The Algorithm for Crossing the N-dimensional Hyperquadric with N-1-dimensional Hyperspace

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Abstract

In descriptive geometry, the problem of finding a surface curve section with a plane is common. One such surface curve is a quadric. Due to the increased demand for tasks related to quadric, the synthetic modeling method becomes relevant. In recent years, geometric constructions of dimensions of more than three began to be studied more and more often. Multidimensional geometric shapes in multidimensional space are typically constructed using geometric modeling software. However, without additional building automation tools, software does not sufficiently facilitate human labor. The larger the dimension of the constructions, the more cumbersome and time consuming the drawing process becomes. The increasing complexity of constructions requires automation of constructions that can be traditimatized. Geometric constructions made using automation tools make us rethink the process of structural geometric modeling in descriptive geometry. Within the framework of the article, the algorithm for crossing the N-dimensional hyperquadric with N-1-dimensional hyperspace is presented. Special cases of this geometric construction are also considered: intersection of a three-dimensional quadric with a plane and intersection of a four-dimensional hyperquadric with a three-dimensional space. The implementation of the developed algorithm is carried out using the Simplex system and the built-in interpreter of the prolog logical programming language.

Keywords

Descriptive geometry, quadric, hyperquadric, multidimensional, algorithms, logical programming, Simplex system, prolog.

1. Introduction

In recent years, there has been an increasing interest in quadrics [1-8]. One can specify a quadric by both the points on its surface and the conics that form the quadrics. This article uses the second method. A quadric of dimension more than three is called a hyperquadric.

Larger dimensions [9] increase the volume and complexity of drawings. Geometric modeling programs [10] speed up the drawing process and open up new opportunities for the development of geometric science. However, the basic tools provided by geometric software are lacking in multidimensional geometric modeling and 3D complex drawings.

Programming languages together with geometric modeling software can act as an effective tool for drawing. To automate geometric constructions, they need to be incompatimized. A formalized logical instruction for solving a geometric problem can be implemented within software that supports logical programming.

In this article, we consider an example of traditimic formalization of such a process as the construction of a hyperquadric section by a hyperspace whose dimension is one less than the dimension of the hyperquadric.

The purpose of the work is to develop an algorithm for constructing the intersection of an N-dimensional quadric N-1 space on a hyperepure for subsequent automation. To do this, it needs to

GraphiCon 2021: 31st International Conference on Computer Graphics and Vision, September 27-30, 2021, Nizhny Novgorod, Russia
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consider cases of small dimension, identify general patterns and describe the algorithm for natural N, starting with three.

2. Algorithm for constructing a quadric intersection with space

The quadric construction performed within the framework of this article is tied to the reference in the form of a bundle of several intersecting lines. On the basis of arbitrarily defined intersecting lines, points are built through which conics pass, forming the surface of the quadric. The space is one dimension lower than the quadric perpendicular to the first projection plane. This article will consider cases with a three-dimensional, four-dimensional and N-dimensional quadric.

2.1. A 3D quadric intersected by a plane

The first case considered is a three-dimensional quadric. For its unambiguous assignment, three conics are needed, two projections for each. To define a plane that intersects a quadric, it needs its first projection as a straight line. Consider a step-by-step construction algorithm. The visualization of the algorithm is shown on Figure 1.

