Plane Tangent to Quasi-Rotation Surface*

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Abstract. The analysis of a surface generated by quasi-rotation of a straight line around a circle is provided in the present paper. The considered case features a straight generatrix belonging to the plane of a circular axis of quasi-rotation and intersecting it in two points. A geometric method of determination of a point belonging to a surface given its projection on the axis plane is demonstrated. Geometric construction of the curves of intersection between the considered surface and a conic surface is presented. A method of determination of points belonging to the considered surface as points belonging to the curve of intersection of two conic surfaces is acquired. Step-by-step constructions illustrating the solution of the problem of determination of a plane tangent to the considered surface in a given point are provided. The problem is solved through the methods of descriptive geometry. Every construction is performed according to an analytic algorithm, not involving approximate methods of determination of the sought points. The construction is carried out in a CAD system through the use of tools "straight line by two points" and "circle by center and point". The presented solution to the defined problem is connected to the solution to the problem of determination of the rays reflected from the considered surface. The results of the paper expose the geometric properties of surfaces of quasi-rotation. The provided constructions can serve as the basis for the research of optical properties of the considered surfaces.

Keywords: Quasi-Rotation, Surface of Quasi-Rotation, Tangent Plane.

1 Introduction

Quasi-rotation is a geometric correspondence between a point located in common plane with a conic, which is the axis of quasi-rotation, and, in general case, four circles located in planes perpendicular to the plane of the conic. If the axis of quasi-rotation constitutes a circle, then the number of circles corresponding to a point is reduced to

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two. This degeneracy is due to the fact that a circle, unlike other conics, has only one real focus. Fig. 1 depicts the initial conditions and the solution to the task of quasi-rotation of a point *L* around an axis *i*.

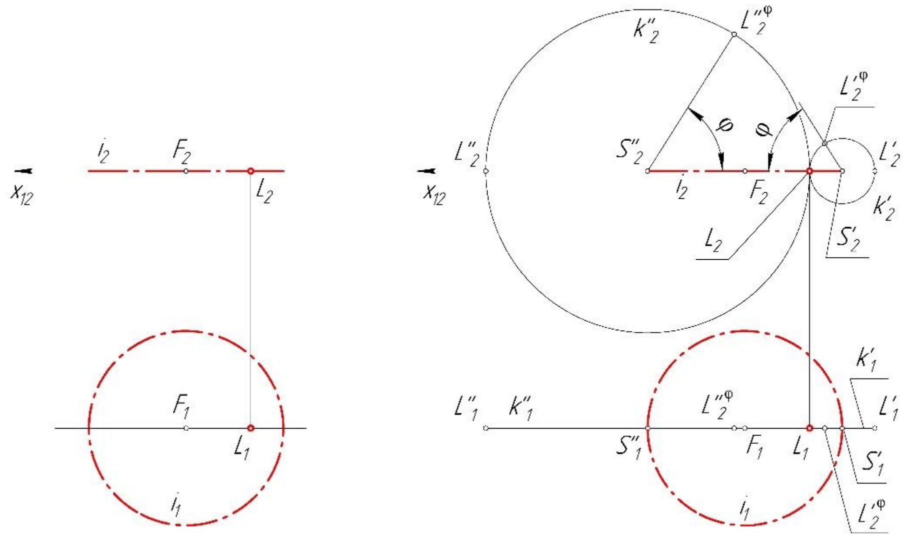

Fig. 1. Quasi-rotation of a point *L* around an axis *i*: the initial conditions, construction of quasi-rotation of the point *L* around *i* on angle *φ* and on 180° angle.

