An Approach for Improving Local Solutions in the Unequal Circles Packing Problem

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Abstract

The problem of packing unequal circles of fixed radius into a circle of minimum radius is considered. An approach to improving local solutions obtained by any of the known methods is proposed. The approach is based on the idea of expanding the space of variables. The dimension of the problem expands if we assume that the radii of the circles are considered as independent variables. In this case, an additional rigid system of constraints is constructed in such a way that the convergence of variable radii to the initial fixed values is ensured in the process of the algorithm. To construct such a system of constraints, the combinatorial structure of the problem is used.The numerical results of solving test problems of packing various numbers of circles are presented and the analysis of the results obtained is carried out.

Keywords [1]

Packing problem, optimization, circle, permutation

1. Introduction

The problems of packing circles and spheres in containers of various shapes are given constant attention from scientists. Problem statements differ in the space dimension, the shape of containers, the specifics of accounting for metrical parameters (sizes) of objects and containers, etc. [1 – 7]. In recent times, the number of publications in this area has increased [8 – 12]. This, on the one hand, is due to the fact that the problems have numerous practical applications. On the other hand, these problems have become a testing benchmark for new methods of global optimization, since they are characterized as multi-extremum and of high dimension.

According to the typology proposed by Wäscher et al [13] the problems can be considered as the Knapsack Problems (KP) or Open Dimension Problems (ODP): 2DCKP, 3DSKP, 2DCODP, 3DSODP. The subject of this article is the problem of packing unequal circles into a circle of minimum radius. However, the approach proposed below can be easily extended to pack both circles and spheres into an arbitrary container and even into multiple containers.

The main idea of the proposed approach is to identify the combinatorial structure of the problem and artificially expand the space of variables (lifting) to create new possible directions for improving the objective function. Note that increasing the dimension of the problem due to the variable radii of the spheres (circles) is used in Jump Algorithm [14], where successively the auxiliary problems of minimizing the unused domain of the container are solved. This allows the transition from one local extremum to another, with the best value of the objective function.

The problems of optimal packing of unequal circles and spheres have wide practical application. Such problems arise in various industries, materials science, powder metallurgy, nanotechnology, radiosurgery, laser coagulation, monitoring systems design, studying various structures in biology, layout problems in logistics systems and space technology, coding, classification, information security, etc. [15 – 21].

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2. Problem Statement

Let be given a set of circles $S_1, S_2, ..., S_n$, the radii of which are equal $\hat{r}_1, \hat{r}_2, ..., \hat{r}_n$ respectively. Without loss of generality, suppose $\hat{r}_1 \le \hat{r}_2 \le ... \le \hat{r}_n$. The problem is to place the circles $S_1, S_2, ..., S_n$ in a circle S_0 of minimum radius r_0 with a center at the point $p^0 = (0,0)$. We denote the coordinates of the centers by $p^i = (u_i, v_i)$, $i \in J_n = \{1,2,...,n\}$. Then the mathematical model of the problem will take the form

$$r_0 \to min, \tag{1}$$

subject to

$$u_i^2 + v_i^2 \le (r_0 - \hat{r}_i)^2, i \in J_n, \tag{2}$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \ge (\hat{r}_i + \hat{r}_j)^2, \quad i \in J_n, j \in J_n, i < j. \tag{3}$$

Thus, we have a nonlinear optimization problem with variables $r_0, u_i, v_i, i \in J_n$, which we call *Problem 1*.

Problem 1 is multiextremal due to the nonconvexity of constraints (2). The use of numerical methods for nonlinear optimization allows finding only its local solutions, depending on the choice of initial values r_0, u_i, v_i, $i \in J_n$. Traditionally, methods of directed search of local solutions are further used. At the same time, multi-start schemes are effective, in which starting points are generated randomly or according to some special rule. This paper proposes a new approach that improves local solutions obtained by known methods. The approach is based on the idea of the method of artificial extension of the dimension of space, first proposed in [22] using the concept of the configuration space of geometric objects [23]. In this case, generalized variables of the configuration space are the metrical and placement parameters of geometric objects. First of all, let us use the combinatorial structure of the problem as a Euclidean combinatorial optimization problem [24].