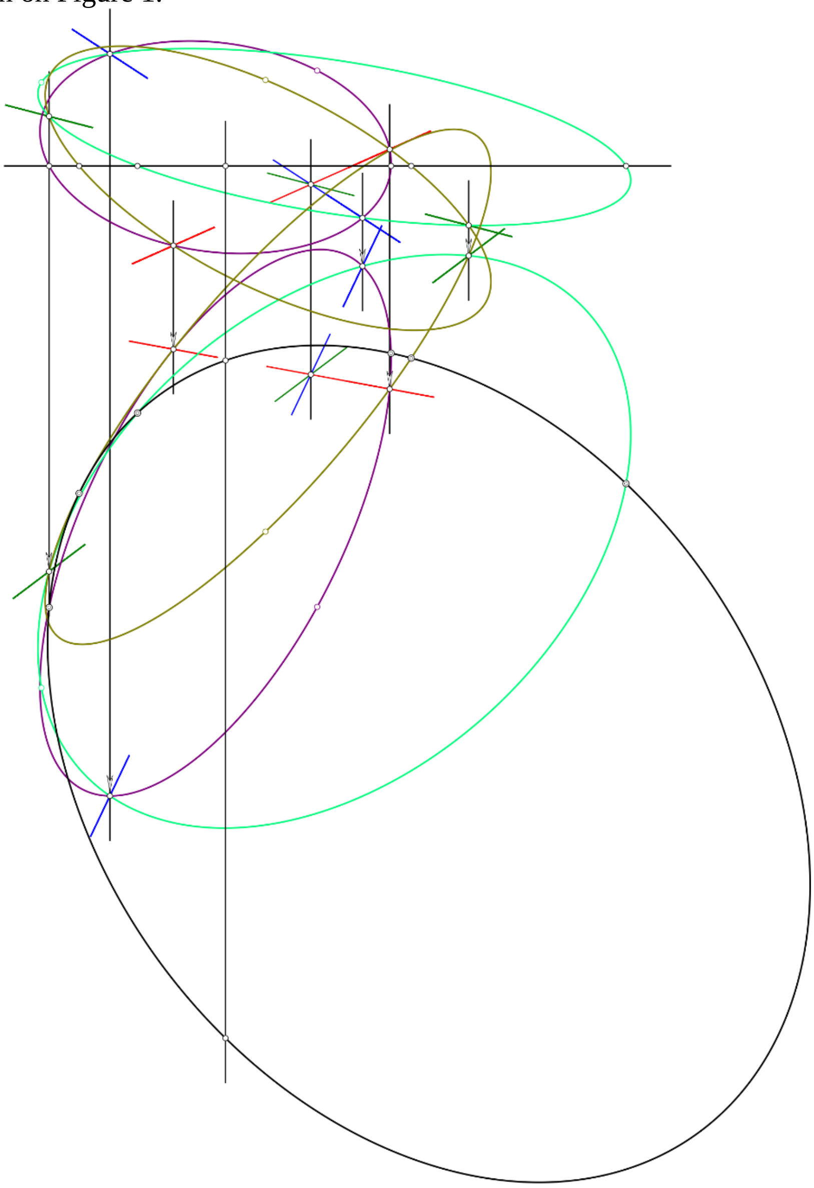

Figure 1: Intersection a 3D quadric with a plane

1. Two projections of the point are built - the rep of the constructions, and their projection line.
2. Two projections of three lines are built (in this case, red, blue and green) passing through the rep. It is assumed that three lines do not lie on the same plane, otherwise they cannot uniquely define a second-order curve surface.
3. On each of the three lines, two points are taken and their first and second projections are built together with a projection connection. Thus, six points are obtained. In the plane formed by red and blue lines - four points. In the plane formed by blue and green, as well as in the plane formed by green and red lines, there are also four points. For a quadric, the generatrix is a conic, second-order curve, five points are needed to construct it. It turns out that to build a conic, it needs another point in each of the three planes.
4. To create three points in three different planes, one specifies three collineations using pairs of two projections of four points in each plane.
5. The first projections of three points are built and the corresponding three second projections are located using three collinearations from the previous point. One gets five points for each of the three planes.
6. Conics are drawn through five points of each plane. Three conics are obtained, two projections for each. Thus, a second-order surface is given - a quadric.
7. It is necessary to create a plane that intersects the quadric. This article discusses an intersecting plane that is orthogonal to the first projection plane. Therefore, a horizontal line is created, which is the first projection of the plane crossing the quadric. The line is selected so as to pass through all three first projections of the conic, otherwise there will be a shortage of points for the construction of the second projection of the section. This point ends the initial constructions and begins the solution of the problem of crossing the quadric with a plane.
8. To create a second projection of the section, there are points at the intersection of the first projection of the plane and the three conics. The first six projection points are obtained. They are matched on the second projections of conics by vertical projection connections. Six second point projections are obtained. The section of a quadric is a conic, five points are necessary for its construction, while in the described constructions of such points six.
9. Through any five of the six available second projections of the points, a conic is drawn. This conic passes through the sixth unused point. The section is constructed.

2.2. A four-dimensional hyperquadric intersected by three-dimensional space

The algorithm for constructing the intersection of a four-dimensional hyperquadric with a three-dimensional space is somewhat more complicated than the previous algorithm. However, the logic of these two algorithms is the same. The visualization of the algorithm is shown on Figure 2.

