Numerical approach for the one stationary nonlinear problem governing the flow of incompressible viscous fluid in L-shaped domain

Alexey V. Rukavishnikov[a], Viktor A. Rukavishnikov[b]

[a]Institute of Applied Mathematics of Far-Eastern Branch, Russian Academy of Sciences, Dzerzhinsky Str., 54, Khabarovsk, 680000, Russia

[b]Computing Center of the Far Eastern Branch of the Russian Academy of Sciences, Kim Yu Chen Str., 65, Khabarovsk, 680000, Russia

Abstract

The steady Navier-Stokes equations governing the flow of an incompressible viscous fluid in the rotation form in L-shaped domain is considered. The weighted finite element method based on the definition of an R_ν-generalized solution is constructed. The advantage of the proposed approach over classical approximations is numerically established. The modern elements of computational technologies to find the optimal parameters of the proposed method are used.

Keywords

corner singularity, weighted finite element method, preconditioning, steady Navier-Stokes equations

1. Introduction

Most of the mathematical models representing natural processes are described using boundary value problems for the systems of partial differential equations with a singularity. The peculiarity of the solution is as follows systems in a bounded, connected domain of the Euclidean space $\mathbf{R}^2$ can be attributed to the presence of obtuse corners on its boundary, to the degeneration of initial data, or to internal characteristics of the solution. If the solution of the boundary value problem does not belong to the Sobolev space $W_2^1(\Omega)$, then it is called strong singular. If the solution of the boundary value problem belongs to $W_2^1(\Omega)$, but does not belong to $W_2^2(\Omega)$, then the boundary value problem is called weakly singular. The generalized solution of such problems in Ω with a boundary containing a reentrant corner $\sigma \in (\pi, 2\pi]$ belongs to the space $W_2^{1+\alpha-\epsilon}(\Omega),\ \alpha < 1$. Moreover, the approximate finite element or finite difference solution by classical method converges to the exact one with a $\mathcal{O}(h^\alpha)$ rate.

In [1], it was proposed to define the solution of elliptic boundary value problems with a singularity as an R_ν-generalized one. The approach allows us to introduce a weight space or a set, depending on the geometry of the domain and input data (right-hand sides, equation coefficients, boundary and initial data) to which an R_ν-generalized solution belongs. In [2, 3, 4], the existence, uniqueness and differential properties of the elliptic problems solution are proved. In [5], a weighted analogue of the Ladyzhenskaya-Babuska-Brezzi condition for the Stokes problem is established. In [6, 7, 8, 9, 10], a weighted finite element method (FEM) for an approximate solution of elliptic problems with a singularity has been developed.

Far Eastern Workshop on Computational Technologies and Intelligent Systems, March 2–3, 2021, Khabarovsk, Russia
78321a@mail.ru (A.V. Rukavishnikov); vark0102@mail.ru (V.A. Rukavishnikov)
0000-0002-7585-4559 (A.V. Rukavishnikov); 0000-0002-3702-1126 (V.A. Rukavishnikov)

In the paper, an R_ν-generalized solution of the steady Navier-Stokes equations governing the flow of an incompressible viscous fluid in the rotation form in L-shaped domain is defined. We use Picard's iterative procedure [11] to find a solution of a nonlinear problem. Then, we construct a weighted finite element scheme based on the definition of an R_ν-generalized solution: 1) the functions of the finite element spaces satisfy the mass conservation law in a strong sense — Scott-Vogelius (SV) pair of 2nd order [12]; 2) basis functions are the product of SV functions and weight functions in some degree. This construction allows us better take into account the behavior of the solution in the δ-neighborhood of the singularity point and increase the convergence rate of the approximate solution to the exact one to the first order with respect to the grid step h, i. e. $\mathcal{O}(h)$ rate in $W^1_{2,\nu}(\Omega)$ norm. Thus, we develop the numerical method overcomes the so-called pollution effect (see [13]). The same advantage for other hydrodynamics problems was achieved in [14, 15, 16]. The optimal values ν, ν^* and δ of the presented weighted FEM using the modern elements of computational technologies were derived numerically.

The paper consists of six sections. Section 2 is devoted to the definition of an R_ν-generalized solution. In Section 3, we present the weighted FEM. The iterative procedure for solving the systems of linear algebraic equations is constructed in Section 4. In Section 5, we shaw and discuss the results of computational experiments. Necessary conclusions are made in last section.

2. The problem statement

Let Ω be a bounded, connected domain in the Euclidean space $\mathbf{R}^2$. Denote by $\bar{\Omega}$ and Γ the closure and boundary of Ω, respectively, $\bar{\Omega} = \Omega \cup \Gamma$. Let $\mathbf{x} = (x_1, x_2)$ be an element of $\mathbf{R}^2$, where $d\mathbf{x} = dx_1\, dx_2$ and $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$ are the measure and norm of $\mathbf{x}$, respectively.

We write the steady Navier-Stokes equations governing the flow of an incompressible viscous fluid in the convection form: find a velocity field $\mathbf{u} = \mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}))$ and a kinematic pressure $p = p(\mathbf{x})$ from

$$-\bar{\nu} \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \alpha\mathbf{u} + \nabla p = \mathbf{f}, \qquad \operatorname{div} \mathbf{u} = \mathbf{0} \quad \text{in} \quad \Omega, \tag{1}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma, \tag{2}$$

where $\bar{\nu} > 0$ is the kinematic viscosity coefficient (inversely proportional to the Reynolds number), $\alpha > 0$, $\mathbf{f} = \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$ and $\mathbf{g} = \mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}))$ are given force field in Ω and boundary data on Γ, respectively. Denote by $\triangle, \nabla$ and div the Laplace, gradient and divergence operators in $\mathbf{R}^2$, respectively.

