# Design of Stable Periodic Regimes for One Class of Hybrid Planar Systems

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**Abstract**

The conditions for the existence of limit regimes for the Liénard equation, the solutions of which are subjected to instantaneous forces of impulse nature at unfixed moments in time, are investigated. For this system, the constructive conditions for the existence of periodic solutions (limit cycles) are obtained such that the phase point of the system when moving along the corresponding trajectory is subjected to $n \in \mathbb{N}$ impulse effects for the period. It is shown that the points that define cycles corresponding to periodic solutions satisfy the Sharkovsky order. The conditions for the existence of at least one impulsive periodic solution and a single discontinuous limit cycle are found. The existence of a single stable limit cycle is proved, the phase point of which will be affected by pulsed forces $n \in \mathbb{N}$ times. It is shown that a stable limit cycle under given conditions will exist despite the presence of the influence of destabilizing forces of impulsed nature.

**Keywords** [1]

differential equation, impulse action, Liénard equation, limit cycle

## 1. Introduction

The rapid development of modern science and technology requires constant attention to the study of nonlinear evolutionary dynamical systems in which there are short-term processes or which are under the action of external forces, the duration of which can be neglected in the preparation of appropriate mathematical models.

Such evolutionary models can be found, for example, in mechanics, chemical technology, medicine and mathematical biology, aircraft dynamics, economics, adaptive control theory and other fields of science and technology, where we have to study systems under the influence of short-term (pulsed) external forces called systems with pulsed action.

In fact, it has been found that the presence of impulse action can significantly complicate the behavior of the trajectories of such systems, even for cases of rather simple differential equations. In the general case, in the presence of impulse action, the qualitative behavior of solutions of differential equations (including linear problems with constant coefficients) can be significantly nonlinear and significantly different from the behavior of such systems in the absence of impulse action.

Studying the nonlinear damping of oscillations in electric circuits, Liénard obtained a natural generalization of the famous van der Paul equation. At the same time, the problem of the existence of periodic regimes is important for oscillating systems in the region. Note that the limit cycle is an isolated closed trajectory of the vector field (in other words, it is a periodic solution in some neighborhood which has no other periodic solutions, respectively, all other trajectories from this region tend to the limit cycle in positive or negative time). Therefore, when modeling many systems of oscillating systems that are affected by destabilizing factors of instantaneous (impulse) nature, it is important to understand the conditions for the existence of stable periodic regimes in them. Obtaining appropriate design conditions allows to develop methods for supporting decision-making on the management of such systems. Special attention should be paid to models that describe the most natural objects - dissipative dynamical systems. A dissipative system (or dissipative structure, from the Latin dissipatio - "disperse,

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destroy") - is an open system that operates in the neighbourhood of equilibrium position. In other words, it characterizes the state that occurs in an nonequilibrium environment under the condition of energy dissipation. A dissipative system is sometimes called a stationary open system or a nonequilibrium open system. The dissipative system is characterized by the spontaneous appearance of a complex, often chaotic structures. Recent studies in the field of dissipative structures allow conclude that self-organization occurs much faster in the presence of external and internal influences in the system. Thus, such effects accelerate the process of self-organization.

In the phase plane, the trajectory corresponding to such a solution is represented by the so-called limit cycle. The limit cycle is an isolated closed curve on the phase plane, to which all integral curves are approached in the limit case at t– "". The limit cycle is a stationary mode with a certain amplitude, which does not depend on the initial conditions, but is determined only by the structure of the system.

In general, if there is some closed domain on the phase plane such that all phase trajectories crossing the boundary of this region enter it and there is an unstable singular point within this domain, then the latter necessarily has at least one limit cycle.

At the same time, when there is a domain on the phase plane from which the phase trajectories do not come out and in which there are no equilibrium positions (special points), then the limit cycle exists in this domain, and the rest of all trajectories are wound on it.

Thus, if we find on the phase plane such a two-connected domain that the directions of the phase trajectories are inverted inside this region on the whole boundary, then we can say that the limit cycle exists inside this domain.

In fact, this work is devoted to the study of the limit cycles existence conditions for a generalized second-order differential equation of the Lénard type under the influence of destabilizing external impulse perturbations.