The geometric objects on Fig. 1 are designated according to the system of designation accepted in paper [1] and applied in description of quasi-rotation. For example, the point *L'* constitutes the result of quasi-rotation of the point *L* around the axis *i* on 180° angle around the closest center of rotation *S'*; *k'* is the circle correspondent to this quasi-rotation centered at the point *S'*. The quasi-rotation correspondence is detailed in papers [1, 2, 3, 4, 5]. Quasi-rotation, just as regular rotation, can be applied in solution to various problems of formation. The geometric properties of surfaces of quasi-rotation were considered in paper [5]. Further to the research of geometrical properties of the surfaces of quasi-rotation and their possible practical applications, the solution to the task of construction of a plane tangent to a surface of quasi-rotation in an arbitrary point is considered in the present paper.

2 Problem Definition

The determinant of a surface of quasi-rotation is of the following general form:

$$\alpha(l, i)[li = QRT_i(l)], \quad (1)$$

where *α* represents the surface of quasi-rotation, *l* represents the generatrix, *i* represents the axis of quasi-rotation, *QRTi* represents the Quasi-Rotation Transformation apparatus. A surface generated upon quasi-rotation of a straight line around a circle as the axis is considered in the present paper. The determinant (1) of such surface is of the following form:

$$\alpha(l(l_1, l_2), i(i_1, i_2))[li \ = \ QRT_i(l)], \tag{2}$$

where l_1 and l_2 represent projections of the line, i_1 and i_2 represent projections of the axis. Fig. 2 depicts two projections of the generatrix *l* and axis *i* belonging to a common plane and the horizontal projection of a point *A*.

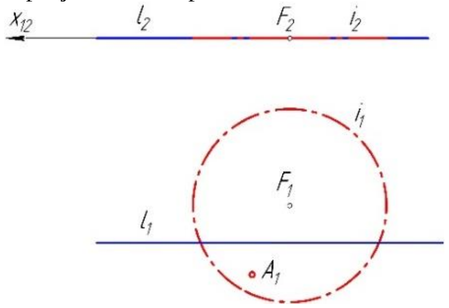

Fig. 2. The initial condition to the problem of generation of a plane tangent to the surface of quasi-rotation in a point *A*.

The problem is to construct a plane tangent to the surface of quasi-rotation in its point *A*. The geometric part of the determinant of the surface α (2) is defined by its two projections (see Fig. 2). The point *A* is defined by its projection and the condition of belonging to the surface α. Let us put the initial conditions of the problem into symbolic form:

Given: $l(l_1, l_2), i(i_1, i_2), A(A_1, A \epsilon \alpha), \alpha \left(l(l_1, l_2), i(i_1, i_2)\right)[li \ = \ QRT_i(l)]$,

Find: $\Gamma =?, \Gamma \overline{\cap} \ \alpha = A$,

where Γ represents the sought tangent plane, symbol «$\overline{\cap}$» represents tangency of the geometric objects. Therefore, «$\Gamma \overline{\cap} \ \alpha = A$» means that the plane Γ is tangent to the surface α in the point *A*.

3 Theory

The apparatus *QRTi* provides two circles correspondent to any point of generatrix *l* (see Fig. 1). Therefore, the surface α defined by the initial condition of the problem consists of two sheets β' and β'' tangent to each other along the line *l*.

$$\alpha(\beta', \beta''); \ \beta' \ \overline{\cap} \ \beta'' = l \tag{3}$$

These sheets also intersect each other and themselves. This makes the surface structure difficult to perceive.

In order to better understand the shape of this surface, it is convenient to consider images of its sheets separately. Figures 3*a* and 3*b* depict halves of the sheets β' and β'' respectively. Judging by the top view, the point *A* belongs to the sheet β'. Let us find the front projection of the point *A* through the condition ($A \in \beta$'). The 3D models depicted on Fig. 3 are acquired constructively by means of a CAD system tool called "surface by a network of curves".