Further, we introduce the necessary notation. Let $\mathbf{w} = (w_1, w_2)$, $\mathbf{v} = (v_1, v_2)$ and b – scalar, then

$$\mathbf{w} \cdot \mathbf{v} = w_1\, v_1 + w_2\, v_2, \qquad b \times \mathbf{w} = (-b w_2,\ b w_1)^T, \qquad \operatorname{curl} \mathbf{w} = -\frac{\partial w_1}{\partial x_2} + \frac{\partial w_2}{\partial x_1}.$$

We have the identity

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = (\operatorname{curl} \mathbf{u}) \times \mathbf{u} + \frac{1}{2}\nabla \mathbf{u}^2. \tag{3}$$

It follows from the equality $(\mathbf{w}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w} = \nabla(\mathbf{w}\cdot\mathbf{u}) + (\operatorname{curl} \mathbf{w}) \times \mathbf{u} + (\operatorname{curl} \mathbf{u}) \times \mathbf{w}$ and assumption that $\mathbf{w} = \mathbf{u}$.

Using (3), with $P = p + \frac{1}{2}\mathbf{u}^2$ for the system (1), (2) we get the rotation form of the steady Navier-Stokes equations: find a velocity field $\mathbf{u}$ and a Bernoulli pressure P such that

$$-\bar{\nu} \triangle \mathbf{u} + (\operatorname{curl} \mathbf{u}) \times \mathbf{u} + \alpha\mathbf{u} + \nabla P = \mathbf{f}, \qquad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{4}$$

$$\mathbf{u} = \mathbf{g} \qquad \text{on} \qquad \Gamma. \tag{5}$$

The system (4), (5) as well as (1), (2) is nonlinear due to the presence of the rotation term $(\operatorname{curl}\mathbf{u})\times\mathbf{u}$ in the momentum equations. The system on (4), (5) and term in particular we linearized by Picard's procedure (see [11]).

Starting with an initial approximation $\mathbf{u}^{(0)}$ for which

$$\operatorname{div}\mathbf{u}^{(0)} = 0 \qquad \text{in} \qquad \Omega \qquad \text{and} \qquad \mathbf{u}^{(0)} = \mathbf{g} \qquad \text{on} \qquad \Gamma \tag{6}$$

Picard's iteration constructs a sequence of solutions $(\mathbf{u}^{(k)}, P^{(k)})$ by solving the linear system:

$$-\bar{\nu} \triangle \mathbf{u}^{(k)} + (\operatorname{curl}\mathbf{u}^{(k-1)}) \times \mathbf{u}^{(k)} + \alpha\mathbf{u}^{(k)} + \nabla P^{(k)} = \mathbf{f}, \quad \operatorname{div}\mathbf{u}^{(k)} = 0 \qquad \text{in} \qquad \Omega, \tag{7}$$

$$\mathbf{u}^{(k)} = \mathbf{g} \qquad \text{on} \qquad \Gamma. \tag{8}$$

Note that the initial Bernoulli pressure in (7) need not be specified. If $\bar{\nu}$ be a not too small and $\mathbf{f}$ be a not too large, the steady Navier-Stokes equations (4), (5) have a unique solution $(\mathbf{u}, P)$ and the iterates $(\mathbf{u}^{(k)}, P^{(k)})$, $k = 1, 2,$ in (7), (8) converge to it as $k \longrightarrow \infty$ for any choice of the arbitrary $\mathbf{u}^{(0)}$ satisfying (6) (see [11]).

Note that for a linearized system (7), (8) the conservation laws of the mass and momentum remain in force.

In the article, we consider the special case of a bounded polygon domain Ω. Let Ω be a L-shaped domain with one reentrant obtuse corner equals to $\frac{3\pi}{2}$ on the boundary and its vertex coincides with the origin. We define an R_ν-generalized solution in each Picard's iteration of the problem (7), (8) and construct the effective weighted FEM. Thus, we solve the nonlinear problem (4), (5) governing the flow of a incompressible viscous fluid in the rotation form and show the advantage of our approximate method over the classical approaches in a L-shaped domain by the computational simulations.

Let us introduce the notation and define necessary spaces of generalized functions. Denote by $\Omega'_\delta = \{\mathbf{x} \in \bar{\Omega} : \|\mathbf{x}\| \le \delta < 1, \delta > 0\}$ a part of a δ-neighborhood of a point $(0,0)$ contained in $\bar{\Omega}$. Let $\rho(\mathbf{x}) = \begin{cases} \|\mathbf{x}\|, \mathbf{x} \in \Omega'_\delta, \\ \delta, \mathbf{x} \in \bar{\Omega} \setminus \Omega'_\delta \end{cases}$ be a weight function.

Denote by $L_{2,\beta}(\Omega)$ and $W^1_{2,\beta}(\Omega)$ the spaces of functions $v(\mathbf{x})$ with a bounded norms

$$\|v\|_{L_{2,\beta}(\Omega)} = \sqrt{\int_\Omega \rho^{2\beta}(\mathbf{x}) v^2(\mathbf{x}) d\mathbf{x}}$$

and

$$\|v\|_{W^1_{2,\beta}(\Omega)} = \sqrt{\|\rho^\beta(\mathbf{x})|v(\mathbf{x})|\|^2_{L_2(\Omega)} + \|\rho^\beta(\mathbf{x})|D^1 v(\mathbf{x})|\|^2_{L_2(\Omega)}},$$

respectively, where $D^m v(\mathbf{x}) = \frac{\partial^{|m|} v}{\partial x_1^{m_1} \partial x_2^{m_2}}$, $|m| = m_1 + m_2$, $m_i \ge 0$ - integer.

Let $W^1_{2,\beta}(\Omega, \delta)$ for $\beta > 0$ be a set of functions from the space $W^1_{2,\beta}(\Omega)$, meets the conditions

$$\int_{\bar{\Omega}\setminus\Omega'_\delta} \rho^{2\beta}(\mathbf{x}) v^2(\mathbf{x}) d\mathbf{x} \ge C_1 > 0, \qquad |D^m v(\mathbf{x})| \le C_2 \left(\frac{\delta}{\rho(\mathbf{x})}\right)^{\beta+m} \qquad \mathbf{x} \in \Omega'_\delta, \tag{9}$$

where $m = 0, 1$ and C_2 a positive constant which is not depend on m, with the norm of a space $W^1_{2,\beta}(\Omega)$. Denote by $L_{2,\beta}(\Omega, \delta)$ a set of functions from the space $L_{2,\beta}(\Omega)$ which subject to conditions (9) (only for $m = 0$) with a norm of a space $L_{2,\beta}(\Omega)$. Let $L^0_{2,\beta}(\Omega, \delta) = \{q \in L_{2,\beta}(\Omega, \delta) : \int_\Omega \rho^\beta q d\mathbf{x} = 0\}$.

