More on Interpolants and Explicit Definitions for Description Logics with Nominals and/or Role Inclusions

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Abstract
It is known that the problems of deciding the existence of Craig interpolants and of explicit definitions of concepts are both 2ExpTime-complete for standard description logics with nominals and/or role inclusions. These complexity results depend on the presence of an ontology. In this article, we first consider the case without ontologies (or, in the case of role inclusions, ontologies only containing role inclusions) and show that both the existence of Craig interpolants and of explicit definitions of concepts become coNExpTime-complete for DLs such as $\mathcal{ALCO}$ and $\mathcal{ALCH}$. Secondly, we make a few observations regarding the size and computation of interpolants and explicit definitions.

Keywords
Craig interpolants, Explicit definitions, Beth definability property, Description logics

1. Introduction

Craig interpolants and explicit definitions have many potential applications in ontology engineering and ontology-based information systems. Examples include the extraction of equivalent acyclic TBoxes from ontologies [1, 2], the computation of referring expressions (or definite descriptions) for individuals [3], concept separability and learning [4, 5], the equivalent rewriting of ontology-mediated queries into concepts or formulas [6, 7, 8, 9, 10], the construction of alignments between ontologies [11], and the decomposition of ontologies [12]. For logics enjoying the Craig interpolation property (CIP) the existence of a Craig interpolant follows from the validity of the defining subsumption and for logics enjoying the projective Beth definability property (PBDP) the existence of an explicit definition of a concept follows from its implicit definability. For such logics, deciding the existence of a Craig interpolant or an explicit definition of a concept are therefore not harder than subsumption and can be decided in ExpTime for DLs such as $\mathcal{ALC}$, $\mathcal{ALCI}$, $\mathcal{ALCQI}$ (which enjoy the CIP/PBDP [2]) if an ontology is present, and in PSpace without ontology.

This paper is a part of a research program with the goal of understanding Craig interpolants and explicit definitions for logics that do not enjoy the CIP/PBDP [13, 14]. The two most basic

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constructors that lead to DLs without the CIP and PBDP are nominals and role inclusions. In fact, it is known that the complexity of deciding the existence of Craig interpolants and explicit definitions are both 2ExpTime complete for standard DLs containing $\mathcal{ALCO}$ or $\mathcal{ALCH}$ and contained in the extension of $\mathcal{ALCHIO}$ with the universal role, in the presence of an ontology [15]. The case without ontology remained open. Note that nothing interesting happens for DLs containing the universal role or both nominals and inverse roles as it is known that then the ontology can be 'internalized', and thus there is no difference between the case with and without ontology. For DLs such as $\mathcal{ALCO}$, $\mathcal{ALCH}$, and $\mathcal{ALCHI}$, however, this is not the case. In fact, it is known that subsumption checking becomes PSpace-complete without ontology while it is ExpTime-complete with ontology. In the first part of this paper we investigate the complexity of deciding the existence of Craig interpolants and explicit definitions without ontologies for these DLs and show that it becomes coNExpTime-complete. Hence we observe again a significant increase in complexity compared to subsumption checking. Note that for $\mathcal{ALCH}$ and $\mathcal{ALCHI}$ we assume an ontology containing role inclusions only as otherwise they cannot be introduced and are not relevant.

In practice, of course, one is interested in the actual interpolants or the explicit definition. Unfortunately, the decision procedures for the existence problems provided in this paper and in [15] are non-constructive in the sense that they do not return an interpolant (an explicit definition) in case it exists. To address this problem, we (slightly) modify the decision procedure from [15] and show how to read off interpolants / explicit definitions from a run of the procedure, at least for DLs with role inclusions. In doing so, we take inspiration from a recent note on a type elimination based computation of interpolants in modal logic [16] which was originally provided for the guarded fragment [17].

For a discusson of further related work on interpolation, Beth definability, interpolant existence, and explicit definition existence we refer the reader to [13, 15]. Detailed proofs for this article are provided in the full version [18].

