the interval $[2\pi k; 2\pi(k+1)]$ it has the form

$$U(x) = \cos(x) + \varepsilon \cos(2\pi k\lambda + \lambda x)\ (x \in [0; 2\pi]). \qquad (\mathbf{26})$$

If λ is rational and $\lambda = \frac{p}{q}$, then there are no more than q different such potentials and for each of them the spectrum is discrete. For irrational λ, the points $2\pi k\lambda$ modulo 2π everywhere densely fill the segment [0; 2π] and their closure coincides with the segment.

Consider on the segment [0; 2π] dynamic system (6) with potential $\overline{U}(x) = \cos(x) + \cos(\mu + x)$, smooth in the parameters ε ≥ 0 and μ ∈ [0; 2π], For $\mu = 2\pi k\lambda \pmod{2\pi}$ for the potential $\overline{U}$ ρ – length $I = I$ (H, ε, μ) of closed trajectory with given energy H and ε > 0 is equal to ρ – length trajectories for the initial potential U (26) with the same H and ε on the interval $[2\pi k; 2\pi(k+1)]$.

As for potential $\overline{U}$ $\frac{dI(H, \varepsilon, \mu)}{dH} > 0$ at ε = 0, then this is also true for sufficiently small $\varepsilon > 0$. Let the inverse function $f(I) = f(I, \mu, \varepsilon)$ associates the ρ - length I of a closed trajectory with its energy level H. According to Lemma 2, for $I = h(n + \frac{1}{2})$ (n = 0, 1, …) all this $H = f(I, \mu, \varepsilon)$ and there only are the eigenvalues (4) with order of accuracy h^2. Since the dependence of f on μ is continuous, then for fixed I and $\varepsilon > 0$ image of the segment [0; 2π] along the μ axis is some segment K_I along the H axis.

The length of the segment K_I smoothly depends on ε, and for sufficiently small h and ε of order h it increases approximately linearly with increasing ε. Therefore, with increasing ε, the union of the segments K_I over all $I = h(n + \frac{1}{2})$ (n = 0, 1, …) will cover all values of energy in the field of oscillatory movements.

All the above considerations remain valid with a further increase in ε, and therefore the spectral picture does not change, only the range of vibrational motions expands. As ε → ∞, this region occupies the entire phase space. Moreover, in the spectral pattern in this region, a process occurs that is opposite to what was when ε changed from zero to h. Namely, the point spectrum narrows to segments converging in the limit to discrete points corresponding to phase curves with ρ – length $I = h(n + \frac{1}{2})$ (n = 0,

1, …) for potential cos λ*x*. This follows from the symmetry of the plots $0 < \varepsilon < h$ and $1/h < \varepsilon < \infty$: divide equation (4) by ε

$$\frac{ih}{\varepsilon}\frac{\partial\psi}{\partial t} = -\frac{h^2}{\varepsilon}\frac{\partial^2\psi}{\partial x^2} + (\frac{1}{\varepsilon}\cos x + \cos\lambda x)\psi \quad .$$

As a result of the replacement of variables $\widetilde{t} = \varepsilon t,\ \widetilde{x} = \sqrt{\varepsilon}x,\ \widetilde{\varepsilon} = \frac{1}{\varepsilon}$, we obtain

$$ih\frac{\partial\psi}{\partial\widetilde{t}} = -h^2\frac{\partial^2\psi}{\partial\widetilde{x}^2} + (\widetilde{\varepsilon}\cos x + \cos\lambda x)\psi \cdot$$

At $\varepsilon \to \infty$, i.e. $\widetilde{\varepsilon} \to 0$ we back to the periodic case with potential $U(x) = \cos\lambda x$, QED.

4 Conclusion

The structural stability of a mathematical model is a necessary condition for its correctness. But after S. Smale's works it became clear that in smooth dynamics the system of a general form is not structurally stable and therefore there is no strict mathematical basis for modeling and computational analysis of systems. The contradiction appeared in science: according to physicists dynamics is simple and universal.

The solution to this problem was proposed based on the construction of dynamic quantum models (DQM). From the assumption that quantum effects are caused by unrecoverable “white noise”, a certain mathematical model of quantum mechanics already follows and is essentially unambiguous. This model is simply connected with the traditional one. Construction of such models can be considered as an asymptotic method for solving spectral problems, for example, for the one-dimensional system with the quasiperiodic potential.

But the definition of DQM is not formally related to Hamiltonian systems; it is defined for any ordinary differential equation or any diffeomorphism on any smooth Riemannian manifold. DQM is defined and constructed universally for both Hamiltonian systems and systems with the truth function. As a result, for example, the point of the DQM spectrum is interpreted exactly as the average value of truth from approximate logical conclusions. Quantization with the Bohr-Sommerfeld condition also extends to systems with a truth function.

Hopefully this will allow a new approach to the problems of dynamical systems that using logical operations: algorithms, theorems, software applications.

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