Let $\overset{o}{W}{}^1_{2,\beta}(\Omega,\delta)\,(\overset{o}{W}{}^1_{2,\beta}(\Omega,\delta) \subset W^1_{2,\beta}(\Omega,\delta))$ be a closure by $W^1_{2,\beta}(\Omega)$ norm of a set of the infinitely-differentiable functions with a compact support in Ω comply with the conditions (9). We will say $\varphi(\mathbf{x}) \in W^{1/2}_{2,\beta}(\Gamma,\delta)$, if exists a function $\Phi(\mathbf{x}) \in W^1_{2,\beta}(\Omega,\delta)$ such that $\Phi(\mathbf{x})|_\Gamma = \varphi(\mathbf{x})$ and $\|\varphi\|_{W^{1/2}_{2,\beta}(\Gamma,\delta)} = \inf\limits_{\Phi|_\Gamma=\varphi} \|\Phi\|_{W^1_{2,\beta}(\Omega)}$.

For the vector field $\mathbf{v} = (v_1, v_2)$ we define sets $\mathbf{L}_{2,\beta}(\Omega,\delta)$ and $\mathbf{W}^1_{2,\beta}(\Omega,\delta)$ such that $v_i \in L_{2,\beta}(\Omega,\delta)$ and $v_i \in W^1_{2,\beta}(\Omega,\delta)$, respectively, with a bounded norms $\|\mathbf{v}\|_{\mathbf{L}_{2,\beta}(\Omega)} = \sqrt{\|v_1\|^2_{L_{2,\beta}(\Omega)} + \|v_2\|^2_{L_{2,\beta}(\Omega)}}$ for the first set and $\|\mathbf{v}\|_{\mathbf{W}^1_{2,\beta}(\Omega)} = \sqrt{\|v_1\|^2_{W^1_{2,\beta}(\Omega)} + \|v_2\|^2_{W^1_{2,\beta}(\Omega)}}$ for the second one. Similarly, we define the sets of vector fields $\mathbf{W}^{1/2}_\beta(\Gamma,\delta)$ and $\overset{o}{\mathbf{W}}{}^1_{2,\beta}(\Omega,\delta)$ on Γ and in Ω, respectively.

We introduce the concept of an R_ν-generalized solution for the linearized problem (7), (8).

Definition 1. The pair $\mathbf{u}^{(k)}_\nu \in \mathbf{W}^1_{2,\nu}(\Omega,\delta)$ and $P^{(k)}_\nu \in L^0_{2,\nu}(\Omega,\delta)$ is called an R_ν-generalized solution of the problem (7), (8), where $\mathbf{u}^{(k)}_\nu$ satisfies a condition (8) on Γ for any pair $\mathbf{v} \in \overset{o}{\mathbf{W}}{}^1_{2,\nu}(\Omega,\delta)$ and $q \in L^0_{2,\nu}(\Omega,\delta)$

$$a_k(\mathbf{u}^{(k)}_\nu, \mathbf{v}) + b(\mathbf{v}, P^{(k)}_\nu) = l(\mathbf{v}), \qquad c(\mathbf{u}^{(k)}_\nu, q) = 0$$

hold, where bilinear and linear forms are as follows

$$a_k(\mathbf{u}^{(k)}_\nu, \mathbf{v}) = \int\limits_\Omega \left[\alpha\rho^{2\nu}\mathbf{u}^{(k)}_\nu \cdot \mathbf{v} + \bar{\nu}\nabla\mathbf{u}^{(k)}_\nu \cdot \nabla(\rho^{2\nu}\mathbf{v}) + \rho^{2\nu}((\operatorname{curl} \mathbf{u}^{(k-1)}_\nu) \times \mathbf{u}^{(k)}_\nu) \cdot \mathbf{v}\right] d\mathbf{x},$$

$$b(\mathbf{v}, P^{(k)}_\nu) = -\int\limits_\Omega P^{(k)}_\nu \operatorname{div}(\rho^{2\nu}\mathbf{v}) d\mathbf{x}, \qquad c(\mathbf{u}^{(k)}_\nu, q) = -\int\limits_\Omega (\rho^{2\nu} q) \operatorname{div} \mathbf{u}^{(k)}_\nu \, d\mathbf{x}, \qquad l(\mathbf{v}) = \int\limits_\Omega \rho^{2\nu} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}$$

and $\mathbf{f} \in \mathbf{L}_{2,\beta}(\Omega,\delta)$, $\mathbf{g} \in \mathbf{W}^{1/2}_{2,\beta}(\Gamma,\delta)$, $\nu \geq \beta \geq 0$.

3. The weighted finite element method

Perform triangulation Υ_h based on the barycentric partition of the elements L_i of the quasi-uniform triangulation T_h of the domain $\overline{\Omega}$. Then, we divide each element $L_i \in T_h$ (macroelement) into three triangles K_{i_j} (finite element), $K_{i_j} \in \Upsilon_h$ (their common vertex is in the barycenter of the macroelement L_i). Let R_l and S_m be the vertices and midpoints of the sides $K_s \in \Upsilon_h$, respectively. Introduce the notation of sets:
1)$Z^{vel} = Z^{vel}_\Omega \cup Z^{vel}_\Gamma = \{R_l \cup S_m\}$, where Z^{vel}_Ω and Z^{vel}_Γ are triangulation nodes subsets for the velocity field components in Ω and on Γ, respectively;
2)$Z^{pres} = \{Q_l\}$ of triangulation nodes for the Bernoulli pressure, where the node Q_l an exact match to the node R_m at the appropriate K_{i_j}.

