Theoretical and numerical analysis of extremum problems for reaction-diffusion model

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Abstract

Boundary and extremum problems for the reaction-diffusion equation, in which the reaction coefficient nonlinearly depends on the concentration of the substance are studied. The maximum principle is stated for solving boundary value problems. Optimality systems are derived for extreme problems. Based on the analysis of these systems, local stability estimates of optimal solutions are derived, numerical algorithms for solving extreme problems are developed, and a stationary analogue of the bang–bang principle is established.

Keywords

nonlinear reaction-diffusion equation, maximum principle, control problem, optility system, local stability estimates, numerical algorithm, bang–bang principle

1. Introduction

In recent years, interest in the study of inverse and control problems for heat and mass transfer models has only increased. Note the works [1, 2, 3, 4, 5, 7, 6, 8, 9, 10, 11, 12, 13] devoted to the theoretical analysis of these problems. In these papers, the solvability of boundary value, inverse and extremum problems for the indicated models was studied, and the questions of uniqueness and stability of solutions were studied. Related problems for models of complex heat transfer and ferroelectric hysteresis are studied in [14, 15, 16, 17].

We also note that applications of control problems are not limited to the search for effective mechanisms for controlling physical fields in continuous media. Within the framework of the optimization approach (see [10, 13]), problems of reconstructing unknown functions in the considered models are reduced to control problems using additional information about solving the corresponding boundary value problems.

This paper is devoted to the theoretical and numerical analysis of boundary value and extremum problems for the reaction-diffusion equation, in which the reaction coefficient nonlinearly depends on the concentration of the substance. In contrast to [7, 6, 8, 9, 10, 13], in this paper the optimality systems obtained for control problems are used not only to study the stability (and uniqueness) of optimal solutions. Using these systems, numerical algorithms for solving extreme problems are constructed and new properties of their solutions are stated.

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2. Boundary value problem

In bounded domain $\Omega \subset \mathbb{R}^3$ with boundary Γ the nonlinear reaction–diffusion equation is considered

$$-\operatorname{div}(\lambda(\mathbf{x})\nabla\varphi) + k(\varphi, \mathbf{x})\varphi = f \ \text{ on } \Omega, \quad \varphi = \psi \text{ on } \Gamma. \tag{1}$$

Here function φ means pollutant substance's concentration, f is a volume density of external sources of substance, $\lambda(\mathbf{x})$ is a diffusion coefficient, function $k = k(\varphi, \mathbf{x})$ is a reaction coefficient, $\mathbf{x} \in \Omega$. This problem (1) will be called Problem 1 below.

In this paper, we prove the global solvability of Problem 1 and the local uniqueness of its solution in the case when the nonlinearity $k(\varphi, \mathbf{x})\ \varphi$ is not monotone in the entire domain Ω, as was assumed in [8]. This allows you to expand the range of mathematical models, the correctness of which we can justify. For concentration φ the principle of minimum and maximum is stated.

Further, for Problem 1, a control problem is formulated, the role of controls in which is played by the functions λ and f, and its solvability is proved in the general case. Optimality systems are derived for extreme problems with specific reaction coefficients. Based on the analysis of these systems, local stability estimates of optimal solutions are derived, numerical algorithms for solving extreme problems are developed.

A one-parameter control problem is considered separately, for which regularization is not used. For this problem, the validity of a stationary analogue of the bang-bang principle is established (see the meaning of this term below or in [14, 15]).

3. Solvability of the boundary value problem

While studying the considered problems we will use Sobolev spaces $H^s(D)$, $s \in \mathbb{R}$. Here D means either the domain Ω or some subset $Q \subset \Omega$, or the boundary Γ. By $\|\cdot\|_{s,Q}, |\cdot|_{s,Q}$ and $(\cdot,\cdot)_{s,Q}$ we will denote the norm, seminorm and the scalar product in $H^s(Q)$. The norms and scalar products $L^2(Q)$, $L^2(\Omega)$ or in $L^2(\Gamma)$ will be denoted correspondingly by $\|\cdot\|_Q$ and $(\cdot,\cdot)_Q$, $\|\cdot\|_\Omega$ and $(\cdot,\cdot)$ or $\|\cdot\|_\Gamma$ and $(\cdot,\cdot)_\Gamma$. Let

$$L^p_+(\Omega) = \{k \in L^p(\Omega) \,:\, k \geq 0\},\ p > 1, \ \ Z = \{\mathbf{v} \in L^4(\Omega)^3 \,:\, \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\},$$

$$H^s_{\lambda_0}(\Omega) = \{h \in H^s(\Omega) \,:\, h \geq \lambda_0 > 0 \text{ in } \Omega\},\ s > 3/2.$$