## 2. Analysis of literature sources

Classical statements of impulsive-perturbed problems, as well as basic notions concerning qualitative behavior of solutions for such systems in the case of impulsive effect at fixed moments of time, were developed in [1-3] as an adequate mathematical tool for describing physical and mechanical phenomena where instantaneous changes of the phase state are present. Global attracting sets for impulsive evolutionary systems, including random noise, with impulses at fixed moments of time, were studied in [4,5]. Robust stability properties for such systems in terms of Input-to-State Stability theory were considered in [6-8]. Limit cycles for finite-dimensional impulsive dynamical systems, that is, systems described by ordinary differential equations whose trajectories undergo instantaneous changes after reaching a certain surface of the phase space, were investigated in [10]. Systematic studying of the qualitative behavior of impulsive dynamical systems infinite-dimensional spaces was carried out in [11-15]. Global attracting sets for abstract infinite-dimensional impulsive dynamical systems were investigated in [16-18]. A modern point of view on systems with mixed types of dynamics, i.e., systems where there exist both continuous dynamics described by systems of differential equations and discrete, described by difference equations, was reflected in [19], where such systems were called hybrid systems.

In [20] the review of the most modern research methods for impulse differential equations solutions stability and their application to problems of impulse adaptive control is carried out. In [21] the problem of design the approximate adaptive control, including the case of impulse control functions, is considered for some classes of infinite-dimensional problems. The well-known method of averaging for obtaining approximate adaptive control is substantiated. The concept of an impulsive non-autonomous evolutionary system is introduced. Questions concerning existence and properties of impulsive attracting sets are investigated. The obtained results are applied to the study of the qualitative behavior of the two-dimensional impulsive-perturbed Navier-Stokes system.

In [22] the recursive properties of almost periodic motions of impulsive-perturbed evolutionary systems are studied. The obtained results are effectively applied to the study of discrete systems qualitative behavior. In [23] the qualitative properties of stability with respect to the external (control) perturbations for differential equations systems with impulse effects at fixed moments of time are studied. The transparent criteria of stability conditions for classes of impulsive systems having a Lyapunov type function are obtained. In [24], non-autonomous evolutionary problems with multi-valued right-hand parts and with impulse influences at fixed moments of time are considered. The corresponding non-autonomous multivalued evolutionary systems are designed, for which the existence of a compact global attractor in phase space is proved.

The problems of control and decision-making in the presence of impulse perturbations were investigated in [25], for periodic solutions were studied in [27]. In [28], the existence of global attracting sets in multi-valued discontinuous infinite-dimensional evolutionary systems, which can have trajectories with an infinite number of impulsive perturbations, was proved. The obtained abstract schemes are applied to the asymptotic behavior study of the weakly nonlinear impulsive-perturbed parabolic equations and inclusions.

In all the above works the basement of the qualitative theory of differential systems with impulsive perturbations (jumps) are designed. In essence, the main issues of the qualitative theory of impulsive systems were investigated with the help of the classical qualitative theory of ordinary differential equations, methods of asymptotic integration for such equations, the theory of difference equations and generalized functions. However, the question of the solutions existence for weakly nonlinear impulse systems has not yet been investigated appropriately.

At the same time, the works in which important results in the field of information technologies and social communications were obtained deserve attention. In particular, in works [29] studied applied control algorithm functionally sustainable production processes industry.

## 3. Conditions for the existence of a limit cycle for the general Liénard equation with impulse action

Let us investigate the problem of the existence of harmonic cycles for the Liénard equation, the solutions of which are subjected to instantaneous forces of impulse nature at unfixed moments in time.

Consider a dynamical system in which motion is described by a generalized Liénard differential equation of the form:

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0, \quad (1)$$

($x \in \mathcal{D} \subset \mathbb{R}^3$, $\mathcal{D} -$ phase space of the system (1), $t \in \mathbb{R} -$ time) and which is affected by instantaneous perturbations determined by some operator $\mathcal{A}_t$, that at the moment of reaching a moving point of some fixed position$x = x_*$ acts according to the rule $(x,\ t) \to (t,\ \mathcal{A}_t x)$. Impulse action in such a system occurs at non-fixed moments of time and increases the amount of motion in the system by a certain amount $I(\dot{x})$, which depends on the speed of the moving point at the time of its passage $x = x_*$. Next we will consider that $I(y)$, де $y = \dot{x}$, as a function of its argument is continuous.