We denote by $\Omega_h = \bigcup\limits_{K_s \in \Upsilon_h} K_s$ the totality of the finite elements with sides of order h. Next, we describe the Scott-Vogelius (SV) element pair (see [12]). For the components of the velocity field, we use polynomials of the second degree (X^h), and for the pressure — the first one (Z^h):
$X^h = \{v_h \in C(\Omega) : v_h|_K \in P_2(K), \forall K \in \Upsilon_h\}, \mathbf{X}_h = X^h \times X^h$;
$Z^h = \{z_h \in L_2(\Omega) : z_h|_K \in P_1(K), \forall K \in \Upsilon_h, \int\limits_\Omega z_h d\mathbf{x} = 0\}$.

The SV pair has useful feature, namely $\operatorname{div} \mathbf{X}^h \subset Z^h$. Next, we represent special basis functions and construct a scheme of the weighted finite element method. To each node $M_m \in Z^{vel}_\Omega (Q_l \in Z^{pres})$ we associate the basis function

$$\theta_m(\mathbf{x}) = \rho^{\nu^*}(\mathbf{x}) \cdot \varphi_m(\mathbf{x}), \ \left(\chi_l(\mathbf{x}) = \rho^{\mu^*}(\mathbf{x}) \cdot \psi_l(\mathbf{x})\right), m = 0, 1, \dots, (\ l = 0, 1, \dots),$$

where $\varphi_m \in X^h$, $\varphi_m(M_j) = \delta_{mj}\ m, j = 0, 1, \dots \left(\psi_l \in Z^h, \psi_l(Q_j) = \delta_{lj},\ l, j = 0, 1, \dots\right)$; $\delta_{lj} = \begin{cases} 1, l = j, \\ 0, l \neq j, \end{cases}$, ν^* and μ^* are real parameters.

The spaces V^h and Q^h for the components of the velocity field and pressure are defined as linear span of the basis functions $\{\theta_m\}_m$ and $\{\chi_l\}_l$, respectively. Let V_0^h be a subspace of $V^h\ :\ V_0^h = \{v^h \in V^h\ :\ v_h(M_m)|_{M_m \in Z_\Gamma^{vel}} = 0\}$. The approximate components of the velocity field $\mathbf{u}_{\nu,h}^{(k)} = (u_{\nu,h,1}^{(k)},\ u_{\nu,h,2}^{(k)})$ and pressure $P_{\nu,h}^{(k)}$ we seek as a

$$u_{\nu,h,1}^{(k)}(\mathbf{x}) = \sum_m d_{2m}^{(k)}\,\theta_m(\mathbf{x}),\ \ u_{\nu,h,2}^{(k)}(\mathbf{x}) = \sum_m d_{2m+1}^{(k)}\,\theta_m(\mathbf{x}),\ \ P_{\nu,h}^{(k)}(\mathbf{x}) = \sum_l e_l^{(k)}\,\chi_l(\mathbf{x}), \tag{10}$$

where $d_j^{(k)} = \rho^{-\nu^*}(M_{[j/2]})\,\tilde{d}_j^{(k)}, e_i^{(k)} = \rho^{-\mu^*}(Q_i)\,\tilde{e}_i^{(k)}$. The coefficients $d_j^{(k)}$ and $e_i^{(k)}$ in (10) are found as a result of solving a system (11), (12) (see below). Let $\mathbf{V}^h = V^h \times V^h$, $\mathbf{V}_0^h = V_0^h \times V_0^h$ and $\mathbf{V}^h \subset \mathbf{W}_{2,\nu}^1(\Omega_h, \delta)$, $\mathbf{V}_0^h \subset \overset{o}{\mathbf{W}}{}_{2,\nu}^1(\Omega_h, \delta)$, $Q^h \subset L_{2,\nu}^0(\Omega_h, \delta)$.

Definition 2. The pair $\mathbf{u}_{\nu,h}^{(k)} \in \mathbf{V}^h$ and $P_{\nu,h}^{(k)} \in Q^h$ is called an approximate R_ν-generalized solution of the problem (7), (8) for any pair $\mathbf{v}^h \in \mathbf{V}_0^h$ and $q^h \in Q^h$ the equalities

$$a_k(\mathbf{u}_{\nu,h}^{(k)}, \mathbf{v}^h) + b(\mathbf{v}^h, P_{\nu,h}^{(k)}) = l(\mathbf{v}^h) \qquad \text{and} \qquad c(\mathbf{u}_{\nu,h}^{(k)}, q^h) = 0 \tag{11}$$

hold, where $\mathbf{f} \in \mathbf{L}_{2,\beta}(\Omega, \delta)$, $\mathbf{g} \in \mathbf{W}_{2,\beta}^{1/2}(\Gamma, \delta)$, $\nu \geq \beta \geq 0$.

Thus, we construct a weighted FEM to find an R_ν-generalized solution for the problem (7), (8). We get a system of linear algebraic equation:

$$\mathbf{A}_k\mathbf{d}^{(k)} + \mathbf{B}\mathbf{e}^{(k)} = \omega \qquad \text{and} \qquad \mathbf{C}^T\mathbf{d}^{(k)} = \mathbf{0}, \tag{12}$$

where $\mathbf{d}^{(k)} = (d_0^{(k)}, d_2^{(k)}, \dots, d_1^{(k)}, d_3^{(k)}, \dots)^T$, $\mathbf{e}^{(k)} = (e_0^{(k)}, e_1^{(k)}, e_2^{(k)}, \dots)^T$ and ω be a vector of values of the linear form $l(\theta_m)$.