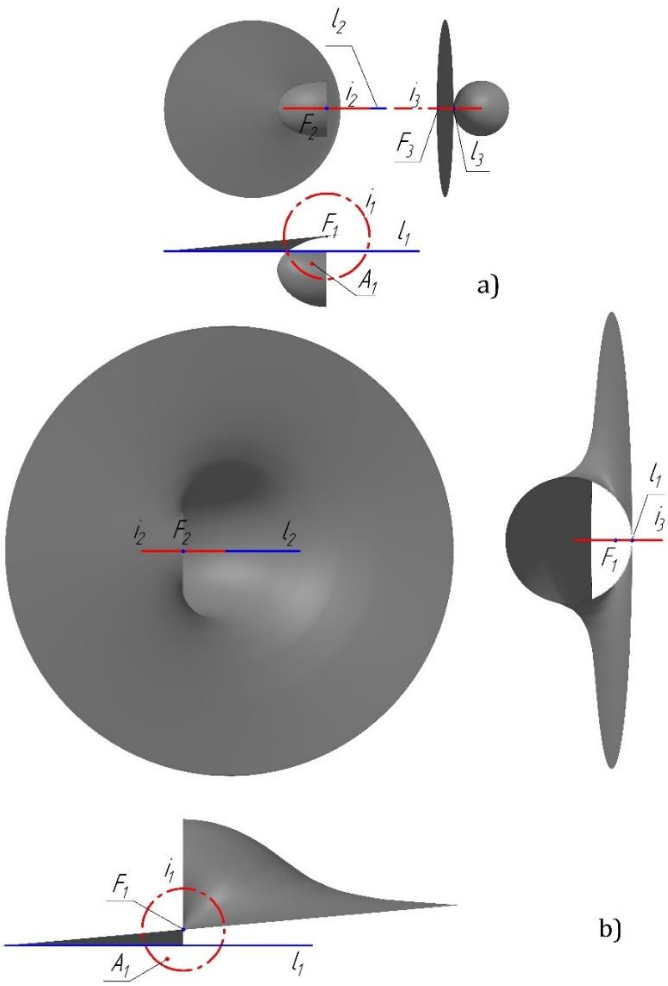

Fig. 3. Rendering of halves of the sheets of the surface α: a) sheet β'; b) sheet β''.

A network of circles is constructed according to Fig. 4. The horizontal projections of the circles k_i are determined using the algorithm realized on Fig. 1. These acquired models are not a part of the solution to the problem, but rather provide visual aid.

As seen from Fig. 3*a*, the point *A* can be located either on the upper, or on the lower part of the sheet β'. Let us consider the locus of point *A*, at which it will be visible on the top view. The constructions for the opposite case are symmetrical to the presented below.

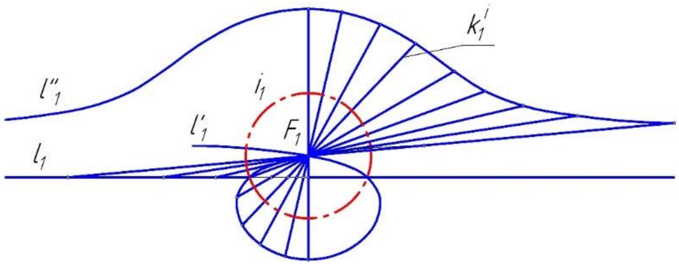

Fig. 4. Construction of horizontal projections of circles required for 3D modeling of the sheets of the surface α.

The solution to the defined problem applies construction operations logically justified by the quasi-rotation apparatus.

Fig. 5 depicts the constructions defining the locus of the point L quasi-rotated on angle φ around the axis i. The sequence of actions is depicted through numeration of projection lines and indication of direction of the respective construction. Unlike Fig.1, on Fig. 5 the point L is quasi-rotated in a plane that is not parallel to a projection plane. Therefore, the point L is rotated until it matches the plane parallel to the projection plane, where the center F of the circle i is located (step 1). The further construction is performed in a similar to Fig. 1 way. Then the backwards rotation is applied (step 6). Any geometric object belonging to the axis plane W upon quasi-rotation on angle φ is located on the surface of the cone ω^{φ}.