If $t_*$ is a certain moment of time at which the moving point reaches position $x = x_*$, when an impulsive action occurs, the impulsive perturbations of the moving point can be written as [27]:

$$\Delta \frac{dx}{dt}\bigg|_{x=x_*} = \frac{dx}{dt}\bigg|_{t=t_*+0} - \frac{dx}{dt}\bigg|_{t=t_*-0} = \mathcal{A}_t x - x = I(\dot{x}). \quad (2)$$

The description of the physical interpretation of the generalized Liénard equation and the characteristics of its phase limit behavior are studied in detail in [30].

Equation (1) is written in an equivalent way as a system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -g(x) - f(x, y)y. \end{cases} \quad (3)$$

In the sequel, we will use the notation

$$F(x) = \int_0^x f(s)\,ds, \qquad G(x) = \int_0^x g(s)\,ds.$$

We will consider further that functions $g(x)$ and $f(x, y)$ provide the condition for the existence and uniqueness of the solution of the system (3). In addition, we will assume that

$$x\,g(x) > 0 \quad \text{при} \quad x \neq 0 \quad (4)$$

and

$$G(\pm\infty) = \int_0^{\pm\infty} g(x)\,dx. \quad (5)$$

In this case, the origin of the phase plane is the only stationary point. It is covered by a family of so-called energy curves

$$\mathscr{w}(x,y) \equiv G(x) + \frac{1}{2}y^2 = C. \tag{6}$$

They are all closed. Under assumption

$$g(-x) = -g(x) \tag{7}$$

these curves are symmetrical about both coordinate axes.

Let's denote $y_* = \sqrt{2G(x_*)}$.

## 3.1. There are periodic solutions that satisfy Sharkovsky's order

Suppose that

$1^0$. The generalized Liénard differential equation (1) is assumed to be satisfied the conditions for the existence and uniqueness of the solution.

$2^0$. The line $x = x_*$ is transversal to flow (1) everywhere except the trajectory for which the line $x = x_*$ is tangent. In this case, we assume that $I(0) = 0$.

$3^0$. Impulsive operator $\mathcal{A}_t$ is assumed to be continuous with respect to its variables.

Calculate the complete derivative of the derivative with respect to the system (3)

$$\left(\frac{d\mathscr{w}(x,y)}{dt}\right)_{(3)} = g(x)\dot{x} + y\dot{y} = -f(x,y)y^2. \tag{8}$$

It follows that

$$f(x,y) \geq 0 \quad \text{для всіх} \quad x, y, \tag{9}$$

then the energy curves can be intersected by phase trajectories only from the outside to the middle. So

$$|x(t)| \leq a, \qquad |y(t)| \leq b \ \text{ для } \ t \geq 0, \tag{10}$$

and with the help of a phase picture it is possible to be convinced that functions $x(t)$ and $y(t)$ or both oscillating (have the property: for either $t_1 > t_0$ there will be a point $t_2 > t_1$, when going through which function $x(t)$ or $y(t)$ change the sign), or both tend to zero at $t \to \infty$, with $x(t)$ it must eventually become monotonous. In the first case, due to condition (7), the amplitudes of the function $x(t)$ decrease monotonically.

If we additionall assume that the equality $f(x,y) = 0$ does not hold on any curve $\mathscr{w}(x,y) = C$ (that is, such curves should not be on the phase plane), it can be stated that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0. \tag{11}$$

Note that condition (4) is sufficient for all bounded solutions of system (3) to have either oscillating coordinates $x(t)$ and $y(t)$, or coordinates that satisfy (11), and the function $x(t)$ monotonic (for large values $t$). This conclusion can be extended to systems

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y,$$
$$\dot{x} = y, \quad \dot{y} = -g(x) - F(y) \quad (F(0) = 0).$$

It is important that the origin of the phase plane is the only stationary point.

The periodic solutions of system (3) correspond to the cycles surrounding the origin.