1. Three projections of the rep of the constructions are built on the projection line.
2. Three projections of four lines passing through the rep are built. Again, none of the three lines should lie on the same plane.
3. To uniquely define a four-dimensional hyperquadric, six conics are needed that form the surface of the hyperquadric. On each of the four lines, two points are taken and three projections are built. Thus, 8 points are obtained, represented in the form of 24 projections. To construct hyperquadric conics, it needs another point in each of the six planes formed by a pair of intersecting lines.
4. To construct six points in three different planes, six collineations are specified to transition from the first projection plane to the second and six more collineations to transition from either the first to the third, or from the second to the third projection plane, a total of 12 collineations. Collineations are also defined using pairs of two projections of four points in each of the planes.
5. The first projections of six points are built and the corresponding six second projections and six third projections are located using 12 collineations from the previous point. One gets five points for each of the six planes.
6. Conics are drawn through five points of each plane. Six conics are obtained, three projections for each. Thus, a four-dimensional hyperquadric is given.

7. It needs to build a three-dimensional that intersects the hyperquadric. This article discusses an intersecting space that is orthogonal to the first projection plane. Therefore, a horizontal line is created, which is the first projection of the plane crossing the hyperquadric.

8. To create the second and third section projections, there are points lying at the intersection of the first projection of the plane and six conics. The first 12 points projections are obtained. They are matched on the second and third projections of conics by vertical projection connections. 12 second and 12 third points projections are obtained. The section of a four-dimensional hyperquadric is a three-dimensional quadric, for its construction it is necessary and enough three conics, two projections for each. There are four sets of six points, one point in each set and one set of points is redundant for constructing a three-dimensional quadric, but the constructed quadric will pass through the fourth set of points.