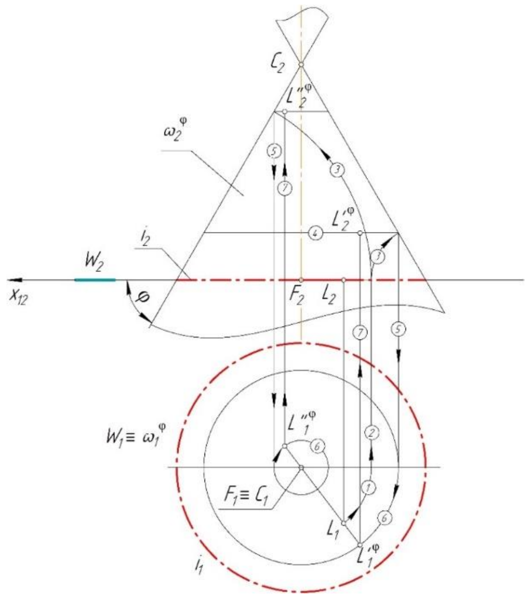

Fig. 5. Quasi-rotation of plane W and its point L on angle φ.

4 Solution to the Problem

Let us acquire the front projection of the point *A* (see Fig. 6). In order to do that, let us project the point *A* on plane π4 ($\pi_4 \perp \pi_1$, $\pi_4 || AF$). The following correlations describe the constructions presented on Fig. 6:

$$\Sigma \perp \pi_1,\ AF \in \Sigma,\ \Sigma \cap i = S',\ \Sigma \cap l = A^l,\ A \in k' \tag{4}$$

The projection A_4 is located on projection of the circle k'_4. The circle *k*' is a result of quasi-rotation of the point A^l. Therefore, by applying the algorithm presented on Fig. 1, we acquire the second projection of the point *A*.

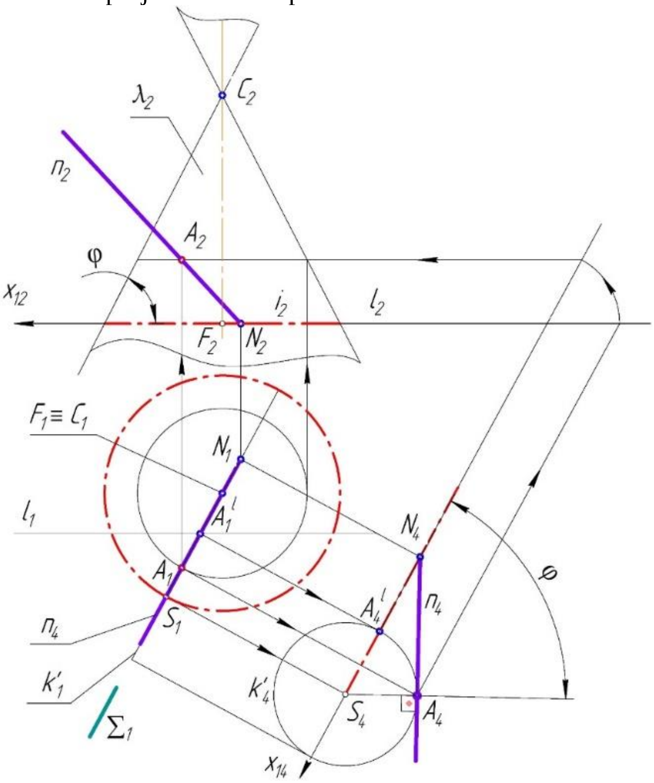

Fig. 6. Determination of the second projection of the point *A* and construction of the tangent *n*.

It follows from the definition of a plane tangent to a surface in any regular point, that in order to solve the considered problem, it is necessary and sufficient to construct the projections of two straight lines *n* and *m* tangent to a certain curve belonging to the surface α and passing through the point *A*. Since $A^l \in l$, the circle *k*' belongs to the surface α generated through quasi-rotation of the line *l*, therefore, line *n* ($n \bar{\cap} k'=A$) (see Fig. 6) belongs to the sought tangent plane Γ ($n \subset \Gamma$).