A continuum of closed trajectories appears, apparently, when a function $f(x,y)$ identically equal to zero in the annular region $C_1 \leq \mathscr{w}(x,y) \leq C_2$, $0 \leq C_1 < C_2$. Here the cycles coincide with the energy curves.

Under such conditions, describing the motion of the phase point of the system (1), (2), we construct a Poincare map for the line $x = x_*$, which is used to study the question of the existence of periodic regimes of problem (1), (2). It is obvious that in this case the problem of the existence of periodic solutions of system (1), (2) is reduced to the problem of the existence of periodic and fixed points of some mapping of a segment into the same segment, which is determined by the formula

$$f(y) = -y + I(-y), \tag{12}$$

where $I(-y) < y, \quad y \neq 0, \quad y = \dot{x}$.

Consider the problem of the existence of periodic solutions of problem (1), (2), when the impulsive function has the form:

$$I(y) = \begin{cases} (\lambda \ -1)y - \lambda \ y_*, & y \geq 0, \\ -(\lambda \ +1)y - \lambda \ y_*, & y < 0, \end{cases} \tag{13}$$

where $y = \dot{x}$, $\lambda$ — some parameter and $0 < \lambda \leq |\min G(x)|$.

The mapping

$$f(y) = -y + I(-y) = \begin{cases} \lambda \ (y_* - y), & y \geq 0, \\ \lambda \ (y_* + y), & y < 0, \end{cases} \tag{14}$$

is continuous for all $y \in \mathbb{R}$ and has the following properties: when $0 < \lambda < 1$ there is only one fixed point that is stable; if $1 < \lambda \leq |\min G(x)|$ then we have two fixed points

$$\left\{\frac{\lambda}{1-\lambda} y_*\right\} \text{ and } \left\{\frac{\lambda - \lambda^2}{1-\lambda^2} y_*\right\},$$

and the periodic point of the period 2:

$$\left\{\frac{\lambda - \lambda^2}{1+\lambda^2} y_*; \quad \frac{\lambda + \lambda^2}{1+\lambda^2} y_*\right\}.$$

The points of period 3 for mapping (8) form two cycles

$$\left\{\frac{\lambda - \lambda^2 - \lambda^3}{1+\lambda^3} y_*; \quad \frac{\lambda + \lambda^2 - \lambda^3}{1+\lambda^3} y_*; \quad \frac{\lambda + \lambda^2 + \lambda^3}{1+\lambda^3} y_*\right\},$$
$$\left\{\frac{\lambda - \lambda^2 + \lambda^3}{1-\lambda^3} y_*; \quad \frac{\lambda + \lambda^2 - \lambda^3}{1-\lambda^3} y_*; \quad \frac{\lambda - \lambda^2 - \lambda^3}{1-\lambda^3} y_*\right\}.$$

Thus, the theorem is valid.

**Theorem 1.** *Suppose that for differential equation* (1) *the function* $f(x,y)$ *identically equal to zero in the region* $C_1 \leq w(x,y) \leq C_2$, $0 \leq C_1 < C_2$. *Then equation* (1) *with impulse action* (2), (14), *where* $y = \dot{x}$, *at* $1 < \lambda \leq |\min G(x)|$, *has* $T(n)$*– periodic regimes such that the phase point of this system when moving along the corresponding trajectory undergoes exactly* $n$ *impulse actions for the period where* $n$ *is an arbitrary natural number. The points defining the cycles corresponding to the periodic regimes of the problem* (1), (14) *satisfy the Sharkovsky's order*.

## 3.2. The case of isolated periodic solutions

Further, let us investigate the case when in some domain there are limit cycles, i.e., there are isolated periodic solutions of the system, so in this domain $f(x,y) \not\equiv 0$. The problem of the existence and uniqueness of limit cycles for system (2) is considered in the following theorems.