4. Iterative procedure

Now, we present an iterative procedure for solving the sequences of systems view (12), $k = 1, 2, 3, \dots$ and thus we will approximately solve the original nonlinear problem in rotation form (4), (5):

1. Let $\mathbf{d}^{(0)}$ and $\mathbf{e}^{(0)}$ be an arbitrary vectors such that $\mathbf{u}_{\nu,h}^{(0)}(M_l)|_{M_l \in Z_\Gamma^{vel}} = \mathbf{g}(M_l)$, $\operatorname{div} \mathbf{u}_{\nu,h}^{(0)}(M_l)|_{M_l \in Z^{vel}} = 0$ (for example $\mathbf{u}_{\nu,h}^{(0)}(M_l)|_{M_l \in Z_\Omega^{vel}} = \mathbf{0}$) and $P_{\nu,h}^{(0)}(M_l)|_{M_l \in Z^{pres}} = 0$.

2. Realize the Picard's procedure $k = 0, 1, 2, \dots$ until the stopping condition is fulfilled:

a) Let $\zeta_0^{(k)} := \mathbf{d}^{(k)}$ and $\eta_0^{(k)} := \mathbf{e}^{(k)}$;

b) We construct an internal convergent iterative process (see [17]). For $n = 0, 1, \dots, N_k - 1\ :$

$$\zeta_{n+1}^{(k)} = \zeta_n^{(k)} + \hat{\mathbf{A}}_k^{-1}(\omega - \mathbf{A}_k\zeta_n^{(k)} - \mathbf{B}\eta_n^{(k)})$$
$$\eta_{n+1}^{(k)} = \eta_n^{(k)} + \hat{\mathbf{S}}_k^{-1}\mathbf{C}^T\zeta_{n+1}^{(k)};$$

c) Let $\mathbf{d}^{(k+1)} := \zeta_{N_k}^{(k)}$ and $\mathbf{e}^{(k+1)} := \eta_{N_k}^{(k)}$,
where $\hat{\mathbf{A}}_k$ and $\hat{\mathrm{S}}_k$ are the preconditioning matrices to $\mathbf{A}_k$ and $\mathrm{S}_k = \mathbf{C}^T\mathbf{A}_k^{-1}\mathbf{B}$, respectively.

At first, we build a preconditioner $\hat{\mathbf{A}}_k$ applying an incomplete LU factorization. We employ the GMRES(5)-method (see [18]). If we have error $\mathbf{r}_0 = \hat{\mathbf{A}}_k^{-1}(\mathbf{s} - \mathbf{A}_k\mathbf{v})$ for the problem $\mathbf{A}_k\mathbf{v} = \mathbf{s}$, then the Arnoldi procedure will build an orthonormal basis of the subspace: Span$\{\mathbf{r}_0, \hat{\mathbf{A}}_k^{-1}\mathbf{A}_k\mathbf{r}_0, \dots, (\hat{\mathbf{A}}_k^{-1}\mathbf{A}_k)^4\mathbf{r}_0\}$.

Further, we construct an auxiliary matrix $\tilde{\mathrm{S}}_k$ to $\hat{\mathrm{S}}_k$, which is the weight mass matrix $\mathbf{M}^{\nu,\mu^*,\bar{\nu}}$, such that on each $L \in \Upsilon_h$:

$$(\mathbf{M}^{\nu,\mu^*,\bar{\nu}})_{lm} = \frac{1}{\bar{\nu}} \int\limits_L \rho^{2(\nu+\mu^*)}\, \psi_l(\mathbf{x})\, \psi_m(\mathbf{x}) d\mathbf{x}, \;\; l, m = 0, 1, \dots .$$

After that, we define a diagonal matrix $\bar{\mathrm{S}}_k = \bar{\mathbf{M}}^{\nu,\mu^*,\bar{\nu}}$, where $\left(\bar{\mathbf{M}}^{\nu,\mu^*,\bar{\nu}}\right)_{ii} = \sum\limits_l \left(\mathbf{M}^{\nu,\mu^*,\bar{\nu}}\right)_{il}$.

It is known (see [19]), that such diagonal lumping $\bar{\mathrm{S}}_k$ is a good preconditioner to matrix $\tilde{\mathrm{S}}_k$. In order to determining the vector $\psi^\diamond := \hat{\mathrm{S}}_k^{-1}\theta$ we need to find a solution of the internal procedure:
1) $\phi_0 = \mathbf{0}$;
2) $\phi_m = \phi_{m-1} + \bar{\mathrm{S}}_k^{-1}(\theta - \tilde{\mathrm{S}}_k\phi_{m-1})\,(m = 1, \dots, M)$;
3) $\psi^\diamond = \phi_M$.
We use the GMRES(5)-method: (Span$\{\bar{\mathbf{r}}, (\bar{\mathrm{S}}_k^{-1}\tilde{\mathrm{S}}_k)\bar{\mathbf{r}}, \dots, (\bar{\mathrm{S}}_k^{-1}\tilde{\mathrm{S}})_k^4\,\bar{\mathbf{r}}\}$, $\bar{\mathbf{r}} = \bar{\mathrm{S}}_k^{-1}(\theta - \tilde{\mathrm{S}}_k\phi_{m-1})$).

5. Results of numerical experiments

Let $\Omega = (-1,1)\times(-1,1)\backslash[0,1]\times[-1,0]$. Then we split $\bar{\Omega}$ by the horizontal and vertical lines $x_1^{(j)} = -1 + j\,h$, and $x_2^{(i)} = -1 + i\,h$, respectively, into elementary squares S_l, where $j, i = 0, \dots, N$, $h = \frac{2}{N}$, N – even number. After that, we divide each S_l by the diagonal (the lower left corner connects to the upper right corner) into two triangles L_m (macroelements). Further, each macroelement L_m is partitioned into three triangles K_s (barycentric partition). Consider a solution $(\mathbf{u},\, P)$ of nonlinear problem (4), (5) which has a singularity in the vicinity of the reentrant corner $\sigma = \frac{3\pi}{2}$ with apex at the origin $(0,0)$:

$$u_1(x_1,\, x_2) = \left(x_1^2 + x_2^2\right)^{\frac{\lambda}{2}} \Big((1+\lambda)\, E(x_1,\, x_2)\cdot\sin(\mathrm{arctg}\frac{x_2}{x_1}) + G(x_1,\, x_2)\cdot\cos(\mathrm{arctg}\frac{x_2}{x_1})\Big),$$