We will consider the radii $r_1, r_2, ..., r_n$ of the circles as independent variables and form the system

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} \hat{r}_i, \tag{4}$$

$$\sum_{i \in W} r_i \ge \sum_{i=1}^{|W|} \hat{r}_i \;\; \forall W \subset J_n, \tag{5}$$

$$\sum_{i=1}^{n} (r_i - \tau)^2 = \sum_{i=1}^{n} (\hat{r}_i - \tau)^2, \tag{6}$$

where elements of subsets $W \subset J_n$ of capacity $|W|$ in increasing order, and

$$\tau = \frac{1}{n} \sum_{i=1}^{n} \hat{r}_i. \tag{7}$$

System (4-6) is such that the set of its solutions coincides with the permutation set of real numbers $\{\hat{r}_1, \hat{r}_2, ..., \hat{r}_n\}$. This property is a partial case and follows from the general concept of continuous representations and functional extensions of Euclidean combinatorial sets. Representation (4-6) is called polyhedral-spherical [25], because, as is easy to see, it describes a set that is the intersection of the permutation polyhedron (5) and the hypersphere (6) with center at point (7). Optimization problems on polyhedral-spherical sets have interesting properties, which are investigated in the theory of convex extensions on vertex-located sets [25].

Let us consider the function optimization problem (1) with restrictions

$$u_i^2 + v_i^2 \le (r_0 - r_i)^2, i \in J_n, \tag{8}$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \geq (r_i + r_j)^2, \\ i \in J_n, j \in J_n, i < j, \tag{9}$$

where variables r_i, $i \in J_n$ satisfy conditions (4-7).

The problem of nonlinear optimization (1), (4-9) with variables r_0, u_i, v_i, r_i, $i \in J_n$ is called *Problem 2*.

Theorem 1. *Both Problem 1* and *Problem 2* are equivalent in the sense of the coincidence of their global solutions.

The proof of the theorem is based on the uniqueness of the polyhedral-spherical representation of the Euclidean set of permutations [24, 25] generated by the set of radii of the circles.

It is known that the equality of global solutions in multiextremal problems does not imply the coincidence of their local solutions. It is this fact that makes it possible to obtain different local solutions for the same initial points when implementing numerical optimization methods.

Note that *Problem 1* has a dimension 2*n*+1, and the dimension of *Problem 2* is equal to 3*n*+1. Thus, we have implemented an approach of artificial expansion of the space of variables, based on the general scheme proposed in [22]. Applied to the problem of packing circles and spheres, this approach will be called the variable radius method.

3. Variable Radius Method and its Application to the Unequal Circles Packing Problem

The method of variable radii allows one to realize the idea of improving solutions of *Problem 1*. Indeed, we choose the local solution obtained as the starting point and consider the radii of the circles as independent variables $r_1, r_2, ..., r_n$ satisfying constraint system (4-9). As a result, it is possible by variables to overcome the neighborhood of attraction of a local extremum and direct to a new, possibly better, local solution. The initial values of the variable radii can be chosen as equal to the initial values, or generated randomly in (r_1, r_n).

The main difficulties in implementing the proposed approach are associated with an increase in the dimension of *Problem 2*. It is easy to see that the number of linear constraints in system (5) is equal to $2^n - 2$, which already at n >20 leads to difficulties in applying modern numerical methods of nonlinear optimization. On the one hand, using the properties of linear and quadratic functions on combinatorial polyhedra allows one to partially overcome the arising difficulties. However, this does not greatly expand the capabilities of methods in large-scale problems.

On the other hand, it is possible to select only $l < n$ circles with variable radii. Let $M' = \{m_1, m_{2,...,} m_l\} \subset J_n$ be the numbers of circles, whose radii $\hat{r}_{m_1} \leq \hat{r}_{m_2} \leq ... \leq \hat{r}_{m_l}$ are fixed, and the set $M'' = J_n \setminus M'$ be of ordered numbers of circles with variables radii $r_i, i \in M''$. Form a system of restrictions

$$u_i^2 + v_i^2 \leq (r_0 - \hat{r}_i)^2, i \in M' \tag{10}$$

$$u_i^2 + v_i^2 \leq (r_0 - r_i)^2, i \in M'' \tag{11}$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \geq (\hat{r}_i + \hat{r}_j)^2, \\ i \in M', j \in M', i < j. \tag{12}$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \geq (r_i + r_j)^2, \\ i \in M'', j \in M'', i < j, \tag{13}$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \geq (\hat{r}_i + r_j)^2, \quad i \in M', j \in M''. \tag{14}$$