**Theorem 2.** *Assume* $f(0,0) < 0$ *and*

$$f(x,y) \geq 0 \quad \text{for} \quad |x| \geq x_0 > 0,$$

*moreover*

$$f(x,y) \geq -\mathcal{F} \ \text{ for } \ |x| \leq x_0. \tag{15}$$

*Assume that there exist* $x_1 > x_0$ *such that*

$$\int_{x_0}^{x_1} f(x,y)\, dx \geq 10\mathcal{F}x_0, \tag{16}$$

*where* $y = y(x) > 0$ *is an arbitrary continuously decreasing function. Under such conditions the system* (3) *has at least one periodic regime.*

*Proof.* We construct a ring domain that satisfies the requirements of Bendixon's theorem [30]. To do this, we use inequality

$$\sqrt{G(x_1) - G(x_0)} \geq \max\left\{20\mathcal{F}x_0, \frac{G(x_0)}{\mathcal{F}x_0}\right\}, \tag{17}$$

We assume that the function $g(x)$ satisfies the conditions $xg(x) > 0$ for $x \neq 0$ and therefore the phase picture on the plane $xy$ has the only stationary point $x = y = 0$.

Since $f(0,0) < 0$ and function $f(x,y)$ continuous, then from (8) we obtain $w \geq 0$ in some neighborhood of the origin. Therefore, as the inner boundary of the annular region, you can choose a curve $w(x,y) = C > 0$ with a sufficiently small value of the parameter $C$.

Because $w \leq 0$ для $|x| \geq x_0$, then in this area to form the outer boundary the curves $w(x,y) =$ const can be used. Let's put $y_0 = \sqrt{G(x_1) - G(x_0)}$ and construct a closed curve $\mathcal{W}_0$: $w(x,y) = w_0$, where (see Fig.1 )

$$w_0 = G(x_0) + \frac{1}{2}y_0^2.$$

Consider also the trajectory coming from the point $A_0(x_0, y_0)$ ($y_0 > 0$) of this curve. For $x \geq x_0$ the trajectory passes inside $\mathcal{W}_0$, approaching the axis $x$ and crosses the vertical for the first time $x = x_1$ at some point $A_1(x_1, y_1)$.

Let $w_1 = w(x_1, y_1)$, then

$$w_1 - w_0 = \int_{x_0}^{x_1} f(x,y)y\,dx.$$

At $y_1 \geq y_0/2$ we will receive

$$w_1 - w_0 \leq -\frac{1}{2}y_0 \int_{x_0}^{x_1} f(x,y)\,dx \leq -5\mathcal{F}x_0y_0.$$

This inequality remains true even when $y_1 < y_0/2$. Indeed, we have

$$w_1 = \frac{1}{2}y_1^2 + G(x_1) < \frac{1}{8}y_0^2 + \frac{1}{4}y_0^2 + G(x_0) =$$

$$= w_0 - \frac{1}{8}y_0^2 = w_0 - \frac{1}{4}x_0\sqrt{G(x_1) - G(x_0)}.$$

Hence, given inequality (16), we obtain $w_1 < w_0 - 5\mathcal{F}x_0y_0$.

If the trajectory is at a point $A_2(x_2, y_2)$ the lower half-plane returns to the vertical $x_0$, then $w_2 \leq w_1$ and, accordingly, $w_2 < w_0 - 5\mathcal{F}x_0y_0$.

Let the trajectory intersect the line $x = -x_0$ on the arc $\widehat{A_2A_3}$, then

$$w_3 - w_2 = \int_{x_0}^{-x_0} f(x,y)y\,dx = -\int_{-x_0}^{x_0} f(x,y)|y|\,dx \leq$$

$$\leq \mathcal{F}\int_{-x_0}^{x_0} |y|\,dx \leq 2\mathcal{F}x_0y_0.$$

If on the set $\widehat{A_2A_3}$ we have $|y| \geq y_0$, and at the first time $y_2' = y_0$ at point $A_2'(x_2', y_2')$, $-x_0 \leq x_2' < x_0$, then

$$w_2' - w_2 \leq \mathcal{F}\int_{x_2'}^{x_0} |y|\,dx \leq 2\mathcal{F}x_0y_0$$

and

$$\frac{1}{2}(y_0^2 - y_2^2) < 2\mathcal{F}x_0y_0 + [G(x_0) - G(x_2')] \leq 2\mathcal{F}x_0y_0 + G(x_0).$$

On the other hand

$$\frac{1}{2}(y_0^2 - y_2^2) > 5\mathcal{F}x_0y_0$$

and, accordingly, using inequality (17), we obtain

$$G(x_0) > 3\mathcal{F}x_0y_0 = 6\mathcal{F}x_0\sqrt{G(x_1) - G(x_0)} \geq 6G(x_0),$$

which is impossible.