$$u_2(x_1,\, x_2) = \left(x_1^2 + x_2^2\right)^{\frac{\lambda}{2}} \Big(G(x_1,\, x_2)\cdot\sin(\mathrm{arctg}\frac{x_2}{x_1}) - (1+\lambda)\, E(x_1,\, x_2)\cdot\cos(\mathrm{arctg}\frac{x_2}{x_1})\Big),$$

$$P(x_1,\, x_2) = \left(x_1^2 + x_2^2\right)^{\frac{\lambda-1}{2}} \Big(\frac{(1+\lambda)^2\, G(x_1,\, x_2) + H(x_1,\, x_2)}{\lambda - 1}\Big),$$

where

$$E(x_1,\, x_2) = \cos((1-\lambda)\,\mathrm{arctg}\frac{x_2}{x_1}) - \cos((1+\lambda)\,\mathrm{arctg}\frac{x_2}{x_1}) +$$

$$+\frac{\sin((1+\lambda)\,\mathrm{arctg}\frac{x_2}{x_1})\cdot\cos(\lambda\,\sigma)}{\lambda+1} + \frac{\sin((1-\lambda)\,\mathrm{arctg}\frac{x_2}{x_1})\cdot\cos(\lambda\,\sigma)}{\lambda-1},$$

$$G(x_1,\, x_2) = (1+\lambda)\sin((1+\lambda)\,\mathrm{arctg}\frac{x_2}{x_1}) - (1-\lambda)\sin((1-\lambda)\,\mathrm{arctg}\frac{x_2}{x_1}) +$$

$$+\cos((1+\lambda)\,\mathrm{arctg}\frac{x_2}{x_1})\cdot\cos(\lambda\,\sigma) - \cos((1-\lambda)\,\mathrm{arctg}\frac{x_2}{x_1})\cdot\cos(\lambda\,\sigma),$$

Table 1
The error between the generalized solution $\mathbf{u}_h$ and exact one $\mathbf{u}$ in the $\mathbf{W}_2^1(\Omega)$ norm.

N=	280	140	70
	1.379e-1	1.988e-1	2.848e-1

Table 2
The error between an R_ν-generalized solution $\mathbf{u}_{\nu,h}$ and exact one $\mathbf{u}$ in the $\mathbf{W}_{2,\nu}^1(\Omega)$ norm, for different values ν, δ and $\nu^*(\mu^* = \nu^*)$.

(ν, ν^*, δ), $N =$	280	140	70
$(2.0, -0.5, 0.015)$	1.627e-5	3.295e-5	6.629e-5
$(2.0, \lambda - 1, 0.015)$	1.394e-5	2.796e-5	5.626e-5
$(2.0, -0.4, 0.015)$	1.181e-5	2.355e-5	4.760e-5
$(1.9, -0.5, 0.016)$	2.785e-5	5.624e-5	1.131e-4
$(1.9, \lambda - 1, 0.016)$	2.331e-5	4.672e-5	9.368e-5
$(1.9, -0.4, 0.016)$	1.869e-5	3.759e-5	7.549e-5

Table 3
The number of grid nodes $M_i \in Z_\Omega^{vel}$ (in percentage of their total number), where the errors β_j^i are more than the given limit values $\varepsilon_l, l = 1, 2$, of the generalized solution $\nu = \nu^* = 0, \delta = 1$.

N=	280	140
$\varepsilon_1 = 10^{-5}$	64.51%	77.82%
$\varepsilon_2 = 10^{-6}$	90.97%	94.81%

Table 4
The number of grid nodes $M_i \in Z_\Omega^{vel}$ (in percentage of their total number), where the errors $\beta_{\nu,j}^i$ are more than the given limit values $\varepsilon_l, l = 1, 2$, of an R_ν- generalized solution $\nu = 1.8, \nu^* = -0.31, \delta = 0.014$.

N=	280	140
$\varepsilon_1 = 10^{-5}$	28.33%	54.30%
$\varepsilon_2 = 10^{-6}$	77.32%	85.76%

$$H(x_1,\ x_2) = (1-\lambda)^3 \sin((1-\lambda)\,\mathrm{arctg}\frac{x_2}{x_1}) - (1+\lambda)^3 \sin((1+\lambda)\,\mathrm{arctg}\frac{x_2}{x_1}) +$$
$$+ (\lambda - 1)^2 \cos((1-\lambda)\,\mathrm{arctg}\frac{x_2}{x_1}) \cdot \cos(\lambda\,\sigma) - (\lambda+1)^2 \cos((1+\lambda)\,\mathrm{arctg}\frac{x_2}{x_1}) \cdot \cos(\lambda\,\sigma).$$

Let $\alpha = \bar{\nu} = 1$ and $\lambda = 0.54448$. The pair of functions $(\mathbf{u}, P)$ is analytic in $\bar{\Omega} \setminus (0,0)$, but $\mathbf{u} \notin \mathbf{W}_2^2(\Omega)$ and $P \notin W_2^1(\Omega)$. It is a typical situation in non-convex polygonal domains.

Numerical experiments were carried out on grids with different steps h. The errors of the generalized (classical FEM with $\nu = 0, \delta = 1, \nu^* = \mu^* = 0$) and R_ν-generalized (presented weighted FEM) solutions were determined using the modulus of the difference between the exact solution and approximate one at the nodes M_i, i. e. $\beta_j^i = |u_j(M_i) - u_{h,j}(M_i)|$ for the generalized solution and $\beta_{\nu,j}^i = |u_j(M_i) - u_{\nu,h,j}(M_i)|$ for an R_ν-generalized one, where $M_i \in Z_\Omega^{vel}, j = 1, 2$, and also in the norms of generalized functions. See Figures 1-2 and Tables 1-4. The optimal values of parameters ν, ν^* and δ were derived numerically.