$$\sum_{i \in M''} r_i = \sum_{i \in M''} \hat{r}_i \tag{15}$$

$$\sum_{i \in W} r_i \geq \sum_{i=1}^{|W|} \hat{r}_{m_i} \quad \forall W \subset M'' \tag{16}$$

$$\sum_{i \in M''} (r_i - \hat{\tau})^2 = \sum_{i \in M''} (\hat{r}_i - \hat{\tau})^2, \tag{17}$$

where

$$\hat{\tau} = \frac{1}{l} \sum_{i=1}^{l} \hat{r}_{m_i}. \tag{18}$$

The problem of nonlinear optimization (1), (10-18) in the space of variables r_0, u_i, v_i, $i \in J_n$, and r_j, $j \in M''$ we call *Problem 3*.

Theorem 2. The sets of global solutions of both *Problem 1* and *Problem 3* are the same for any $M' = \{m_1, m_2, ..., m_l\} \subset J_n$. The proof is based on the uniqueness of the polyhedral-spherical representation of the set of permutations generated by the radii of the circles with numbers from $M'' = J_n \setminus M'$. Let us expand the above reasoning. Consider a set $Q = \{r_1, r_2, ..., r_n\}$ and part it onto $q+1$ pairwise disjoint subsets as follows. Let be

$$\bigcup_{k=0}^{q} M^k = J_n, \tag{19}$$

where

$$M^k = \{m_1, m_2, ..., m_{l_k}\}, \; m_1 < m_2 < ... < m_{l_k}, \; k \in J_q^0 = \{0\} \bigcup J_q, \; \sum_{k=0}^{q} l_k = n, \; l_0 \geq 0.$$

Let us consider the set

$$Q = \bigcup_{k=0}^{q} Q^k \tag{20}$$

where

$$Q^0 = \left\{ \hat{r}_{m_1}, \hat{r}_{m_2}, ..., \hat{r}_{m_{l_0}} \right\}, \; Q^k = \left\{ r_{m_1}, r_{m_2}, ..., r_{m_{l_k}} \right\}, k \in J_q.$$

Form a system of restrictions

$$u_i^2 + v_i^2 \leq (r_0 - \hat{r}_i)^2, i \in M^0, \tag{21}$$

$$u_i^2 + v_i^2 \leq (r_0 - r_i)^2, i \in J_n \setminus M^0, \tag{22}$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \geq (\hat{r}_i + \hat{r}_j)^2, \quad i \in M^0, j \in M^0, i < j, \tag{23}$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \geq (r_i + r_j)^2, \quad i \in J_n \setminus M^0, j \in J_n \setminus M^0, i < j, \tag{24}$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \geq (\hat{r}_i + r_j)^2, \quad i \in M^0, j \in J_n \setminus M^0. \tag{25}$$

For each set Q^k, $k \in J_q$, we write

$$\sum_{i \in M^k} r_i = \sum_{i \in M^k} \hat{r}_i \tag{26}$$

$$\sum_{i \in W} r_i \geq \sum_{i=1}^{|W|} \hat{r}_{m_i}, \forall W \subset M^k \tag{27}$$

$$\sum_{i \in M^k} (r_i - \tau_k)^2 = \sum_{i \in M^k} (\hat{r}_i - \tau_k)^2, \tag{28}$$

where

$$\tau_k = \frac{1}{l_k} \sum_{i=1}^{l_k} \hat{r}_{m_i}. \tag{29}$$

The problem of nonlinear optimization (1), (21-29) in the space of variables r_0, u_i, v_i, $i \in J_n$, and r_j, $j \in J_n \setminus M^0$ we call *Problem 4*.

Let us generalize the statements of **Theorem 1** and **Theorem 2**.

Theorem 3. The sets of global solutions of both *Problem 1* and *Problem 4* are the same for any $M^0 = \{m_1, m_2, ..., m_{l_0}\} \subset J_n$. The choice of the method of partitioning $Q = \{r_1, r_2, ..., r_n\}$ onto a sets $Q^k = \left\{r_{m_1}, r_{m_2}, ..., r_{m_{l_k}}\right\}$, $k \in J_q^0$ with the subsequent formation of constraints (12-14), forms a family of modifications of the method of variable radius.