Fig. 7 depicts section of the surface α by a cone λ ($\alpha \cap \lambda = p$). It is also correct that the spatial curve *p* is a result of quasi-rotation of the line *l* on angle φ. The projection of the curve p on π_1 constitutes conchoid of Nicomedes. The projection of the curve p on π_3 constitutes a hyperbola with asymptotes d and q. The cone λ constitutes an image of plane π_1 upon its quasi-rotation on angle φ around the circle i. The angle φ was found trough construction of the second projection of the point A, therefore $A \in \lambda$, $p \in \lambda$, $A \in p$.

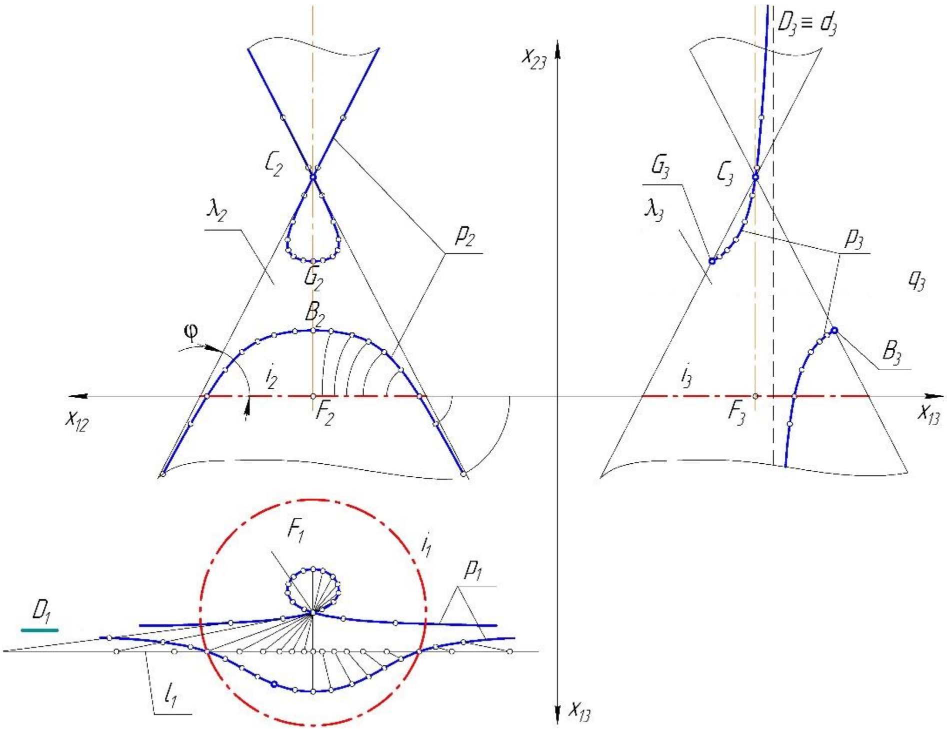

Fig. 7. Quasi-rotation of the straight line *l* around axis *i* on angle φ.

In order to solve the problem, it is required to find line m tangent to the curve p in the point A. The sought straight line m belongs to the plane tangent to the cone λ in the point A.

A curve of the fourth order belonging to surface of a cone is a result of intersection between such cone and another surface of the second order. Analysis of the images presented on Fig. 7 allows us to assume that the curve p is a curve of intersection between two cones.

Fig. 8 depicts a pair of cones λ and ξ, curve of intersection of which visually resembles the curve p. The axes of both cones belong to a common plane with points B and G of to the curve p. Generatrix b of the cone ξ passes through the points B and G. Generatrix u of the cone ξ intersects the curve p in the vertex C of the cone λ. The projections of the generatrix u and the curve p on plane CGB are tangent to each other in point C. The sought straight line m belongs to the plane tangent to the cone ξ in the point A. Therefore, the sought line m is the result of intersection between two planes that are tangent to respective cones λ and ξ in the point A.