2. Preliminaries

We first introduce standard DL definitions and notation [19]. Let $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ be mutually disjoint and countably infinite sets of *concept*, *role*, and *individual names*. A *role* is a role name s or an *inverse role* s^-, with s a role name and $(s^-)^- = s$. We use u to denote the *universal role*. A *nominal* takes the form $\{a\}$, with $a \in \mathsf{N_I}$. An $\mathcal{ALCIO}^u$*-concept* is defined by the syntax rule

$$C, D ::= \top \mid A \mid \{a\} \mid \neg C \mid C \sqcap D \mid \exists r.C$$

where $a \in \mathsf{N_I}$, $A \in \mathsf{N_C}$, and r is a role. We use $C \sqcup D$ as abbreviation for $\neg(\neg C \sqcap \neg D)$, $C \rightarrow D$ for $\neg C \sqcup D$, and $\forall r.C$ for $\neg\exists r.\neg C$. We also consider the following fragments of $\mathcal{ALCIO}^u$: $\mathcal{ALCIO}$, obtained by dropping the universal role; $\mathcal{ALCO}^u$, obtained by dropping inverse roles; $\mathcal{ALCO}$, obtained from $\mathcal{ALCO}^u$ by dropping the universal role; and $\mathcal{ALC}$, obtained from $\mathcal{ALCO}$ by dropping nominals. If $\mathcal{L}$ is any of the DLs defined above, then an $\mathcal{L}$*-concept inclusion ($\mathcal{L}$-CI)* takes the form $C \sqsubseteq D$, with C and D $\mathcal{L}$-concepts. An $\mathcal{L}$*-ontology* is a finite set of $\mathcal{L}$-CIs. We also consider DLs with *role inclusions (RIs)*, expressions of the form $r \sqsubseteq s$, where r and s are roles. As usual, the addition of RIs is indicated by adding the letter $\mathcal{H}$ to the name of the DL, where inverse roles occur in RIs only if the DL admits inverse roles. Thus, for example, $\mathcal{ALCH}$-*ontologies* are finite sets of $\mathcal{ALC}$-CIs and RIs not using inverse roles and $\mathcal{ALCHIO}^u$-*ontologies* are finite sets of $\mathcal{ALCIO}^u$-CIs and RIs. In the following, we use $\mathsf{DL_{nr}}$ to denote the set of DLs $\mathcal{ALCO}$, $\mathcal{ALCIO}$, $\mathcal{ALCH}$, $\mathcal{ALCHO}$, $\mathcal{ALCHIO}$, and their extensions with the universal role. To simplify notation we do not drop the letter $\mathcal{H}$ when speaking about the concepts and CIs of a DL with RIs. Thus, for example, we sometimes use the expressions $\mathcal{ALCHO}$-concept and $\mathcal{ALCHO}$-CI to denote $\mathcal{ALCO}$-concepts and CIs, respectively.

The semantics is given in terms of *interpretations* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, defined as usual [19]. An interpretation $\mathcal{I}$ *satisfies* an $\mathcal{L}$-CI $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and an RI $r \sqsubseteq s$ if $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$. We say that $\mathcal{I}$ is a *model* of an ontology $\mathcal{O}$ if it satisfies all inclusions in it. We say that an inclusion α follows from an ontology $\mathcal{O}$, in symbols $\mathcal{O} \models \alpha$, if every model of $\mathcal{O}$ satisfies α. We write $\mathcal{O} \models C \equiv D$ if $\mathcal{O} \models C \sqsubseteq D$ and $\mathcal{O} \models D \sqsubseteq C$. We write $\models C \sqsubseteq D$ if $\mathcal{O} \models C \sqsubseteq D$ for the empty ontology $\mathcal{O}$. A concept C is *satisfiable* w.r.t. an ontology $\mathcal{O}$ if there is a model $\mathcal{I}$ of $\mathcal{O}$ with $C^{\mathcal{I}} \neq \emptyset$.