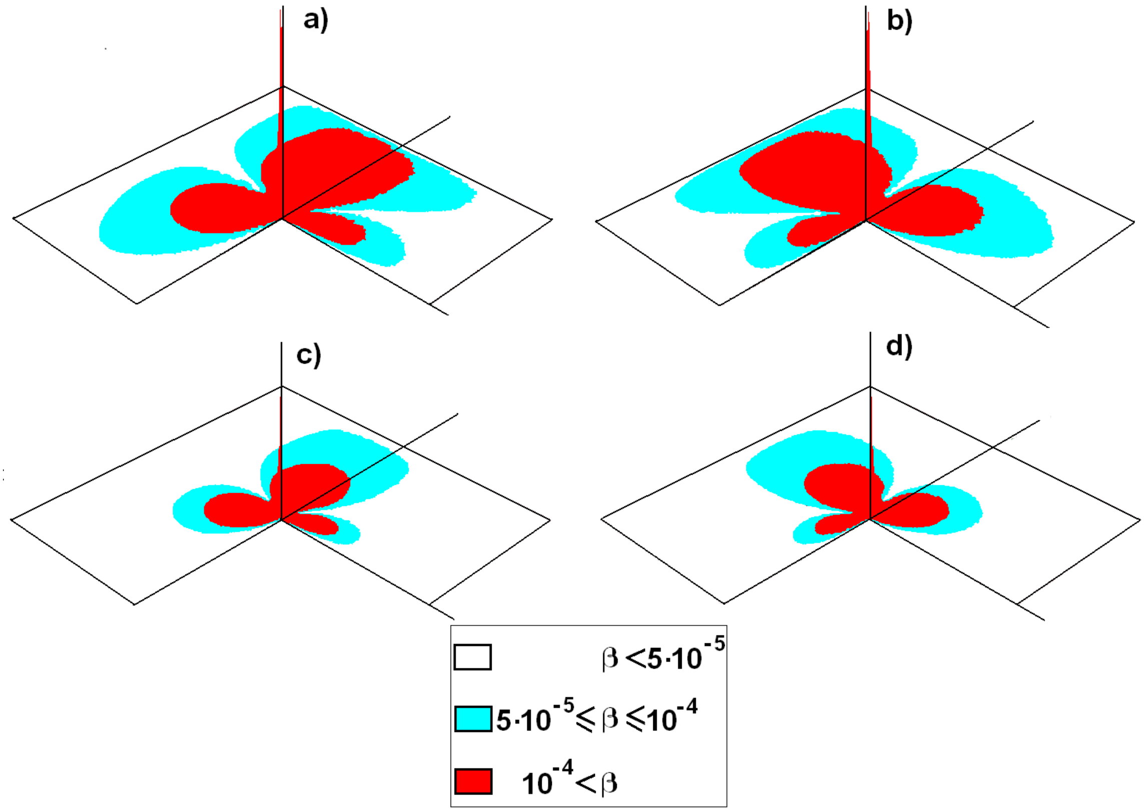

Figure 1: Distribution of the points M_i with errors β of the generalized solution ($\nu = 0, \delta = 1, \nu^* = \mu^* = 0$): *a)* $N = 140$, *c)* $N = 280$ and *b)* $N = 140$, *d)* $N = 280$ for the 1st and 2nd components of $\mathbf{u}_h$, respectively.

6. Conclusions

The results of computational experiments for the steady Navier-Stokes equations (4), (5) lead to the following conclusions:

1) An approximate R_ν-generalized solution by the weighted FEM converges to the exact one with a $\mathcal{O}(h)$ rate in the $\mathbf{W}^1_{2,\nu}(\Omega)$ norm (see Table 2), while the approximate generalized solution by the classical FEM converges to the exact one with a $\mathcal{O}(h^{0.54})$ rate in the $\mathbf{W}^1_2(\Omega)$ norm (see Table 1). In other words, the proposed method suppresses the so-called pollution effect [13].

2) For all values of δ, ν and ν^* from the range of optimal values ($\delta \sim h, \nu \sim 2$ and $\nu^* \sim 1 - \lambda$) an approximate R_ν-generalized solution converges to the exact one with a $\mathcal{O}(h)$ rate in the $\mathbf{W}^1_{2,\nu}(\Omega)$ norm.

3) The number of nodes and their surroundings by using a weighted FEM, in which the values of the absolute errors $\beta^i_{\nu,j}, j = 1, 2$, do not exceed the given values, increases with N and is much more then by using the classical FEM (see Tables 3-4) and Figures 1-2.

Acknowledgments

The reported study was supported by RSF according to the research project No. 21-11-00039. Computational resources for the numerical experiments were provided by the Shared Services Center "Data Center of FEB RAS".