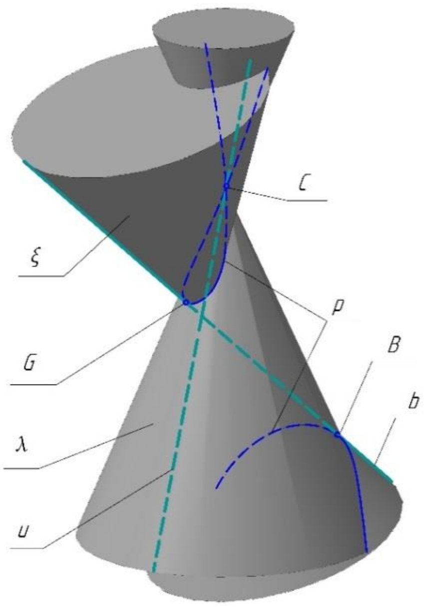

Fig. 8. A pair of cones intersecting each other along the curve *p*.

Let us find the cone ξ (see Fig. 9). In order to do that, it is sufficient to find projections of its outline generatrices *b* and *u*. The straight line *b* contains the points *G* and *B*. Its construction is not a problem of concern. The curve *u* is constructed with the condition that its profile projection is tangent to profile projection of the curve *p* in the point C_3 ($u_3 \overline{\cap} p_3 = C_3$). In other words, it is required to construct a straight line tangent to p_3 in point C_3. In order to do that, let us interpret the curve p_3 as a projection of a flat hyperbola p^τ located on the surface of the cone τ. Outline generatrices of the cone τ are asymptotes *d* and *q* of hyperbola p^τ, axis of the cone τ is the line *s*. The plane of base *v* of the cone τ contains a point *T* ($T \in p$, $T \in v$) that was precisely found through quasi-rotation of the point T^I. The radius of the base of the cone τ is equal to the distance between the point *J* and the line *s*. Let us now find the projection of the cone τ on plane π_5 ($\pi_5 \perp \pi_3$, $\pi_5 \perp s$). The line tangent to the hyperbola p^τ belongs to the plane *W* ($p^\tau \in W$) as well as the plane *CVQ* ($CVQ \overline{\cap} \tau = QC$). Construction of a plane tangent to a cone is a problem of basic course of descriptive geometry. Intersection between the planes *W* and *CVQ* results in a straight line that is tangent to the hyperbola p^τ in the point *C* ($W \cap CVQ = CU$). The line *CU* is the second outline generatrix *u* of the cone ξ.

As a result, the problem of construction of the straight line *m* is reduced to construction of a line that passes through the point *A* and is tangent to both of the cones λ and ξ ($m \overline{\cap} \lambda = \mathrm{A}$, $m \overline{\cap} \xi = A$). Obviously, the straight line *m* is a result of intersection of planes tangent to the respective cones in point *A*. Fig. 10 depicts the construction of such planes ($CHM \overline{\cap} \lambda$, $EKM \overline{\cap} \xi$). These planes intersect each other along the line *m* ($AM = CHM \cap EKM = m$). The line *m* belongs to the sought plane Γ ($m \subset \Gamma$).