So the point $A_2'$ does not exist everywhere on the arc $\widehat{A_2A_3}$ we have $|y| \leq y_0$. From here we find

$$w_3 - w_0 = (w_3 - w_2) + (w_2 - w_0) < 2\mathcal{F}x_0y_0 - 5\mathcal{F}x_0y_0 = -3\mathcal{F}x_0y_0.$$

Since the function $w(x,y)$ falls along the arc $\widehat{A_3A_4}$, where $A_4$ — next for $A_3$ the point of intersection of the trajectory of the line $x = -x_0$, then fair inequality $w_4 - w_0 < -3\mathcal{F}x_0y_0$.

Let the trajectory point $A_5$ lies vertically $x = x_0$. Then for the arc $\widehat{A_4A_5}$ we get the inequality $y \le y_0$, similar to how it was for the arc $\widehat{A_2A_3}$.

From here $w_5 - w_4 \le 2\mathcal{F}x_0y_0$ and, accordingly, $w_5 - w_0 < -\mathcal{F}x_0y_0$. So the point $A_5$ should be below the point $A_0$.

If the phase trajectory does not reach the vertical $x = -x_0$, then it crosses the axis $x$ at some point $\xi$, where $-x_0 < \xi < 0$. Then, accordingly, we get

$$G(\xi) < w_2 + 2\mathcal{F}x_0y_0 < w_0 - 3\mathcal{F}x_0y_0$$

and

$$w_5 < G(\xi) + 2\mathcal{F}x_0y_0 < w_0 - \mathcal{F}x_0y_0.$$

Complementing the arc $\widehat{A_0A_5}$ segment $\overline{A_5A_0}$ to a closed loop, construct the desired upper boundary of the annular region.

If the phase trajectory we are studying ends at some point in the segment $0 < x < x_0$ axis $x$, then you just need to increase $y_0$, that such an end point was a point $x = x_0$. Placing $A_5 = (x_0, 0)$, we obtain, as before, the outer boundary of the annular region. The theorem is proved.

Now let us consider the uniqueness result. We denote by $\mathcal{R}_+$ and $\mathcal{R}_-$ domains in the phase plane $xy$, in which the function $f(x,y)$ is positive or negative. The part of the curve $w(x,y) = w$, that belongs to $\mathcal{R}_\pm$ we will denote by $\mathcal{R}_\pm(w)$.

**Теорема 3.** *Assume conditions of Theorem 2, and assume that the function $f(x,y)$ has continuous derivatives of the first order. Additionally, we assume that for every value of parameter $w$, for which the set $\mathcal{R}_\pm(w)$exists, the infimum of the function*

$$F(x,y) = \frac{1}{y^2} + \frac{1}{yf(x,y)}\frac{\partial}{\partial y} f(x,y) \tag{18}$$

*on $\mathcal{R}_+(w)$ is positive and no less than its supremum on $\mathcal{R}_-(w)$. Then system* (3) *has a unique limit regime.*

## 3.3. Existence of limit cycle under impulsive perturbations

Let us consider the behavior of system (3) under assumptions of theorem 2,3 in domain

$$w_0 = G(x_0) + \frac{1}{2}y_0^2.$$

Assume that

$1^0$. Generalized Liénard differential equation (1) is assumed to be satisfied the conditions of existence and uniqueness of solution.

$2^0$. The line $x = x_*$ is transversal to the flow (1) everywhere everywhere except the trajectory for which the line $x = x_*$ is tangent. And in this case we assume that $I(0) = 0$.

$3^0$. The impulsive operator $\mathcal{A}_t$ is assumed to be continuous with respect to veriables $(x, \dot{x})$.