A *signature* Σ is a set of *symbols*, i.e., concept, role, and individual names. As standard in the literature, the universal role is not regarded as a symbol, but as a logical connective, and as such it is not contained in any signature. We use $\mathsf{sig}(X)$ to denote the set of symbols used in any syntactic object X such as a concept or an ontology. An $\mathcal{L}(\Sigma)$-*concept* is an $\mathcal{L}$-concept C with $\mathsf{sig}(C) \subseteq \Sigma$, and a Σ-*role* is a role r such that r or r^- is in Σ.

We require a model-theoretic characterization of when nodes are indistinguishable by $\mathcal{L}(\Sigma)$-concepts. A pair $\mathcal{I}, d$ with $\mathcal{I}$ an interpretation and $d \in \Delta^{\mathcal{I}}$ is called a *pointed interpretation*. For pointed interpretations $\mathcal{I}, d$ and $\mathcal{J}, e$ and a signature Σ, we write $\mathcal{I}, d \equiv_{\mathcal{L},\Sigma} \mathcal{J}, e$ and say that $\mathcal{I}, d$ and $\mathcal{J}, e$ are $\mathcal{L}(\Sigma)$-*equivalent* if $d \in C^{\mathcal{I}}$ iff $e \in C^{\mathcal{J}}$, for all $\mathcal{L}(\Sigma)$-concepts C. An $\mathcal{L}(\Sigma)$-*bisimulation* S is a relation $S \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ satisfying the standard back-and-forth conditions required by the constructors of $\mathcal{L}$, we refer the reader to [20, 21]. We write $\mathcal{I}, d \sim_{\mathcal{L},\Sigma} \mathcal{J}, e$ and call $\mathcal{I}, d$ and $\mathcal{J}, e$ $\mathcal{L}(\Sigma)$-*bisimilar* if there exists an $\mathcal{L}(\Sigma)$-bisimulation S such that $(d, e) \in S$. Then the following holds for all ω-saturated interpretations $\mathcal{I}$ and $\mathcal{J}$ (for the “if”-direction, the ω-saturatedness condition can be dropped):[1] $\mathcal{I}, d \equiv_{\mathcal{L},\Sigma} \mathcal{J}, e$ if and only if $\mathcal{I}, d \sim_{\mathcal{L},\Sigma} \mathcal{J}, e$.

3. Basic Notions and Results

Let $\mathcal{L}$ be a DL, let $\mathcal{O}_1, \mathcal{O}_2$ be $\mathcal{L}$-ontologies, and let C_1, C_2 be $\mathcal{L}$-concepts. We set $\mathsf{sig}(\mathcal{O}, C) = \mathsf{sig}(\mathcal{O}) \cup \mathsf{sig}(C)$, for any ontology $\mathcal{O}$ and concept C. Following [2], an $\mathcal{L}$-concept D is called an $\mathcal{L}$-*interpolant* for $C_1 \sqsubseteq C_2$ under $\mathcal{O}_1 \cup \mathcal{O}_2$ if: (i) $\mathsf{sig}(D) \subseteq \mathsf{sig}(\mathcal{O}_1, C_1) \cap \mathsf{sig}(\mathcal{O}_2, C_2)$; (ii) $\mathcal{O}_1 \cup \mathcal{O}_2 \models C_1 \sqsubseteq D$; (iii) $\mathcal{O}_1 \cup \mathcal{O}_2 \models D \sqsubseteq C_2$. $\mathcal{L}$-*interpolant existence* is the problem to decide the existence of an interpolant for $C_1 \sqsubseteq C_2$ under $\mathcal{O}_1 \cup \mathcal{O}_2$. In logics with the Craig Interpolation Property (CIP) (such as, for instance, $\mathcal{ALC}$ and $\mathcal{ALCI}$ [2]) the existence of an $\mathcal{L}$-interpolant for $C_1 \sqsubseteq C_2$ under $\mathcal{O}_1 \cup \mathcal{O}_2$ is equivalent to the entailment $\mathcal{O}_1 \cup \mathcal{O}_2 \models C_1 \sqsubseteq C_2$ and thus reduces to standard subsumption checking (which is, for instance, ExpTime-complete for $\mathcal{ALC}$ and $\mathcal{ALCI}$). This is not the case for the DLs considered here; in fact the following increase in complexity by one exponential is shown in [15].