It will be assumed that the following conditions hold:

(i) Ω is a bounded domain in the space $\mathbb{R}^3$ with boundary $\Gamma \in C^{0,1}$;

(ii) $f \in L^2(\Omega)$, $\psi \in H^{1/2}(\Gamma)$;

(iii) For any function $v \in H^1(\Omega)$ the embedding $k(v,\cdot) \in L^p_+(\Omega)$ is true for some $p \geq 5/3$, which doesn't depend on v, and on any sphere $B_r = \{v \in H^1(\Omega) \,:\, \|v\|_{1,\Omega} \leq r\}$ of radius r the inequality takes place

$$\|k(v_1,\cdot) - k(v_2,\cdot)\|_{L^p(\Omega)} \leq L_1\|v_1 - v_2\|_{L^5(\Omega)} \ \ \forall v_1, v_2 \in B_r.$$

Here L is a constant, which depends on r, but doesn't depend on $v_1, v_2 \in B_r$.

(iv) Let $\Omega_1 \subset \Omega$ be a subdomain of Ω, such that $\overline{\Omega}_1 \subset \Omega$. Put $\Omega_2 = \Omega \setminus \overline{\Omega}_1$.

The function $k(\varphi,\cdot)\varphi$ is monotone in the subdomain Ω_2 in the following sense:

$$(k(\varphi_1,\cdot)\varphi_1 - k(\varphi_2,\cdot)\varphi_2, \varphi_1 - \varphi_2)_{\Omega_2} \geq 0 \ \ \forall \varphi_1, \varphi_2 \in H^1(\Omega) \tag{2}$$

and bounded in the sense that there exist positive constants A_1, B_1, depending on k, such that

$$\|k(\varphi,\cdot)\|_{L^p(\Omega_2)} \le A_1\|\varphi\|_{1,\Omega}^t + B_1,\ p \ge 5/3,\ t \ge 0. \tag{3}$$

In subdomain Ω_1 for function $k(\varphi,\cdot)$ with constant $C_1 > 0$ the following inequality is true:

$$\|k(\varphi,\cdot)\|_{L^p(\Omega_1)} \le C_1 \ \ \forall\varphi \in H^1(\Omega).$$

Let us mention that the conditions (iii), (iv) describe an operator, acting from $H^1(\Omega)$ to $L^p(\Omega)$, where $p \ge 5/3$, which give an opportunity to take into consideration the dependence of the reaction coefficient on either the solution φ or the spatial variable $\mathbf{x}$. For example,

$$k = \frac{1}{1+\varphi^2} \text{ in } \Omega_1 \text{ and } k = \varphi^2 \text{ in } Q \subset \Omega_2,\ k = k_0(\mathbf{x}) \in L_+^{5/3}(\Omega_2 \setminus \overline{Q}) \text{ in } \Omega_2 \setminus \overline{Q},$$

where Q is a subdomain of Ω_2.

Let us also remind that on the strength of the Sobolev embedding theorem the space $H^1(\Omega)$ is embedded into the space $L^s(\Omega)$ continuously at $s \le 6$ and compactly at $s < 6$, with some constant C_s, which depend on s and Ω, and the estimate is true

$$\|\varphi\|_{L^s(\Omega)} \le C_s\|\varphi\|_{1,\Omega} \ \ \forall\varphi \in H^1(\Omega). \tag{4}$$

The following lemmas hold (see., for example, [8]).

Lemma 1.1.

If condition (i), (ii) hold and $\lambda \in H^s_{\lambda_0}(\Omega)$, $s > 3/2$, $k_1 \in L^p_+(\Omega)$, $p \ge 5/3$ then there are such positive constants C_0, δ_0, γ_p, depending on Ω or Ω and p, such that the following inequalities hold:

$$|(\lambda\nabla\varphi,\nabla\eta)| \le C_0\|\lambda\|_{s,\Omega}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega},$$

$$|(k_1\varphi,\eta)| \le \gamma_p\|k_1\|_{L^p(\Omega)}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega},\ \forall\varphi,\eta \in H^1(\Omega), \tag{5}$$

$$(\lambda\nabla\varphi,\nabla\varphi) \ge \lambda_*\|\varphi\|^2_{1,\Omega},\ (\lambda\nabla\varphi,\nabla\varphi) + (k_1\varphi,\varphi) \ge \lambda_*\|\varphi\|^2_{1,\Omega}\ \forall\varphi \in H^1_0(\Omega),\ \lambda_* \equiv \delta_0\lambda_0. \tag{6}$$

Lemma 3.1.