4. Numerical Results and Their Discussion

The ability to control the variable radii of the circles in the process of solving the problem allows us to propose various strategies for the formation of initial points and the sequential search of local decisions in order to improve them. Suppose that at the initial stage (0-th iteration) a local solution was obtained $r_0^{(0)}$, $u_i^{(0)}, v_i^{(0)}$, $r_i^{(0)} = \hat{r}_i$, $i \in J_n$. Strategies for improving this solution are associated both with the method of forming the parameters for placing the circles and with the rule for choosing the radii, which we will consider as variables. Classical approaches are associated with the choice of a new starting point from an extended neighborhood of the placement parameters $u_i^{(0)}, v_i^{(0)}$ with fixed radii $r_i^{(0)} = \hat{r}_i$ and the search for a new local solution $r_0^{(1)}$, $u_i^{(1)}, v_i^{(1)}$, $r_i^{(1)} = \hat{r}_i$, $i \in J_n$ (1-st iteration). At the *k*-th iteration, the initial values for the placement parameters are chosen, corresponding to the best current value of the local solution for fixed values of the radii of the circles. In this paper, it is proposed at each iteration to fix the placement parameters of the circles, setting them equal to the corresponding best current local solution. In this case, the formation of new local solutions is carried out by considering the radii of the circles as variables. We conducted the following computational experiments. For local optimization, the IPOPT software package was used (*https://projects.coin-or.org/Ipopt*), which implements an internal point method for continuous nonlinear programming problems. Computer with following characteristics was used to perform the computation: i3/8G/SSD 256G.

At first, test problems with the number of circles less than 15 were considered. A series of 30 problems was formed in which the radii of the circles were randomly generated uniformly in the interval (1, 15). For each test, the coordinates of the circle centers from a square (-100,100) x (-100,100) were generated randomly. The values obtained were chosen as initial for the implementation of the local optimization method in *Problem 1*. Then, with the same initial data, *Problem 2* was solved, in which the radii of the circles are variables. The initial values for the variable radii r_i, $i \in J_n$ in *Problem 2* were randomly generated from the interval (1, 15). Thus, we compared the solutions of both *Problem 1* and *Problem 2* using the same local optimization method for the same starting point.

In 80% of test problems, local solutions of *Problem 2* turned out to be better than in *Problem 1*. The average time to complete *Problem 2* was only 1.5 times higher. Note that initially we intended to improve local solutions. Therefore, in the future, solutions to *Problem 1* were chosen as the starting point for the implementation of *Problem 2*. In all cases, improvements to local solutions were obtained. However, when we tried to improve solutions to *Problem 2,* we succeeded in 86% of cases. An attempt to improve the new best solution at the next iteration was successful only in 43% of cases. As a rule, there were no improvements after the 7-th iteration. This is natural, since the better the initial local solution, the more difficult it is to improve.

In the problem of packing circles for $n \geq 15$, the numerical experiment was carried out using three strategies, depending on the rules for forming a set of circles with fixed and variable radii. The first strategy corresponds to the case when we fixed the radii of the circles (*Problem 1*). For the second strategy, the radii of $l = 7$ circles were assumed to be variables (*Problem 3*).

At the same time, a series of 10 test tasks was formed in which the radii of circles were uniformly randomly generated in the interval (1,100). For each test, textit problem 1 was solved, in which the initial values of the coordinates of the centers of the circles were also randomly generated from the set $(-500,500)\times(-500,500)$. Then the solutions obtained were improved using *Strategies 1,2*. Improvements were received for all tests reviewed. The average runtime for *Strategy 2* was 1.6 times higher, but the average solution quality was 4.3% better on average.

The third strategy was used for large-scale problems and involved decomposing a set Q into subsets.

The choice of the set Q decomposition method has a significant impact on the result. Experimentally, the best results corresponded to the groupings obtained as a result of partition of a set $Q = \{\hat{r}_1, \hat{r}_2, ..., \hat{r}_n\}$, in non-decreasing order of $\hat{r}_1 \leq \hat{r}_2 \leq ... \leq \hat{r}_n$ (*Strategy 3*).