[1] See [22] for the definition of ω-saturated interpretations.

Theorem 1. *Let $\mathcal{L} \in \mathsf{DL}_{\mathsf{nr}}$. Then $\mathcal{L}$-interpolant existence is* 2ExpTime*-complete.*

In this article we consider interpolant existence with either empty ontologies or ontologies containing RIs only. In detail, *ontology-free $\mathcal{L}$-interpolant existence* is the problem to decide $\mathcal{L}$-interpolant existence for empty ontologies. Note that for logics with the CIP ontology-free interpolant existence reduces to checking $\models C_1 \sqsubseteq C_2$ and hence is PSpace-complete for DLs such as $\mathcal{ALC}$ and $\mathcal{ALCI}$. If $\mathcal{L}$ admits RIs, then we consider *ontology-free $\mathcal{L}$-interpolant existence with RIs*, the problem to decide $\mathcal{L}$-interpolant existence for ontologies containing RIs only. We observe that DLs in $\mathsf{DL}_{\mathsf{nr}}$ do not enjoy the CIP, even without ontologies (ontologies containing RIs only, respectively).

Example 1. *Consider $C_1 = \{a\} \sqcap \exists r.\{a\}$ and $C_2 = \{b\} \rightarrow \exists r.\{b\}$. Then $\models C_1 \sqsubseteq C_2$ but there does not exist any $\mathcal{ALCO}$-interpolant for $C_1 \sqsubseteq C_2$ (see Example 5 for a proof). An example using RIs instead of nominals can be constructed from Example 3 below.*

We next introduce explicit definitions. We call an $\mathcal{L}(\Sigma)$-concept D an *explicit $\mathcal{L}(\Sigma)$-definition of C_0 under an ontology $\mathcal{O}$* if $\mathcal{O} \models C_0 \equiv D$. *$\mathcal{L}$-explicit definition* is the problem to decide the existence of an $\mathcal{L}(\Sigma)$-definition of an $\mathcal{L}$-concept under an $\mathcal{L}$-ontology. In logics with the appropriate projective Beth Definability Property (PBDP) [2, 15] the existence of an explicit $\mathcal{L}(\Sigma)$-definition of a concept follows from its implicit definability according to which the extension of the concept is determined by the extension of symbols in Σ. The latter condition can be decided using subsumption checking and is therefore ExpTime-complete for DLs with the PBDP such as $\mathcal{ALC}$ and $\mathcal{ALCI}$ [2]. Similarly to the interpolant existence problem, this is not the case for the DLs considered here and we have again an increase in complexity by one exponential [15].

Theorem 2. *Let $\mathcal{L} \in \mathsf{DL}_{\mathsf{nr}}$. Then explicit definition existence is* 2ExpTime*-complete.*

In this article we consider explicit definition existence without ontologies and ontologies containing RIs only. If C and C_0 are concepts and Σ a signature, then we call D an *explicit $\mathcal{L}(\Sigma)$-definition of C_0 under C* if $\models C \sqsubseteq (C_0 \leftrightarrow D)$.

Remark 2. *Explicit definitions under a concept C can be regarded as a 'local' version of explicit definitions under ontologies. If $\mathcal{O}$ is an ontology, then let N_n be the concept stating that $\mathcal{O}$ is true in all nodes reachable in at most n steps. Then a concept D is an $\mathcal{L}(\Sigma)$-definition of C_0 under $\mathcal{O}$ iff there exists an $n \geq 0$ such that D is an $\mathcal{L}(\Sigma)$-definition of C_0 under N_n.*

