On Some Weakened Forms of Transitivity in the Logic of Norms

(Extended Abstract)

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Abstract

The paper investigates the impact of weakened forms of transitivity of the betterness relation on the logic of conditional obligation, originating from the work of Hansson, Lewis, and others. These weakened forms of transitivity come from the rational choice literature, and include: quasi-transitivity, Suzumura consistency, a-cyclicity, and the interval order condition. The first observation is that plain transitivity, quasi-transitivity, acyclicity and Suzumura consistency make less difference to the logic of $\bigcirc(-/-)$ than one would have thought. The axiomatic system remains the same whether or not these conditions are introduced. The second observation is that unlike the others the interval order condition corresponds to a new axiom, known as the principle of disjunctive rationality. These two observations are substantiated further through the establishment of completeness (or representation) theorems.

Keywords

Deontic conditional, betterness, transitivity, quasi-transitivity, Suzumura consistency, acyclicity, interval order

1. Introduction

The present paper ([1], under review) continues a project started in [2] and pursued further in [3, 4]. It deals with the problem of axiomatizing the logic of conditional obligation (*aka* dyadic deontic logic) with respect to preference models. Two types of consideration are thoroughly investigated: the choice of properties of the betterness (or preference) relation in the models, and the choice of the evaluation rule for the conditional obligation operator. Here my focus is on weakened forms of transitivity discussed in the related area of rational choice theory: quasi-transitivity, Suzumura consistency, a-cyclicity and the interval order condition [5, 6].

An important task in Knowledge Representation and Reasoning (KRR) is to understand what new axiom corresponds to a given semantic property in the models (as identified by the expert of the domain). This is relevant for the design of the reasoner itself: the conclusions this one will be able to draw from a KB vary depending on the logical system being used. This paper focuses on the property of transitivity of betterness and its weakenings thereof. Transitivity seems entrenched in our conceptual scheme, if not analytically true. However, the question of whether it holds, in what form, and in what context, has been much debated over the years [5, 6, 7, 8, 9, 10].

Reference is made to Åqvist [11]'s system **E**, the weakest known (preference-based) dyadic deontic logic. **E** corresponds to the most general case, involving no commitment to any structural property of the betterness relation in the models. **E** offers a simple solution to the contrary-to-duty paradoxes and allows to represent norms with exceptions. As is well-known (e.g. [12]), deontic logicians have struggled with the problem of giving a formal treatment to contrary-to-duty (CTD) obligations. These are obligations that come into force when some other obligation is violated. According to Hansson [13], Lewis [14] and others, the problems raised by CTDs call for an ordering on possible worlds in terms of preference (or relative goodness, or betterness), and Kripke-style models fail in as much as they do not allow for grades of ideality. The use of a preference relation has also been advocated for the analysis of defeasible conditional obligations. In particular, Alchourrón [15] argues that preference models provide a better treatment of this notion than the usual Kripke-style models do. Indeed, a defeasible conditional obligation leaves room for exceptions. Under a preference-based approach, we no longer have the deontic analogue of two laws, the failure of which constitutes the main formal feature expected of defeasible conditionals: "deontic" modus-ponens; and Strengthening of the Antecedent. $\bigcirc(B/A)$ may be read as "B is obligatory, given A". The first is the law: $\bigcirc(B/A)$ and A imply $\bigcirc B$. The second is the law: $\bigcirc(B/A)$ entails $\bigcirc(B/A \wedge C)$.

2. Framework

The syntax is generated by adding the following primitive operators to the syntax of propositional logic: $\Box$ (for historical necessity); $\bigcirc(-/-)$ (for conditional obli-