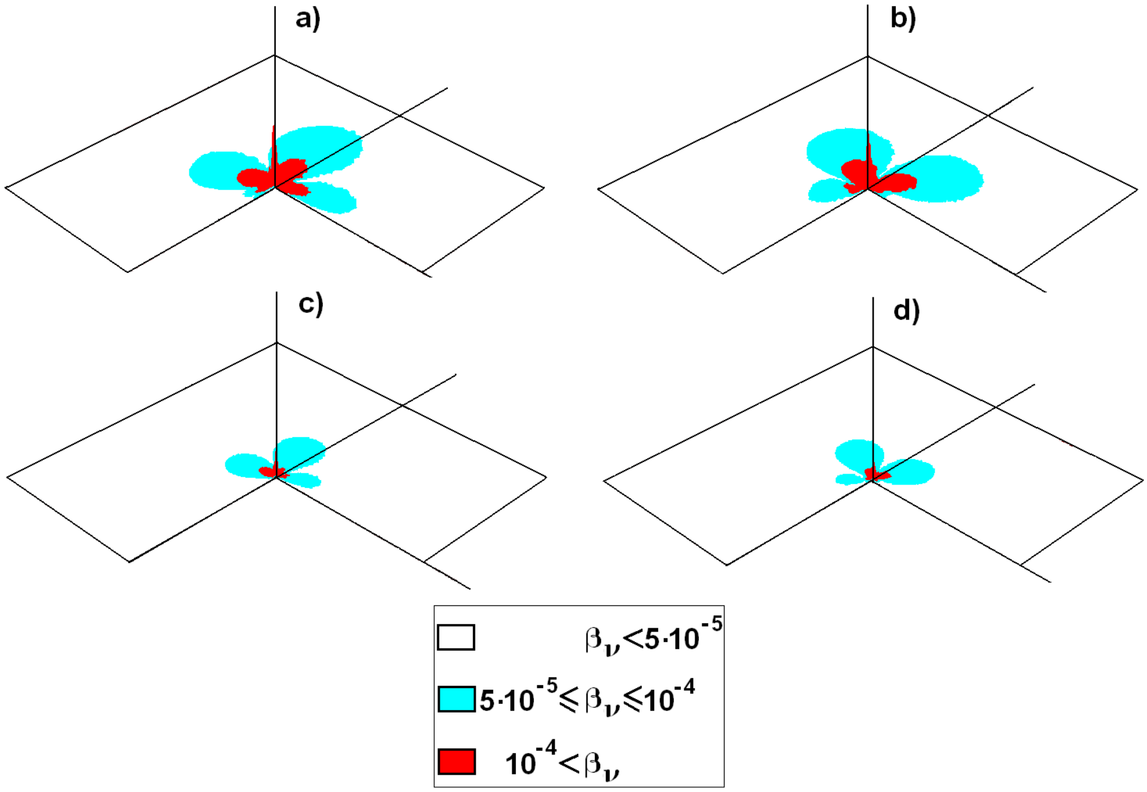

Figure 2: Distribution of the points M_i with errors β_ν of the R_ν-generalized solution ($\nu = 1.9, \delta = 0.014, \nu^* = \mu^* = -0.35$): *a*) $N = 140$, *c*) $N = 280$ and *b*) $N = 140$, *d*) $N = 280$ for the 1st and 2nd components of $\mathbf{u}_{\nu,h}$, respectively.

References

[1] V. A. Rukavishnikov, On the differential properties of R_ν-generalized solution of Dirichlet problem, Dokl. Akad. Nauk SSSR 309 (1989) 1318–1320.

[2] V. A. Rukavishnikov, On the existence and uniqueness of an R_ν-generalized solution of a boundary value problem with uncoordinated degeneration of the input data, Doklady Mathematics 90 (2014) 562–564. doi:10.1134/S1064562414060155.

[3] V. A. Rukavishnikov, E. V. Kuznetsova, The R_ν-generalized solution of a boundary value problem with a singularity belongs to the space $W^{k+2}_{2,\nu+\beta/2+k+1}(\Omega, \delta)$, Differential Equations 45 (2009) 913–917. doi:10.1134/S0012266109060147.

[4] V. A. Rukavishnikov, S. G. Nikolaev, On the R_ν-generalized solution of the Lamé system with corner singularity, Doklady Mathematics 92 (2015) 421–423. doi:10.1134/S1064562415040080.

[5] V. A. Rukavishnikov, A. V. Rukavishnikov, Weighted finite element method for the Stokes problem with corner singularity, Journal of Computational and Applied Mathematics 341 (2018) 144–156. doi:10.1016/j.cam.2018.04.014.

[6] V. A. Rukavishnikov, A. Y. Bespalov, An exponential rate of convergence of the finite element method for the Dirichlet problem with a singularity of the solution, Doklady Mathematics 62 (2000) 266–270.

[7] V. A. Rukavishnikov, E. V. Kuznetsova, A finite element method scheme for boundary value

problems with noncoordinated degeneration of input data, Numerical Analysis and Applications 2 (2009) 250–259. doi:10.1134/S1995423909030069.

[8] V. A. Rukavishnikov, E. I. Rukavishnikova, Numerical method for Dirichlet problem with degeneration of the solution on the entire boundary, Symmetry 11 (2019) 1455. doi:10.3390/sym11121455.

[9] V. A. Rukavishnikov, A. O. Mosolapov, E. I. Rukavishnikova, Weighted finite element method for elasticity problem with a crack, Computers and Structures 243 (2021) 106400. doi:10.1016/j.compstruc.2020.106400.

[10] V. A. Rukavishnikov, O. P. Tkachenko, Dynamics of a fluid-filled curvilinear pipeline, Applied Mathematics and Mechanics 39 (2018) 905–922. doi:10.1007/s10483-018-2338-9.

[11] M. Benzi, H. Golub, J. Liesen, Numerical solution of saddle point problems, Acta Numerica 14 (2005) 1–137. doi:10.1017/S0962492904000212.

[12] L. R. Scott, M. Vogelius, Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials, Mathematical Modelling and Numerical Analysis 19 (1985) 111–143.

[13] H. Blum, The influence of reentrant corners in the numerical approximation of viscous flow problems, volume 30 of *Numerical Treatment of the Navier-Stokes Equations*, Springer, 1990. doi:110.1007/978-3-663-14004-7-4.

[14] V. A. Rukavishnikov, A. V. Rukavishnikov, New approximate method for solving the Stokes problem in a domain with corner singularity, Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programming & Computer Software 11 (2018) 95–108. doi:10.14529/mmp180109.

[15] V. A. Rukavishnikov, A. V. Rukavishnikov, New numerical method for the rotation form of the Oseen problem with corner singularity, Symmetry 11 (2019) 54. doi:10.3390/sym11010054.

[16] V. A. Rukavishnikov, A. V. Rukavishnikov, The method of numerical solution of the one stationary hydrodymics problem in convective form in L-shaped domain, Computer Research and Modeling 12 (2020) 1291–1306. doi:10.20537/2076-7633-2020-12-6-1291-1306.

[17] J. H. Bramble, J. E. Pasciak, A. T. Vassilev, Analysis of the inexact Uzawa algorithm for saddle point problems, SIAM Journal on Numerical Analysis 34 (1997) 1072–1092. doi:10.1137/S0036142994273343.

[18] Y. Saad, Iterative methods for sparse linear systems, 2nd. ed., SIAM, Philadelphia, 2003.

[19] M. Olshanskii, A. Reusken, Analysis of a Stokes interface problem, Numerische Mathematik 103 (2006) 129–149. doi:10.1007/S00211-005-0646-X.