Then *ontology-free $\mathcal{L}$-definition existence* is the problem to decide for $\mathcal{L}$-concepts C and C_0, and a signature Σ whether there exists an explicit $\mathcal{L}(\Sigma)$-definition of C_0 under C. If $\mathcal{L}$ admits RIs, then *ontology-free $\mathcal{L}$-definition existence with RIs* is the problem to decide for an ontology $\mathcal{O}$ containing RIs only, $\mathcal{L}$-concepts C and C_0, and a signature Σ whether there exists an explicit $\mathcal{L}(\Sigma)$-definition D of C_0 under $\mathcal{O}$ and C, that is $\mathcal{O} \models C \sqsubseteq (C_0 \leftrightarrow D)$. For DLs with the PBDP such as $\mathcal{ALC}$ and $\mathcal{ALCI}$ ontology-free $\mathcal{L}$-definition existence reduces to subsumption checking without ontologies and is thus PSpace-complete. We next observe that the DLs in $\mathsf{DL}_{\mathsf{nr}}$ do not enjoy the PBDP without ontologies (ontologies containing RIs only).

Example 3. *Consider* $\mathcal{O} = \{r \sqsubseteq r_1, r \sqsubseteq r_2\}$ *and let* C *be the conjunction of* $(\neg\exists r.\top \sqcap \exists r_1.A) \rightarrow \forall r_2.\neg A$ *and* $(\neg\exists r.\top \sqcap \exists r_1.\neg A) \rightarrow \forall r_2.A$. *Let* $\Sigma = \{r_1, r_2\}$. *Then there does not exist an explicit* $\mathcal{ALC}(\Sigma)$*-definition of* $\exists r.\top$ *under* $\mathcal{O}$ *and* C *(see Example 6 below for a proof). The concept* $\exists r_1 \cap r_2.\top$*, however, is an explicit definition of* $\exists r.\top$ *under* $\mathcal{O}$ *and* C *in the extension of* $\mathcal{ALC}$ *with role intersection (with semantics defined in the obvious way). As any concept with an explicit definition in FO is implicitly definable,* $\exists r.\top$ *is implicitly definable.*

We conclude this section with a few observations on the relationship between the existence problems introduced above. It has been observed in [15] already that $\mathcal{L}$-explicit definition existence is polyomial time reducible to $\mathcal{L}$-interpolant existence. This also holds for the ontology-free versions.

Lemma 3. *Let* $\mathcal{L} \in \mathsf{DL}_{\mathsf{nr}}$. *Then ontology-free* $\mathcal{L}$*-definition existence (with RIs) can be reduced in polynomial time to ontology-free* $\mathcal{L}$*-interpolant existence (with RIs).*

By applying a standard encoding of ontologies into concepts one can show that for DLs in $\mathsf{DL}_{\mathsf{nr}}$ containing the universal role or both inverse roles and nominals dropping the ontology does not affect the complexity of explicit definition existence.

Lemma 4. *Let* $\mathcal{L} \in \mathsf{DL}_{\mathsf{nr}}$ *contain the universal role or both inverse roles and nominals. Then* $\mathcal{L}$*-explicit definition existence can be reduced in polynomial time to ontology-free* $\mathcal{L}$*-definition existence (with RIs if* $\mathcal{L}$ *admits RIs).*

We obtain the following complexity result as a consequence of Theorems 1-2 and Lemmas 3-4.

Theorem 5. *Let* $\mathcal{L} \in \mathsf{DL}_{\mathsf{nr}}$ *contain the universal role or both inverse roles and nominals. Then ontology-free interpolant existence (with RIs) and ontology-free explicit definition existence (with RIs) are both* 2ExpTime*-complete.*