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gation). The main ingredient of a preference model is a preference relation $\succeq \subseteq W \times W$, where W is a non-empty set of worlds. Intuitively, $\succeq$ is a betterness or comparative goodness relation; "$a \succeq b$" can be read as "world a is at least as good as world b". a and b are equally good (indifferent), if $a \succeq b$ and $b \succeq a$. a is strictly better than b (notation: $a \succ b$) if $a \succeq b$ and $b \not\succeq a$. In that framework, $\bigcirc(B/A)$ is true if the best A-worlds are all B-worlds. There is variation among authors regarding the definition of "best". Here I assume "best" is cast in terms of maximality or—following Bradley [16]—strong maximality. A world a is maximal if it is not (strictly) worse than any other worlds. And a is strongly maximal if no world equally good as a is worse than any other worlds. The role of strong maximality is to ensure that the agent's choice meets the natural requirement of (as Bradley calls it) "Indifference based choice" (IBC): two alternatives that are equally good should always either both be chosen or both not chosen. Such a requirement can be violated, if $\succeq$ is no longer assumed to be transitive. Consider three worlds a, b and c with $a \succeq b$, $b \succeq c$ and $c \succeq b$. b and c are equally good; c is maximal (and hence chosen), but not b. Maximality and strong maximality coincide when $\succeq$ is transitive.

The weakened forms of transitivity mentioned above may be defined thus:

- $\succeq$ is quasi-transitive, if $\succ$ is transitive;
- $\succeq$ is acyclic, if $a \succ^\star b$ implies $b \not\succ a$ ($\succ^\star$ is the transitive closure of $\succ$);
- $\succeq$ is Suzumura consistent, if $a \succeq^\star b$ implies $b \not\succ a$;
- $\succeq$ is an interval order, if $\succeq$ is reflexive and Ferrers ($a \succeq b$ and $c \succeq d$ imply $a \succeq d$ or $c \succeq b$).

Intuitively, quasi-transitivity demands that the strict part of the betterness relation be transitive. A-cyclicity rules out the presence of strict betterness cycles. Suzumura consistency rules out the presence of cycles with at least one instance of strict betterness. The interval order condition makes room for the idea of non-transitive equal goodness relation due to discrimination thresholds.

The relationships between these conditions may be described thus (an arrow represents implication):

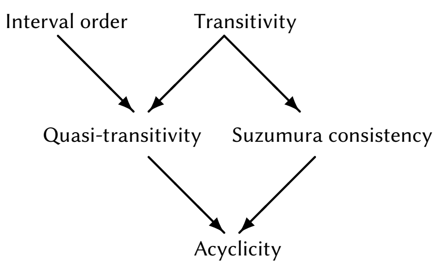

Figure 1: Implication relations

These conditions are studied in relation with four systems of increasing strength. The base system is Åqvist's system **E**, shown in Fig. 2 (labels are from [2]). Next we have Åqvist's system **F**; it is obtained by supplementing **E** with the law (D*): $\Diamond A \rightarrow (\bigcirc(B/A) \rightarrow P(B/A))$. Then comes **F**+(CM); it is obtained by supplementing **F** with the principle of cautious monotony (CM): $(\bigcirc(B/A) \wedge \bigcirc(C/A)) \rightarrow \bigcirc(C/A \wedge B)$. Finally, we have **F**+(DR); it is obtained by supplementing **F** with the principle of disjunctive rationality: $\bigcirc(C/A \vee B) \rightarrow (\bigcirc(C/A) \vee \bigcirc(C/B))$. We have $\mathbf{E} \subset \mathbf{F} \subset \mathbf{F}+(\mathrm{CM}) \subset \mathbf{F}+(\mathrm{DR})$.[1] (D*) rules out the possibility of conflicting obligations for a "consistent" context A. (CM) tells us that complying with an obligation does not modify our other obligations arising in the same context. (DR) tells us that if a disjunction of state of affairs triggers an obligation, then at least one disjunct triggers this obligation. It is noteworthy that (CM) is a theorem of **F**+(DR).

Suitable axioms for propositional logic	(PL)
S5 schemata for $\Box$ and $\Diamond$	(S5)
$\bigcirc(B \rightarrow C/A) \rightarrow (\bigcirc(B/A) \rightarrow \bigcirc(C/A))$	(COK)
$\bigcirc(B/A) \rightarrow \Box \bigcirc(B/A)$	(Abs)
$\Box A \rightarrow \bigcirc(A/B)$	(O-nec)
$\Box(A \leftrightarrow B) \rightarrow (\bigcirc(C/A) \leftrightarrow \bigcirc(C/B))$	(Ext)
$\bigcirc(C/A \wedge B) \rightarrow \bigcirc(B \rightarrow C/A)$	(Sh)
$\bigcirc(A/A)$	(Id)
If $\vdash A$ and $\vdash A \rightarrow B$ then $\vdash B$	(MP)
If $\vdash A$ then $\vdash \Box A$	(Nec)