Let condition (i) holds. Then for any function $\psi \in H^{1/2}(\Gamma)$ there exists function $\varphi_0 \in H^1(\Omega)$ such that $\varphi_0 = \psi$ on Γ and with some constant C_Γ, depending on Ω and Γ, the following estimate is true:

$$\|\varphi_0\|_{1,\Omega} \le C_\Gamma\|\psi\|_{1/2,\Gamma}.$$

Let us multiply (1) by $h \in H^1_0(\Omega)$ and integrate over Ω. Using the Green's formula, we are coming to the weak formulation of Problem 1. It consists in finding function $\varphi \in H^1(\Omega)$ from condition

$$(\lambda\nabla\varphi,\nabla h) + (k(\varphi)\varphi,h) = (f,h) \quad \forall h \in H^1_0(\Omega), \quad \varphi|_\Gamma = \psi. \tag{7}$$

The next theorem follows from results [11]

Theorem 3.1

If conditions (i)–(iv) hold. then a weak solution $\varphi \in H^1(\Omega)$ of Problem 1 exists and the following estimate takes place:

$$\|\varphi\|_{1,\Omega} \le M_\varphi \equiv C_*(\|f\|_\Omega + C_\Gamma(C_0\|\lambda\|_{s,\Omega} + \gamma_1\|\mathbf{u}\|_{L^4(\Omega)^3} + \gamma_p C_1)\|\psi\|_{1/2,\Gamma}+$$
$$+ C_*\gamma_p C_\Gamma(C_\Gamma^r A_1\|\psi\|_{1/2,\Gamma}^r + B_1)\|\psi\|_{1/2,\Gamma} + C_\Gamma\|\psi\|_{1/2,\Gamma}. \tag{8}$$

If, besides, this condition is met

$$\gamma_p L M_\varphi < \lambda_*, \tag{9}$$

where constants γ_p and λ_* are specified in Lemma 1.1, constant L in introduced in condition (iii), then Problem 1's solution is unique.

Let, in addition to (i)–(iv), the following conditions be satisfied:

(v) $\psi_{\min} \le \psi \le \psi_{\max}$ on Γ, $\lambda_0 \le \lambda \le \lambda_{\max}$ on Ω, $f_{\min} \le f \le f_{\max}$ on Ω_2 and $f = 0$ on Ω_1 (or $\Omega_1 = \emptyset$).

Here $\psi_{\min}$, $\psi_{\max}$, $f_{\min}$, $f_{\max}$ are nonnegative numbers, $\lambda_{\max} > \lambda_0 > 0$.

(vi) $k(\varphi, \mathbf{x})\varphi$ satisfies the inequality (2), while $k(\varphi, \mathbf{x}) = a(\mathbf{x})k_1(\varphi)$, where $0 < a_{\min} \le a(\mathbf{x}) \le a_{\max} < \infty$ a.e. in Ω, $k_1(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}_+$ is a continuous function and the functional equations for M and m:

$$k_1(M_1)M_1 = f_{\max}/a_{\min} \quad \text{and} \quad k_1(m_1)m_1 = f_{\min}/a_{\max} \tag{10}$$

have at least one solution.

Lemma 3.2.

Under conditions (i)–(vi) for the solution $\varphi \in H^1(\Omega)$ of Problem 1 the following principle of maximum and minimum is valid:

$$m \le \varphi \le M \text{ a.e. in } \Omega,\ M = \max\{\psi_{\max}, M_1\},\ m = \min\{\psi_{\min}, m_1\}. \tag{11}$$

Here M_1 is the minimum root of the first equation in (10) and m_1 is the maximum root of the second equation in (10).

Remark 3.1

For power-law reaction coefficients, the parameters M_1 and m_1 are easily calculated. For example, if $k(\varphi) = |\varphi|$ then $M_1 = (f_{\max}/a_{\min})^{1/2}$ and $m_1 = (f_{\min}/a_{\max})^{1/2}$.

4. Statement of optimal control problem

Let us formulate an optimal control problem for Problem 1. For this purpose the whole set of initial data will be divided into two groups: the group of fixed functions, in which functions ψ and $k(\varphi, \cdot)$ are included, and the group of controlling functions, in which λ and f will be included. We assume that λ and f can be changed in subsets K_1 and K_2, respectively, which satisfy the following condition:

(j) $K_1 \subset H^s_{\lambda_0}(\Omega)$, $s > 3/2$, $K_2 \subset L^2(\Omega)$ are nonempty convex closed sets.

Define functional space

$$Y = H^{-1}(\Omega) \times H^{1/2}(\Gamma),$$

and set $u = (\lambda, f)$, $K = K_1 \times K_2$.