Let for the initial data $(x_0, \dot{x}_0)$ of the problem (3), (12) the following property holds:

$\mathbb{M}$: $x_0 < x_*$ and $\dot{x}_0 > x_*$, or $x_0 > x_*$ and $\dot{x}_0 < x_*$, or $(x_0, \dot{x}_0) \in w_*$ i $|x_0| > x_*$
and conditions of theorems 2,3 hold for all $(x,y)$. Then the phase point $(x_0, \dot{x}_0)$ moves along trajectory $x(t_*, x_0, \dot{x}_0, t_0) = x_*$, when it is affected by impulsive perturbation (12). Let

$$t_1 = \min_{t_* > t_0}\{t_*: x(t_*, x_0, \dot{x}_0, t_0) = x_*\}.$$

Let us consider coordinates of the phase point $\big(x(t), \dot{x}(t)\big)$, where $x(t) = x(t, x_0, \dot{x}_0, t_0)$ for $t = t_1 + 0$, i.e., afterimpulsive perturbation. Then

$$x(t_1 + 0) = x_*,$$
$$\dot{x}(t_1 + 0) = \dot{x}(t_1, x_0, \dot{x}_0, t_0) + I\big(\dot{x}(t_1, x_0, \dot{x}_0, t_0)\big). \quad (19)$$

Then, if $(x_1, \dot{x}_1)$, when $x_1 = x_*$, and $\dot{x}_1 = \dot{x}(t_1 + 0)$ is defined by (19), as a new initial data for the problem (3), (12), the property $\mathbb{M}$ holds, i.e., $(x_*, \dot{x}_1) \in w_*$, then there exists a moment of time $t_*$ such that $x(t_*, x_*, \dot{x}_0, t_0) = x_*$, when the phase point of (3), (12) is again affected by impulsive perturbations (12). We denote

$$t_2 = \min_{t_* > t_1}\{t_*: x(t_*, x_*, \dot{x}_1, t_1) = x_*\}.$$

Note, that if the property $\mathbb{M}$ does not hold, i.e., $(x_*, \dot{x}_1) \notin w_*$, then the phase point of the system (3), (12) for $t > t_1$ will not have impulsive perturbations. This situation is possible only if $x_* = x'_*$, when the hyperplane does not intersect the limit cycle $\mathcal{W}_0$ at any point. In such a case the system (3), (12) has only one impulsive influences. After that we get a new initial data for (3) which has a limit cycle due to theorem 2,3, and the phase point in the sequel will move without any impulses.

Assume that we have constructed $n$ members of the sequence $\{t_k, (x_k, \dot{x}_k)\}$, $k = \overline{1, n}$, where

$$t_1 = \min_{t_* > t_{k-1}}\{t_*: x(t_*, x_{k-1}, \dot{x}_{k-1}, t_{k-1}) = x_*\}, \quad (20)$$

$$x_k = x(t_k, x_{k-1}, \dot{x}_{k-1}, t_{k-1}) = x_*, \quad k = \overline{1, n}, \quad (21)$$

$$\dot{x}_k = \dot{x}(t_k, x_{k-1}, \dot{x}_{k-1}, t_{k-1}) + I\big(\dot{x}(t_k, x_{k-1}, \dot{x}_{k-1}, t_{k-1})\big). \quad (22)$$

It is clear that $(x_k, \dot{x}_k) = (x_*, \dot{x}_k) \in w_*$, where $k = \overline{1, n-1}$. Under condition $(x_n, \dot{x}_n) = (x_*, \dot{x}_n) \in w_*$, it is possible to construct $(n+1)$-th member of this sequence. Otherwise, it consists of only $n$ members. P- In general case, for arbitrary values of initial data $(x_0, \dot{x}_0)$ the sequence $t_1, t_2, \ldots$ can be infinite, or it can be finite, in particular, it may consists of only one element, or it can be empty.

If the sequence of moments of time consists of one point, which is possible, for example, for the case when for all $x \notin w_*$, then the system (4), (12) undergoes an impulse action only once and for it there is a limit cycle in the requirements of Theorems 2, 3.

If the sequence has a finite (not empty) number of points containing, for example, exactly k≥1 elements, then the condition $|\dot{x_k} + I(\dot{x_k})| \notin w_*$, $I(w_*) = 0$, and the system has exactly k times the action of impulses, and there is a limit cycle in the requirements of Theorems 2,3.