4. Joint Consistency

The first main concern of the present paper is to study the computational complexity of the ontology-free interpolant and explicit definition existence problems. We show the upper bound for a generalization of interpolant existence. *Generalized* $\mathcal{L}$*-interpolant existence* is the problem to decide for an $\mathcal{L}$ ontology $\mathcal{O}$, $\mathcal{L}$-concepts C_1, C_2 and signature Σ whether there exists an $\mathcal{L}(\Sigma)$*-interpolant* for $C_1 \sqsubseteq C_2$ under $\mathcal{O}$, that is, an $\mathcal{L}(\Sigma)$-concept D such that $\mathcal{O} \models C_1 \sqsubseteq D$ and $\mathcal{O} \models D \sqsubseteq C_2$. The ontology-free version and the version with ontologies containing RIs only are defined in the obvious way. Note that $\mathcal{L}$-interpolant existence is indeed a special case of generalized $\mathcal{L}$-interpolant existence by setting $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ and $\Sigma = \text{sig}(\mathcal{O}_1, C_1) \cap \text{sig}(\mathcal{O}_2, C_2)$. As a preliminary step, we provide model-theoretic characterizations in terms of bisimulations as captured in the following central notion.

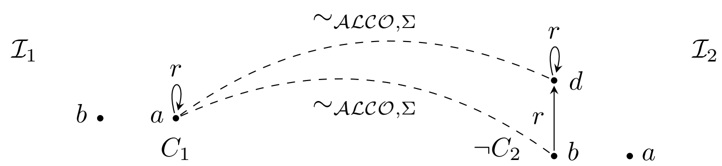
Definition 4 (Joint consistency)**.** *Let* $\mathcal{L} \in \mathsf{DL}_{\mathsf{nr}}$*,* $\mathcal{O}$ *be an* $\mathcal{L}$*-ontology,* C_1, C_2 *be* $\mathcal{L}$*-concepts, and* $\Sigma \subseteq sig(\mathcal{O}, C_1, C_2)$ *be a signature. Then* C_1, C_2 *are called* jointly consistent under $\mathcal{O}$ modulo $\mathcal{L}(\Sigma)$-bisimulations *if there exist pointed models* $\mathcal{I}_1, d_1$ *and* $\mathcal{I}_2, d_2$ *such that* $\mathcal{I}_i$ *is a model of* $\mathcal{O}$*,* $d_i \in C_i^{\mathcal{I}_i}$*, for* $i = 1, 2$*, and* $\mathcal{I}_1, d_1 \sim_{\mathcal{L},\Sigma} \mathcal{I}_2, d_2$*.*

The associated decision problem, *joint consistency modulo $\mathcal{L}$-bisimulations*, is defined in the expected way. The following result characterizes the existence of interpolants using joint consistency modulo $\mathcal{L}(\Sigma)$-bisimulations and is proved in [15].

Theorem 6. *Let $\mathcal{L} \in \mathsf{DL_{nr}}$. Let $\mathcal{O}$ be an $\mathcal{L}$-ontology, C_1, C_2 be $\mathcal{L}$-concepts, and $\Sigma \subseteq \text{sig}(\mathcal{O}, C_1, C_2)$. Then the following conditions are equivalent:*

1. *there is no $\mathcal{L}(\Sigma)$-interpolant for $C_1 \sqsubseteq C_2$ under $\mathcal{O}$;*
2. *$C_1, \neg C_2$ are jointly consistent under $\mathcal{O}$ modulo $\mathcal{L}(\Sigma)$-bisimulations.*

Example 5. *From Example 1, let $C_1 = \{a\} \sqcap \exists r.\{a\}$, $C_2 = \{b\} \rightarrow \exists r.\{b\}$, $\Sigma = \{r\}$. Interpretations $\mathcal{I}_1, \mathcal{I}_2$ below show that C_1 and $\neg C_2$ are jointly consistent modulo $\mathcal{ALCO}(\Sigma)$-bisimulations.*

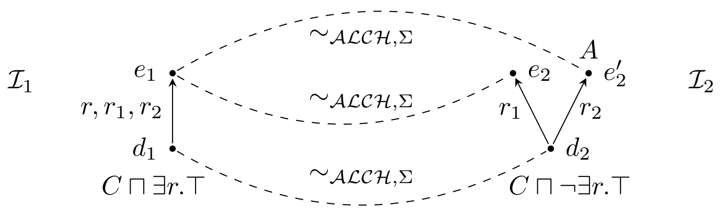


The existence of explicit definitions can be characterized as follows.