Introduce an operator $F = (F_1, F_2) : H^1(\Omega) \times K \longrightarrow Y$ by formulae:

$$\langle F_1(\varphi, u), h \rangle = (\lambda \nabla \varphi, \nabla h) + (k(\mathbf{x}, \varphi)\, \varphi, h) + (\mathbf{u} \cdot \nabla \varphi, h) - (f, h),$$

$$F_2(\varphi) = \varphi|_\Gamma - \psi$$

and rewrite a weak form (7) of Problem 1 in the form of the operator equation $F(\mathbf{x}, u) = 0$.

Let $I : X \longrightarrow \mathbb{R}$ be a weakly lower semicontinuous functional. Consider the following multiplicative control problem:

$$J(\varphi, u) \equiv \frac{\mu_0}{2} I(\varphi) + \frac{\mu_1}{2} \|\lambda\|^2_{s,\Omega} + \frac{\mu_3}{2} \|f\|^2_\Omega \longrightarrow \inf,$$
$$F(\varphi, u) = 0,\ (\varphi, u) \in H^1(\Omega) \times K. \tag{12}$$

The set of possible pairs for the problem (12) is denoted by

$$Z_{ad} = \{(\mathbf{x}, u) \in X \times K : F(\mathbf{x}, u) = 0, J(\mathbf{x}, u) < \infty\}.$$

Let, in addition to 4, the following condition hold:

(jj) $\mu_0 > 0$, $\mu_i \geq 0$, $i = 1, 2$ and $K = K_1 \times K_2$ is a bounded set in $H^s(\Omega) \times L^2(\Omega)$, $s > 3/2$, or $\mu_i > 0$, $i = 0, 1, 2$ and a functional I is bounded from below.

We use the following cost functionals:

$$I_1(\varphi) = \|\varphi - \varphi^d\|^2_Q = \int_Q |\varphi - \varphi^d|^2 d\mathbf{x}, \quad I_2(\varphi) = \|\varphi - \varphi^d\|^2_{1,Q}, \tag{13}$$

Here a function $\varphi^d \in L^2(Q)$ denotes a desired concentration field, which is given in a subdomain $Q \subset \Omega$.

Theorem 4.1.

Assume that the assumptions (i)–(vi) and 4, 4 take place. Let $I : X \longrightarrow \mathbb{R}$ be a weakly semicontinuous below functional and let $Z_{ad} \neq 0$. Then there is at least one solution $(\varphi, u) \in X \times K$ of the control problem (12)

Let us note that he solution $(\hat{\varphi}, \hat{u})$ of the extremum problem (12) can be treated as *an approximate* solution of the inverse problem of recovering unknown functions λ, f and χ with the help of measured concentration $\varphi_d \in Q \subset \Omega$ in case, when $I_1(\hat{\varphi})/\|\varphi_d\|^2_Q$ is small enough. For example, it can be within the limits of the measurment error of φ_d. However, the only possibility to show this is numerical in a general case. Sometimes it is possible to construct the exact solutions of the corresponding boundary value problem and to get the upper estimate of the value I_1 with the their help (see, for example, the papers [19], which are dedicated to the use of the optimization method for solving inverse problems of heat cloaking).

5. Optimality system and stability estimates

The next step in the study of the extreme problem is the derivation of the optimality system, which provides valuable information about additional properties of optimal solutions. Based on its analysis, one can establish, in particular, the uniqueness and stability of optimal solutions, and also to construct numerical algorithms for finding optimal solutions to extreme problems.

We will further assume that $k(\varphi) = |\varphi|$, which corresponds to a slowly decaying pollutant.

We introduce a dual space $Y^* = H_0^1(\Omega) \times H^{-1/2}(\Gamma)$ to Y. It is easy to show that Fréchet derivative of an operator

$$F = (F_1, F_2) : H^1(\Omega) \times K \longrightarrow Y$$

with respect to φ at any point

$$(\hat{\varphi}, \hat{u}) = (\hat{\varphi}, \hat{\lambda}, \hat{f}, \hat{\chi})$$

is a linear continuous operator

$$F'_\varphi(\hat{\varphi}, \hat{u}) : H^1(\Omega) \longrightarrow Y,$$

that maps each element $\tau \in H^1(\Omega)$ into an element $F'_\varphi(\hat{\varphi}, \hat{u})(\tau) = (\hat{y}_1, \hat{y}_2) \in Y$. Here the elements $\hat{y}_1 \in H^{-1}(\Omega)$ and $\hat{y}_2 \in H^{1/2}(\Gamma)$ are defined by $\hat{\varphi}$ and τ and by the following relations:

$$\langle \hat{y}_1, \tau \rangle = (\hat{\lambda} \nabla \tau, \nabla h) + 2(|\hat{\varphi}|\, \tau, h) \quad \forall h \in H^1(\Omega), \;\; y_2 = \tau|_{\Gamma_D}. \tag{14}$$

By $F'_\varphi(\hat{\varphi}, \hat{u})^* : Y^* \longrightarrow H^1(\Omega)^*$ we denote an operator adjoint to $F'_\varphi(\hat{\varphi}, \hat{u})$.