It easy to see that if for some $n$ and for all $y \in [-y_*, y_*] \cup (-\infty, -y_*) \cup (y_*, \infty)$ we have $|f^n(y)| = w_*$, where $f^n(y)$ is the $n$-th iteration of the function $f(y) = -y - I(-y)$, $y = \dot{x}$, then the sequence $\{t_n, (x_n, \dot{x}_n)\}$, $n \in \mathbb{N}$ has an infinite number of points. Moreover, $(x_n, \dot{x}_n) = (x_*, \dot{x}_n) = w_*$ for all $k$.

It follows from the analysis that problem (3), (12) will have a single limit cycle under the conditions of Theorem 3, where the phase point will be affected by impulse perturbation when the sequence $\{t_n\}$, $n = 1,2, \ldots$ is infinite. The following theorem takes place

**Theorem 4.** *Assume that*

*1) Function $f(x, y)$ has continuous derivatives of the first order, and, moreover, there exist positive $x_1$, $x_2$ such that*

$$f(x, y) < 0, \quad \inf f(x, y) = -\mathcal{F} \ \text{ on } \ (x_1, x_2)$$

*and*

$$f(x, y) \geq 0 \ \text{ otherwise;}$$

*2) $y\frac{\partial y}{\partial x} \geq 0$ and*

$$\int_{x_0}^{x_2} f(x, y)dx \geq 10\mathcal{F}x_0,$$

*where $x_0 = \min(x_1, x_2)$ is sufficiently large and $y = y(x)$ is an arbitrary nonincreasing function;*

*3) $G(-x_1) = G(x_2)$.*

*Then the impulsive system (3), (12) has a unique limit cycle, and the phase point has $n \in \mathbb{N}$ impulsive perturbations*

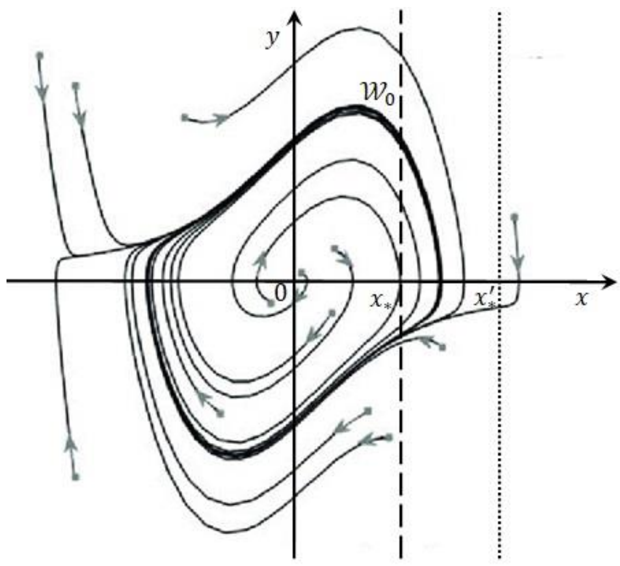


**Figure 1:** limit cycle with the hyperplane of impulsive effect

The detailed analysis of the qualitative behavior of the system (1), (2), (12) demonstrates the complex behavior of the generalized Lénard equation (1) with impulse action (2). The effective criteria for the existence of a single stable limit impulsive regime for such an equation are investigated.

## 4. Conclusions

The paper investigates the conditions for the existence of boundary cycles for the Lienard equation, the solutions of which are affected by instantaneous forces of momentum nature at unfixed moments in time.

For this system, the constructive conditions for the existence of $T(n)$-periodic regimes are proved such that the phase point of the system when moving along the corresponding trajectory undergoes exactly $n$ impulsive disturbances for the period where $n$ is an arbitrary positive integer. The points that define cycles that correspond to periodic regimes satisfy the Sharkovsky's order.

The conditions for the existence of at least one periodic regime and a single limit cycle are found — the only one with precision to shift in time of the periodic regime.

For system (3) with impulsive action (12), the existence of a single stable limit regime is proved, the phase point of which will be affected by pulsed forces $n \in \mathbb{N}$ times. It is shown that a stable limit regime under given conditions will exist despite the influence of impulse forces according to the law (12).

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