Theorem 7. *Let $\mathcal{L} \in \mathsf{DL_{nr}}$. Let $\mathcal{O}$ be an $\mathcal{L}$-ontology, C and C_0 $\mathcal{L}$-concepts, and $\Sigma \subseteq sig(\mathcal{O}, C)$ a signature. Then the following conditions are equivalent:*

1. *there is no explicit $\mathcal{L}(\Sigma)$-definition of C_0 under $\mathcal{O}$ and C;*
2. *$C \sqcap C_0$ and $C \sqcap \neg C_0$ are jointly consistent under $\mathcal{O}$ modulo $\mathcal{L}(\Sigma)$-bisimulations.*

Example 6. *Consider $\mathcal{O}$, C, and Σ from Example 3. The interpretations $\mathcal{I}_1, \mathcal{I}_2$ depicted below show that $C \sqcap \exists r.\top$ and $C \sqcap \neg\exists r.\top$ are jointly consistent under $\mathcal{O}$ modulo $\mathcal{ALCH}(\Sigma)$-bisimulations.*



5. Complexity

We formulate our main complexity result about the problem of deciding the existence of interpolants and explicit definitions.

Theorem 8. *Let $\mathcal{L} \in \mathsf{DL_{nr}}$ not contain the universal role and not contain both inverse roles and nominals simultaneously. Then ontology-free generalized $\mathcal{L}$-interpolant existence (with RIs), ontology-free $\mathcal{L}$-interpolant existence (with RIs), and ontology-free $\mathcal{L}$-definition existence (with RIs) are all* coNExpTime*-complete.*

We show the upper bound for generalized $\mathcal{L}$-interpolant existence by proving that joint consistency is in NExpTime (Theorem 6) and we show the lower bound by proving NExpTime-hardness for the version of joint consistency formulated in Theorem 7 (with empty ontology or, respectively, ontologies containing RIs only).

To show these results, we first require the following definitions. The *depth* of a concept C is the number of nestings of restrictions in C. For instance, a concept name B has depth 0 and $\exists r.\exists r.B$ has depth 2. Given an ontology $\mathcal{O}$ and concepts C_1, C_2, let $\Xi = \text{sub}(\mathcal{O}, C_1, C_2)$ denote the closure under single negation of the set of subconcepts of concepts in $\mathcal{O}, C_1, C_2$. A Ξ-*type* t is a subset of Ξ such that there exists an interpretation $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$ with $t = \text{tp}_{\Xi}(\mathcal{I}, d)$, where $\text{tp}_{\Xi}(\mathcal{I}, d) = \{C \in \Xi \mid d \in C^{\mathcal{I}}\}$ is the Ξ-*type realized at* d *in* $\mathcal{I}$. For a signature $\Sigma \subseteq \text{sig}(\mathcal{O}, C_1, C_2)$ and $i \in \{1, 2\}$, the *mosaic* defined by $d \in \Delta^{\mathcal{I}_i}$ in $\mathcal{I}_1, \mathcal{I}_2$ is the pair $(T_1(d), T_2(d))$ such that $T_j(d) = \{\text{tp}_{\Xi}(\mathcal{I}_j, e) \mid e \in \Delta^{\mathcal{I}_j}, \mathcal{I}_i, d \sim_{\mathcal{L},\Sigma} \mathcal{I}_j, e\}$, for $j = 1, 2$. We say that a pair (T_1, T_2) of sets T_1, T_2 of types is a *mosaic defined by* $\mathcal{I}_1, \mathcal{I}_2$ if there exists $d \in \Delta^{\mathcal{I}_1} \cup \Delta^{\mathcal{I}_2}$ such that $(T_1, T_2) = (T_1(d), T_2(d))$.