9. Through any three sets of four available are three conics, two projections for each. The section is constructed.

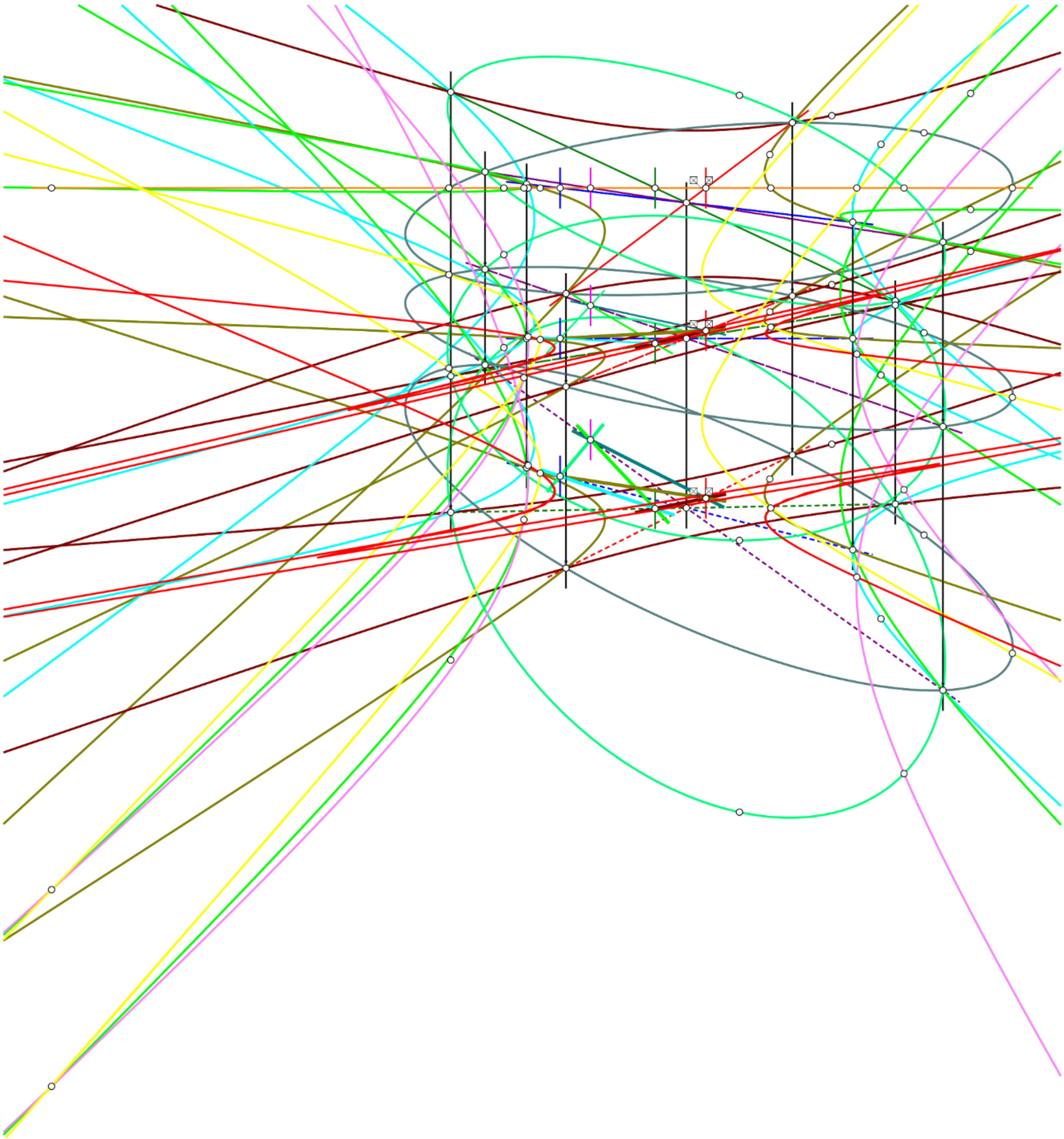

Figure 2: Intersection a 4D quadric with a space

2.3. N-dimensional hyperquadric crossed by N-1-dimensional hyperspace

A common case of the algorithm described in this article will be the algorithm for crossing the hyperquadric (in the particular case, quadric) dimension of N hyperspace (in the special case, space or plane) dimension of N-1.

The logic of constructions continues the logic of the algorithms of the previous two points. The basic conditions of the task are left the same. The hyperquadric is given by forming conics, the conics are tied to the reference - a bundle of lines and planes that are formed by these lines. Hyperspace is orthogonal to the first projection plane.

1. The N-1 projections of the central and their projection line are built.
2. N lines (N-1 projections of each of them) are drawn through the central point, defining the planes in which the forming conics will lie. The number of planes is according to the formula of combinations from combinatorics. $C_n^m = \frac{n!}{(n-m)! \cdot m!}$, where m=2, because two intersecting lines are needed to define the plane, and n = N. Thus, the number of planes will be calculated using the formula $C_N^2 = \frac{N!}{2 \cdot (N-2)!}$.
3. On each of the N lines, two points are taken.
4. $(N-2) \cdot C_N^2$ collineations are specified for each pair of lines and for each transition between projections.
5. In each of the C_N^2 planes of the lines for constructing conics, one more point is taken and using collineations their correspondence is found first for the second field, then for the third, etc. In total, for each point (N-2), projections are made according to the corresponding (N-2) collinearations.
6. Conics are drawn through five points of each plane. The C_N^2 conics are obtained, according to (N-1) projections for each.
7. It needs to construct a (N-1) dimensional space that intersects the quadric. The space must be orthogonal to the first projection plane. Therefore, the first projection of the intersecting space is built - a straight line. Next, the intersection problem is solved.
8. On the first projection are $(2 \cdot C_N^2)$ intersections of conics with a given line. According to projection links, their correspondences are found in other (N-2) planes of projections.
9. It is obtained C_N^3 sets of six points, of which C_{N-1}^2 sets of five points are necessary to construct C_{N-1}^2 conics, which specify the section of the N-dimensional quadric (N-1) with a dimensional space. Conics are built from these sets of five points. The problem is solved.

This algorithm for solving the intersection problem serves as the basis for the prolog program. The principles of logical programming allow us to automatically synthesize structural geometric models to solve problems of any dimension.

3. Conclusion

As a result of the performed work, an algorithm for constructing the intersection of the N-dimensional quadric N-1 space on hyperepure was compiled. The algorithm is described in natural language and is ready for implementation in logical programming languages. The environment for implementing the algorithm can be a Simplex [11-14] geometric modeling system that contains a prolog interpreter.

The program in the prolog logical programming language, compiled on the basis of the presented algorithm, contributes to the development of the use of logical programming and information technologies in geometry.

The automation of geometric constructions is a new direction in structural geometric modeling. The development of this direction provides new opportunities for the development of the science of geometry. Complex multidimensional constructions are no longer necessary to perform manually, which can significantly accelerate the work of scientists to identify new geometric patterns and solve problems related to geometry.

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