In accordance with a general theory of smooth-convex extremum problems [18], we introduce an element $\mathbf{y}^* = (\theta, \zeta) \in Y^*$, to which we will refer as to an adjoint state and will define the Lagrangian $\mathcal{L} : H^1(\Omega) \times K \times \mathbb{R} \times Y^* \longrightarrow \mathbb{R}$ by formula

$$\mathcal{L}(\varphi, u, \mathbf{y}^*) = J(\varphi, u) + \langle \mathbf{y}^*, F(\varphi, u) \rangle_{Y^* \times Y} \equiv J(\varphi, u) + \langle F_1(\varphi, u), \theta \rangle + \langle \zeta, F_2(\varphi, u) \rangle_{\Gamma_D}, \tag{15}$$

where $\langle \zeta, \cdot \rangle_{\Gamma_D} = \langle \zeta, \cdot \rangle_{H^{-1/2}(\Gamma_D) \times H^{1/2}(\Gamma_D)}$.

Since $|\hat{\varphi}| \in L^6_+(\Omega)$, then from [8] it follows that for any $f \in \mathcal{T}$ and $\psi \in H^{1/2}(\Gamma_D)$ there is a unique solution $\tau \in H^1(\Omega)$ of linear problem

$$(\hat{\lambda} \nabla \tau, \nabla h) + 2(|\hat{\varphi}|\, \tau, h) = \langle f, h \rangle \; \forall h \in \mathcal{T}, \quad \tau|_\Gamma = \psi. \tag{16}$$

Then operator $F'_\varphi(\hat{\varphi}, \hat{u}) : H^1(\Omega) \longrightarrow Y$ is an isomorphism and from [18] it follows

Theorem 5.1

Assume that assumptions (i)–(iv), (vi) and 4, 4 take place and let an element $(\hat{\varphi}, \hat{u}) \in X \times K$ be a local minimizer for the problem (12). Suppose also that a cost functional $I : X \longrightarrow \mathbb{R}$ is continuously Frechet differentiable with respect to the state $\mathbf{x}$ in a point $\hat{\mathbf{x}}$. Then there is a unique nonzero Lagrange multiplier $\mathbf{y}^* = (\theta, \zeta) \in Y^*$ such that the Euler–Lagrange equation takes place

$$F'_\varphi(\hat{\varphi}, \hat{u})^* \mathbf{y}^* = -J'_\varphi(\hat{\varphi}, \hat{u}) \text{ in } H^1(\Omega)^*,$$

which is equivalent to the relation

$$(\hat{\lambda} \nabla \tau, \nabla \theta) + 2(|\hat{\varphi}|\, \tau, \theta) + \langle \zeta, \tau \rangle_{\Gamma_D} = -(\mu_0/2) \langle I'_\varphi(\hat{\varphi}), \tau \rangle \quad \forall \tau \in H^1(\Omega), \tag{17}$$

and a minimum principle

$$\mathcal{L}(\hat{\varphi}, \hat{u}, \mathbf{y}^*) \le \mathcal{L}(\hat{\varphi}, u, \mathbf{y}^*) \; \forall u \in K,$$

which is equivalent to the inequalities

$$\mu_1(\hat{\lambda}, \lambda - \hat{\lambda})_{s,\Omega} + ((\lambda - \hat{\lambda})\nabla\hat{\varphi}, \nabla\theta) \geq 0 \quad \forall \lambda \in K_1, \tag{18}$$

$$\mu_2(\hat{f}, f - \hat{f})_{\Omega} - (f - \hat{f}, \theta) \geq 0 \ \ \forall f \in K_2. \tag{19}$$

Let us formulate a theorem on the local stability of optimal solutions to problem (12) for $I(\varphi) = \|\varphi - \varphi^d\|_Q^2$, which is proved according to the scheme suggested in [7].

Theorem 5.2.

Let in addition to the conditions (i), (ii) and 4, K be a bounded set and let the pair $(\varphi_i, u_i) \in H^1(\Omega) \times K$ be the solution of problem (12), which corresponds to the specified function $\varphi_i^d \in L^2(Q)$, $i = 1, 2$, where $Q \subset \Omega$ is an arbitrary open bounded set. Let us suppose that $\mu_0 > 0$ and the following conditions

$$\beta_1^2 \mu_0 \leq (1 - \varepsilon)\mu_1, \quad \beta_2^2 \mu_0 \leq (1 - \varepsilon)\mu_2, \tag{20}$$

are satisfied, where $\varepsilon \in (0, 1)$ is an arbitrary number and parameters β_1 and β_2, monotonically depend on the norms of the initial data of the problem (12).