Figure 2: Åqvist's system **E**

A few comments on the axioms of **E** are in order. (COK) is the conditional analogue of the familiar distribution axiom K. (Abs) is the absoluteness axiom of [14], and reflects the fact that the ranking is not world-relative. (O-nec) is the deontic counterpart of the familiar necessitation rule. (Ext) permits the replacement of necessarily equivalent sentences in the antecedent of deontic conditionals. (Sh) is named after Shoham [17, p. 77], who seems to have been the first to discuss it. (Id) is the deontic analogue of the identity principle. The question of whether (Id) is a reasonable law for deontic conditionals has been much debated. A defence of (Id) can be found in [13, 18] (see also [19]).

For an automation of reasoning tasks in **E** in Isabelle/HOL, see [20, 21].

[1] **F**+(CM) corresponds to the KLM system P supplemented with the principle of consistency preservation (if $A \not\vdash \bot$, then $A \not\mid\!\sim \bot$).

3. Quasi-transitivity, Suzumura consistency and a-cyclicity

The completeness result below is shown to hold under a rule of interpretation in terms of maximality and of strong maximality.[2] Such a result tells us that transitivity, quasi-transitivity, acyclicity and Suzumura consistency make less difference to the logic of $\bigcirc(-/-)$ than one would have thought. The axiomatic system remains the same whether or not these conditions are introduced.

Theorem 1. **E** *is sound and complete with respect to the following classes of preference models:*
(i) *The class of all preference models;*
(ii) *The class of those in which* $\succeq$ *is transitive;*
(iii) *The class of those in which* $\succeq$ *is quasi-transitive;*
(iv) *The class of those in which* $\succeq$ *is Suzumura consistent;*
(v) *The class of those in which* $\succeq$ *is quasi-transitive and Suzumura consistent;*
(vi) *The class of those in which* $\succeq$ *is acyclic.*

An analogous result is shown to hold for

- Åqvist's system **F** with respect to models in which $\succeq$ meets the condition of max-limitedness. It says: if the set of worlds that satisfy A is non-empty, then there is a world that is (strongly) maximal in this set.
- **F**+(CM) with respect to models in which $\succeq$ meets the so-called (strong-)max-smoothness condition. It says: if a satisfies A, then either a is (strongly) maximal in the set of worlds that satisfy A, or it is worse than some b that is (strongly) maximal in the set of worlds that satisfy A.

The paper also points out that Th.1 carries over to models with a reflexive betterness relation.

4. Interval order

A model is said to be finite, if its universe has finitely many worlds. The following result is established for a rule of interpretation in terms of maximality.[3] This result may fruitfully be compared to the representation result reported by [24] for models with a strict preference relation.

Theorem 2 (Weak completeness, finite preference models). *Under the max rule* **F***+(DR) is weakly sound and complete with respect to the class of finite preference models* $M = (W, \succeq, v)$ *in which* $\succeq$ *is an interval order.*

The assumption of finiteness is used in the arguments for both soundness and completeness. For soundness, finiteness comes into play as follows. If the model is finite, then given the interval order condition $\succeq$ is max-smooth and hence max-limited. Hence (D*)–the distinctive axiom of **F**–is validated, and so is (CM).

As a spin-off, one gets that the theoremhood problem in **F**+(DR) is decidable.

5. Wrap-up

Th.1 tells us that plain transitivity, quasi-transitivity, acyclicity and Suzumura consistency make less difference to the logic of $\bigcirc(-/-)$ than one would have thought. The determined logic is **E** whether or not these conditions are introduced. Th. 2 tells us that (in the finite case) the interval order condition boosts the logic to **F**+(DR), obtained by supplementing **F** with the principle of disjunctive rationality (DR).

Topics for future research include the following: to study the interval order condition in conjunction with the other candidate weakenings of transitivity; to study the effect of using variant evaluation rules for the conditional, like maximality-in-the-limit or variations thereof, where there are no best worlds, but (non-empty) sets of ever-better ones, which approximate the ideal (see, *e.g.*, [25, 26, 22]).

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