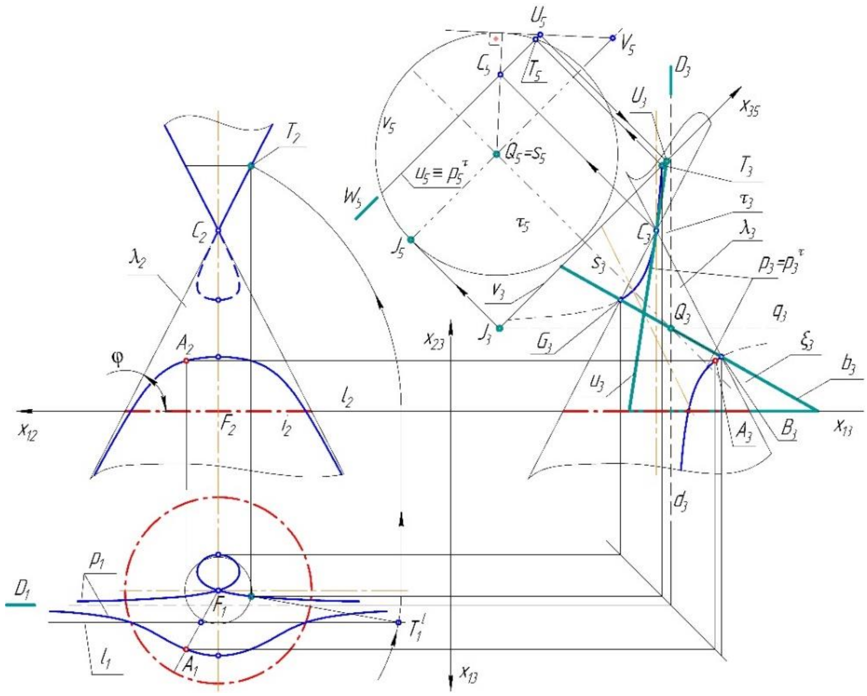

Fig. 9. Projectional determination of the cone ξ.

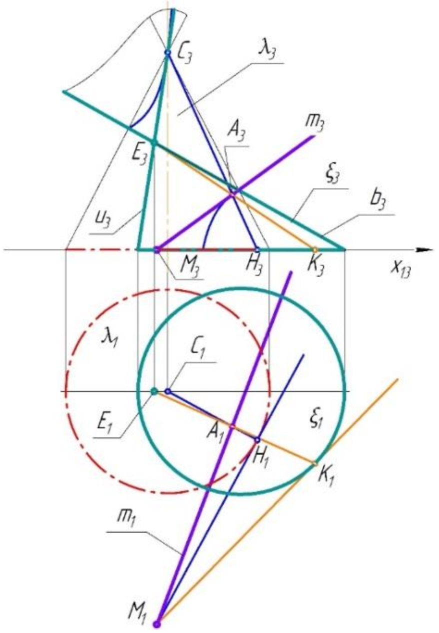

Fig. 10. Projectional determination of the straight line m.

Fig. 11 depicts the sheet *β*' of the surface *α* and the lines *m* and *n* defining the plane Γ tangent to it. The problem was solved from the condition that the point *A* is visible from the top view. If we accept that the point *A* is located on the lower part of the sheet *β*', then the plane tangent to *β*' in the point *A* is located symmetrically to the found plane Γ with respect to plane π_1.