Then the following stability estimates hold:

$$\|\lambda_1 - \lambda_2\|_{s,\Omega} \leq \sqrt{\mu_0/(\varepsilon\mu_1)}(0.5 + \beta_3)\|\varphi_1^d - \varphi_2^d\|_Q, \tag{21}$$

$$\|f_1 - f_2\|_{\Omega} \leq \sqrt{\mu_0/(\varepsilon\mu_2)}(0.5 + \beta_3)\|\varphi_1^d - \varphi_2^d\|_Q, \tag{22}$$

$$\|\varphi_1 - \varphi_2\|_{1,\Omega} \leq C_*(C_0 M_\varphi \sqrt{\mu_0/(\varepsilon\mu_1)} + \sqrt{\mu_0/(\varepsilon\mu_2)})(0.5 + \beta_3)\|\varphi_1^d - \varphi_2^d\|_Q. \tag{23}$$

Here parameter β_3 depends on the initial data of the problem (12), C_*, C_0 are constants from Lemma 1.1 and M_φ is introduced in (8).

The stability estimates (21)–(23) are interesting by themselves, as they clearly characterize the local stability of multiplicative control problem's solution, and the problem has a strong nonlinearity. Moreover, the optimization approach gives an opportunity to reduce the inverse coefficient problems to the problems of multiplicative control (see [10]). Let us note that the method, which was used to obtain these estimates, can be applied also for the studying of convergence of numerical algorithms, which are used for obtaining an approximate solution of extremum problems and which are based on using of optimality systems as in [20].

6. Numerical algorithm

The optimality system (7), (17)–(19) can be used to design efficient numerical algorithms for solving control problem (12). The simplest one for I_1 can be obtained by applying the fixed point iteration method to the optimality system. The m-th iteration of this algorithm consist of finding unknown values φ^m, θ^m, ζ^m, λ^{m+1} and f^{m+1} for given (λ^m, f^m), $m = 0, 1, 2, ...$ beginning with given initial values λ^0 and f^0 by sequentially solving following problems:

$$(\lambda\nabla\varphi^m, \nabla h) + (|\varphi^m|\varphi^m, h) = (f^m, h) \ \ \forall h \in H_0^1(\Omega), \ \ \varphi^m|_\Gamma = \psi, \tag{24}$$

$$(\hat{\lambda}\nabla\tau, \nabla\theta^m) + 2(|\varphi^m|\, \tau, \theta^m) + \langle\zeta^m, \tau\rangle_\Gamma = -\mu_0(\varphi^m - \varphi^d, \tau)_Q \quad \forall \tau \in H^1(\Omega), \tag{25}$$

$$\mu_1(\lambda^{m+1}, \lambda - \lambda^m)_{s,\Omega} + ((\lambda - \lambda^{m+1})\nabla\varphi^m, \nabla\theta^m) \geq 0 \quad \forall \lambda \in K_1, \tag{26}$$

$$\mu_2(f^{m+1}, f - f^m)_\Omega - (f - f^{m+1}, \theta^m) \geq 0 \quad \forall f \in K_2. \tag{27}$$

For discretization and solving variational problem (24), (25), one can use open software freeFEM++ (www.freefem.org) based on using the finite element method. For discretization of variation inequalities (26), (27), it is comfortable to look for solutions λ^{m+1} and f^{m+1} as

$$\lambda^{m+1}(\mathbf{x}) = \sum_{j=1}^{N} \lambda_j^{m+1} l_j(\mathbf{x}), \quad f^{m+1}(\mathbf{x}) = \sum_{k=1}^{N} f_j^{m+1} g_k(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{28}$$

Here, N is an integer, $l_j(\mathbf{x}) \in H^s_{\lambda_0}(\Omega)$, are nonnegative basic functions in $H^s_{\lambda_0}(\Omega)$, $s > 3/2$, $g_k(\mathbf{x}) \in L^2(\Omega)$ are basic functions in $L^2(\Omega)$, $\lambda_j^{m+1} \geq 0$ and $f_k^{m+1} \in \mathbb{R}$ are unknown coefficients.

7. Bang–bang principle for a one parameter control problem

In this section, we will state additional properties of the optimal solution to the following control problem:

$$J(\varphi) \equiv (1/2)I(\varphi) \longrightarrow \inf, \ \mathcal{F}(\varphi, f) = 0, \ (\varphi, f) \in H^1(\Omega) \times K_2. \tag{29}$$

The role of control in the problem (29) is played only by the function f, which can change in the subset K_2. Whereas the function λ is considered to be given.