The proposed approach was tested on the placing of 60 circles, the radii of which were chosen randomly and ordered. In this way $Q = \{\hat{r}_1, \hat{r}_2, ..., \hat{r}_{60}\}$=6, 6, 8, 9, 10, 10, 13, 13, 15, 17, 18, 20, 23, 25, 25, 26, 26, 30, 31, 32, 32, 33, 34, 34, 35, 37, 38, 39, 41, 42, 42, 45, 48, 49, 49, 51, 55, 55, 56, 58, 58, 59, 61, 62, 63, 65, 66, 66, 69, 69, 70, 73, 73, 75, 76, 79, 80, 82, 82, 83. First, a local solution of *Problem 1* was obtained with fixed radii from Q. The starting point for the numerical optimization method was randomly generated uniformly from the square $(-500,500)\times(-500,500)$. Radius $r_0^{(0)} = 450.88$ was obtained in 362 sec. Then, the set Q was partitioned in accordance with *Strategy 3* and *Problem 4* was solved. At each subsequent iteration, the coordinates of the centers of the circles corresponding to the best local solution were chosen as the starting point for the numerical local optimization method. In this case, the set $Q = \{\hat{r}_1, \hat{r}_2, ..., \hat{r}_n\}$ was partitioned into 10, 12, 15, 20, 30 groups of 6, 5, 4, 3, 2 elements, respectively. At subsequent iterations were obtained $r_0^{(1)} = 434.72$, $r_0^{(2)} = 433.15$, $r_0^{(3)} = 431.22$, $r_0^{(4)} = 430.90$, $r_0^{(5)} = 430.90$. Finally, a solution was reached $r_0^{(6)} = 426.74$ and the coordinates u_i, v_i, $i \in J_n$ of the circle centers are shown in Table 1. The runtime was 968 sec.

5. Conclusions

This article proposed a new approach for improving local solutions to the problem of packing unequal circles in a circular container of minimum radius. The basic idea is to expand the variable space of the optimization problem, assuming that the radii of the circles are considered as variables. At the same time, an additional constraint system for variable radii is being constructed using the permutation structure of the problem. It is guaranteed that, as a result of the decision at the last stage, the radii will take their initial values. This approach differs from the known ones in that it allows overcoming the attraction zones of local extrema of the problem not only by changing the coordinates of the centers of the circles, but also by partially varying their radii. This provides additional directions for improving the objective function, which ultimately improves the result. The method of variable radii can be extended to containers of various shapes and even to several containers. In this case, of course, the restrictions on the placement of circles inside the container change. Moreover, the proposed approach can be used in solving problems of packing unequal spheres into an arbitrary container. In these cases, systems of restrictions on one or more metric parameters are formed by analogy with round objects. The combinatorial structure of the problem will be generated by the same system of constraints as for the radii of circles.Thus, the field of application of the variable radius method for improving the obtained solutions can be significantly expanded.

Table 1
Local solution at the final iteration

r_i	u_i	v_i	r_i	u_i	v_i
6	-280,16	230,82	42	15,42	-228,05
6	-303,31	-287,32	45	-311,72	-220,35
8	-271,70	247,70	48	-147,12	-348,99
9	15,60	50,27	49	-227,14	-179,32
10	339,56	-235,67	49	-118,14	358,54
10	-175,93	17,74	51	75,76	19,43
13	-411,64	41,57	55	-81,88	255,59
13	-288,63	296,43	55	-371,50	-13,32
15	-333,40	47,14	56	-322,57	182,74
17	289,49	35,70	58	-232,51	-286,19
18	23,63	-287,49	58	-347,36	-123,71
20	400,88	61,96	59	-19,46	-120,32
23	-189,86	240,44	61	-257,03	5,46
25	-304,08	261,71	62	233,50	280,19
25	187,64	354,83	63	332,47	-147,55
26	-150,06	211,86	65	108,66	256,86
26	-2,01	-28,98	66	176,63	126,77
30	-259,36	-100,62	66	-204,92	121,28
31	316,29	237,83	69	-30,36	-356,44
32	-162,54	288,18	69	357,29	-17,90
32	-197,66	341,68	70	251,46	-253,03
33	-239,29	214,12	73	-77,78	49,46
34	54,97	-63,47	73	38,11	138,26
34	235,13	-135,13	75	5,67	351,69
35	7,05	241,70	76	-150,45	-80,62
37	-303,58	91,70	79	-105,57	-228,98
38	-238,90	285,12	80	322,63	127,01
39	-378,96	82,02	82	117,16	-324,22
41	121,82	365,29	82	119,23	-160,23
42	-104,60	161,29	83	205,29	-19,45

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