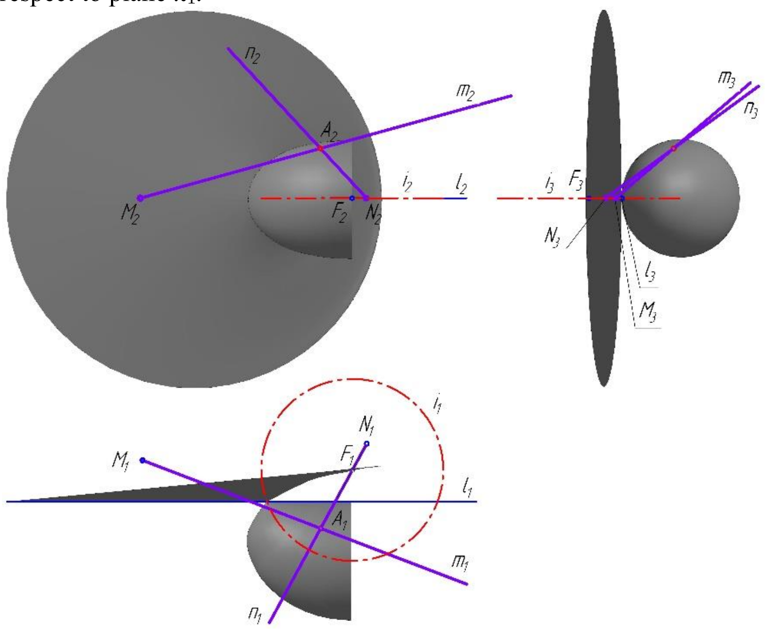

Fig. 11. The sheet *β*' of the surface *α* and lines *n* and *m*, defining the tangent plane.

5 Conclusion

In the present paper a method of determination of a plane tangent to a surface of quasi-rotation is acquired. The construction is based on correspondence induced by quasi-rotation of plane π_1 around a curvilinear axis belonging to said plane. A plane tangent to one of the sheets of a two-sheet surface is determined. The proposed method can be applied to any surface defined by determinant (2), where *l* represents a straight line and *i* represents a circle. Construction of a plane tangent to a surface of quasi-rotation is similar for both of its sheets. The pair of cones *λ* and *ξ* is invariable for any point of surface *α* acquired through rotation of the generatrix on angle *φ*.

Construction of a plane tangent to a surface in a given point is a part of applied tasks related to determination of optical properties of surfaces including analysis of images of the surrounding objects [6]. It is known that a ray is reflected form a surface in the same direction as if it was reflected from a surface tangent to such surface at the point of incidence. Such optical properties of surfaces are vital in architecture [7].

The solutions to the problems requiring determination of aerodynamic properties of a surface are also tied to the tangent planes. Sections of surfaces of quasi-rotation can find application in design of products of required aerodynamic properties.

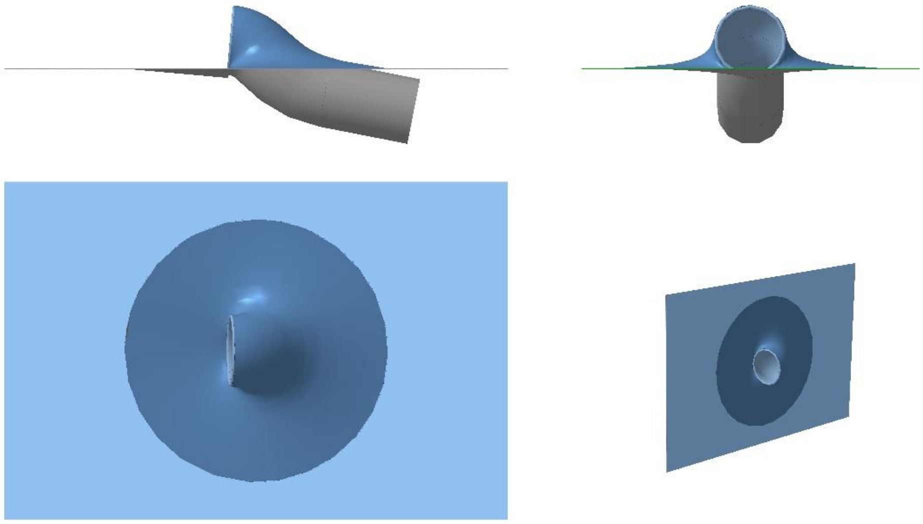

Fig. 12. Air inlet opening. A section of the sheet β'' of the surface α is highlighted with darker color.

Fig. 13. Examples of air inlet openings.

The capabilities of design of technical surfaces on the basis of non-ruled transformations of plane and space are described in sources [8, 9]. Fig. 12 depicts an example of application of the sheet β'' of the surface α (see Fig. 3) in design of an air inlet opening. Air inlets are common elements of constructions in various areas of engineering and technology (see Fig. 13).

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