The operator

$$\mathcal{F} = (\mathcal{F}_1, F_2) \, : \, H^1(\Omega) \times K_2 \longrightarrow Y$$

is defined by formulas:

$$\langle \mathcal{F}_1(\varphi, u), h \rangle = (\lambda \nabla \varphi, \nabla h) + (|\varphi|\varphi, h) - (f, h),$$
$$F_2(\varphi) = \varphi|_\Gamma - \psi.$$

Let us denote by

$$\mathcal{Z}_{ad} = \{(\varphi, f) \in H^1(\Omega) \times K_2 \, : \, \mathcal{F}(\varphi, f) = 0, J(\varphi, f) < \infty\}$$

the set of admissible pairs for the problem (29) and assume that the condition

(jjj) $K_2 \subset L^2(\Omega)$ is a nonempty convex, closed and bounded set.

Theorem 7.1.

Assume that the assumptions (i), (ii) and 7 take place. Let $I \, : \, X \longrightarrow \mathbb{R}$ be a weakly semicontinuous below functional and let $\mathcal{Z}_{ad} \neq 0$. Then there is at least one solution $(\varphi, f) \in H^1(\Omega) \times K_2$ of the control problem (29).

It clear, for the problem (29) an analog of Theorem 3.1 and the minimum principle takes the following form:

$$(f - \hat{f}, \theta) \leq 0 \quad \forall f \in K_2. \tag{30}$$

Let a more stringent condition be satisfied instead of 7:

(jjj‘) $f_{\min} \leq f \leq f_{\max}$ a.e. in Ω for all $f \in K_2$, where $f_{\min}$ and $f_{\max}$ are positive numbers.

It is clear that conditions 7 define a special case of a convex, bounded, and closed set K_2 introduced in 7.

Let us show that the optimal control $\hat{f}(\mathbf{x})$ of the problem (29) has the bang-bang property, according to which it takes one of two values $f_{\min}$ or $f_{\max}$, respectively, depending on the sign of the function $\theta(\mathbf{x})$ at the point $\mathbf{x} \in \Omega$.

Lemma 7.1.

Under the conditions 7 the inequality (30) is equivalent to the following inequality

$$(f - \hat{f})\theta \leq 0 \text{ a.e. in } \Omega \quad \forall f \in K_2. \tag{31}$$

Proof.

Let us show that (30) implies (31). Suppose that there is a function $f_1 \in K_2$, with which on the set $D_0 \subset \Omega$, meas $D_0 > 0$, the inequality holds

$$(f_1 - \hat{f})\,\theta > 0 \text{ a.e. in } D_0.$$

Consider a f_2, such that $f_2 = \hat{f}$ if $\mathbf{x} \notin D_0$ and $f_2 = f_1$ if $\mathbf{x} \in D_0$. It clear, that $f_2 \in K_2$ and the inequality is true for it

$$(f_2 - \hat{f}, \theta) = (f_1 - \hat{f}, \theta)_{D_0} > 0,$$

which contradicts (30). ■

Corollary 7.1.

From (31) it follows that if $\theta < 0$ in D_1, then $\hat{f} = f_{\min}$ in D_1 and $\hat{f} = f_{\max}$ in D_2, if $\theta > 0$ in D_2. Note that interest in the bang–bang property is due to the study of control problems in which, for practical reasons, regularization is not used. In particular, such a formulation of control problems is used in the study of applied problems of thermal and electromagnetic cloaking (see, for example, [19]).

8. Conclusion

It is interesting to note that, on the one hand, a well-developed numerical algorithm for solving the extremal problem should show that the maximum principle for the concentration φ and the bang–bang principle for the optimal control f are satisfied. On the other hand, these properties can serve as a criterion for checking numerical algorithms, since they have been correctly proven theoretically. Of particular interest is the study of the convergence of a numerical algorithm based on the optimality system from Section 6. In this case, the method for deriving estimates of the local stability of optimal solutions from Section 5, which is also based on the analysis of the optimality system, can be applied (see [20]).

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References

[1] Yu. Yu. Belov, T. N. Shipina, The problem of determining a coefficient in the parabolic equation and some properties of its solution, J. Inv. Ill-Posed Probl. 1 (2001) 31–48. doi:10.1515/jiip.2001.9.1.31.

[2] G. V. Alekseev, D. A. Tereshko, On solvability of inverse extremal problems for stationary equations of viscous heat conducting fluid, J. Inv. Ill-Posed Probl. 6 (1998) 521–562. doi:10.1134/S096554251109003X.

[3] G. V. Alekseev, E. A. Adomavichus, Theoretical analysis of inverse extremal problems of admixture diffusion in viscous fluids, J. Inv. Ill-Posed Probl. 9 (2001) 435–468. doi:10.1515/jiip.2001.9.5.435.

[4] G. V. Alekseev, O. V. Soboleva, D. A. Tereshko, Identification problems for a steady-sate model of mass transfer, J. Appl. Mech. Tech. Phys. 49 (2008) 537–547. doi:10.1007/s10808-008-0071-x.

[5] Yu. Yu. Belov, V. G. Kopylova, On some identification problem for source function to one semievolutionary system, J. Inv. Ill-Posed Probl. 20 (2012) 723–743. doi:10.1007/s10808-008-0071-x.

[6] R. V. Brizitskii, Zh. Yu. Saritskaya, A. I. Byrganov, Multiplicative control problems for nonlinear convection–diffusion–reaction equation, Sib. El. Math. Rep. 13 (2016) 352–360. doi:10.17377/semi.2016.13.030.

[7] G. V. Alekseev, R. V. Brizitskii, Zh. Yu. Saritskaya, Stability estimates of solutions to extremal problems for a nonlinear convection-diffusion-reaction equation, J. Appl. Industr. Math. 10 (2016) 155–167. doi:10.1134/S1990478916020010.

[8] R. V. Brizitskii, Zh. Yu. Saritskaya, Stability of solutions to extremum problems for the nonlinear convection–diffusion–reaction equation with the Dirichlet condition, Comp. Math. Math. Phys. 56 (2016) 2011–2022. doi:10.1134/S096554251612006X.

[9] R. V. Brizitskii, Zh. Yu. Saritskaya, Stability of solutions of control problems for the convection–diffusion–reaction equation with a strong nonlinearity, Dif. Eq. 53 (2017) 485–496. doi:10.1134/S0012266117040073.

[10] R. V. Brizitskii, Zh. Yu. Saritskaya, Optimization analysis of the inverse coefficient problem for the nonlinear convection-diffusion-reaction equation, J. Inv. Ill-Posed Probl. 26 (2018) 821–833. doi:10.1515/jiip-2017-0011.

[11] R. V. Brizitskii, P. A. Maksimov, Boundary value and extremum problems for a nonlinear reaction–diffusion–convection equation under the Dirichlet condition, Comp. Math. Math. Phys. 61 (2021) 1–15.

[12] G. V. Alekseev, Stability estimates in the problem of cloaking material bodies for Maxwell's equations, Comp. Math. Math. Phys. 54 (2014) 1788–1803. doi:10.1134/S0965542514120069.

[13] G. V. Alekseev, Analysis and optimization in problems of cloaking of material bodies for the Maxwell equations, Dif. Eq. 52 (2016) 361–372. doi:10.1134/S0012266116030101.

[14] A. E. Kovtanyuk, A. Yu. Chebotarev, N. D. Botkin, K. H. Hoffmann, Solvability of P1 approximation of a conductive-radiative heat transfer problem, Appl. Math. Comp. 249 (2014) 247–252. doi:10.1016/j.amc.2014.10.054.

[15] A. E. Kovtanyuk, A. Yu. Chebotarev, N. D. Botkin, K. H. Hoffmann, Optimal boundary control of a steady-state heat transfer model accounting for radiative effects, J. Math. Anal. Appl. 439 (2016) 678–689. doi:10.1016/j.jmaa.2016.03.016.

[16] G. V. Grenkin, A. Yu. Chebotarev, A. E. Kovtanyuk, N. D. Botkin, K. H. Hoffmann, Boundary optimal control problem of complex heat transfer model, J. Math. Anal. Appl. 433 (2016) 1243–

1260. doi:10.1016/j.jmaa.2016.03.016.

[17] A. G. Maslovskaya, L. I. Moroz, A. Yu. Chebotarev, A. E. Kovtanyuk, Theoretical and numerical analysis of the Landau–Khalatnikov model of ferroelectric hysteresis, Communications in Nonlinear Science and Numerical Simulation (2021) doi:10.1016/j.cnsns.2020.105524.

[18] A. V. Fursikov, Optimal control of distributed systems. Theory and applications, American Mathematical Society (2000)

[19] G. V. Alekseev, D. A. Tereshko, Particle swarm optimization-based algorithms for solving inverse problems of designing thermal cloaking and shielding devices, International Journal of Heat and Mass Transfer 135 (2019) 1269–1277. doi:10.1016/j.ijheatmasstransfer.2019.02.072.

[20] R. V. Brizitskii, A. S. Savenkova, Asymptotic behavior of solutions to multiplicative control problems for elliptic equations, Computational Mathematics and Mathematical Physics 48 (2008) 1570–1580. doi:10.1134/S0965542508090078.