Example 7. *From Example 5, consider C_1, C_2, as well as $\mathcal{I}_1, \mathcal{I}_2$. The set Ξ consists of the concepts $\{a\}$, $\exists r.\{a\}$, $\{b\}$, $\exists r.\{b\}$, C_1, C_2, and negations thereof. We have that:*

- $\text{tp}_{\Xi}(\mathcal{I}_1, a^{\mathcal{I}_1}) = \{\{a\}, \exists r.\{a\}, \neg\{b\}, \neg\exists r.\{b\}, C_1, C_2\}$;
- $\text{tp}_{\Xi}(\mathcal{I}_2, b^{\mathcal{I}_2}) = \{\neg\{a\}, \neg\exists r.\{a\}, \{b\}, \neg\exists r.\{b\}, \neg C_1, \neg C_2\}$;
- $\text{tp}_{\Xi}(\mathcal{I}_2, d) = \{\neg\{a\}, \neg\exists r.\{a\}, \neg\{b\}, \neg\exists r.\{b\}, \neg C_1, C_2\}$.

The mosaic defined by $a^{\mathcal{I}_1}$ in $\mathcal{I}_1, \mathcal{I}_2$ is $(T_1(a^{\mathcal{I}_1}), T_2(a^{\mathcal{I}_1}))$, where $T_1(a^{\mathcal{I}_1}) = \{\text{tp}_{\Xi}(\mathcal{I}_1, a^{\mathcal{I}_1})\}$ and $T_2(a^{\mathcal{I}_1}) = \{\text{tp}_{\Xi}(\mathcal{I}_2, b^{\mathcal{I}_2}), \text{tp}_{\Xi}(\mathcal{I}_2, d)\}$.

A mosaic is *nominal generated* if some type in it contains a nominals. Consider $p = (T_1(d), T_2(d))$ and $q = (T_1(d'), T_2(d'))$ such that there exists a role name $r \in \Sigma$ with $(d, d') \in r^{\mathcal{I}_i}$, for some $i \in \{1, 2\}$. Then define, for every role name s and $i \in \{1, 2\}$, relations $R^{s,i}_{p,q} \subseteq T_i(d) \times T_i(d')$ by setting $(t, t') \in R^{s,i}_{p,q}$ if there exist e, e' realizing t and t', respectively, with $(T_1(e), T_2(e)) = p$ and $(T_1(e'), T_2(e')) = q$, such that $(e, e') \in s^{\mathcal{I}_i}$.

The upper bound follows from the following exponential size model property result.

Lemma 9. *Let $\mathcal{L} \in \mathsf{DL_{nr}}$ not contain the universal role and not contain both inverse roles and nominals simultaneously. Let $\mathcal{O}$ be a set of RIs , C_1, C_2 $\mathcal{L}$-concepts, and Σ a signature. If C_1 and $\neg C_2$ are jointly consistent under $\mathcal{O}$ modulo $\mathcal{L}(\Sigma)$-bisimulations, then there exist models of exponential size witnessing this; in more detail, there exist pointed models $\mathcal{I}, d$ and $\mathcal{J}, e$ of $\mathcal{O}$ of at most exponential size such that $d \in C_1^{\mathcal{I}}$, $e \notin C_2^{\mathcal{J}}$, and $\mathcal{I}, d \sim_{\mathcal{L},\Sigma} \mathcal{J}, e$.*

Proof. Assume that C_1 and $\neg C_2$ are jointly consistent under $\mathcal{O}$ modulo $\mathcal{L}(\Sigma)$-bisimulations. By definition, there exist pointed models $\mathcal{I}_1, d_1$ and $\mathcal{I}_2, d_2$ of $\mathcal{O}$ such that $d_1 \in C_1^{\mathcal{I}_1}$, $d_2 \notin C_2^{\mathcal{I}_2}$, and $\mathcal{I}_1, d_1 \sim_{\mathcal{L},\Sigma} \mathcal{I}_2, d_2$. Let k be the maximum depth of C_1, C_2.

We consider the case involving nominals and without inverse roles. We construct exponential size $\mathcal{J}_1, \mathcal{J}_2$ with the same properties of $\mathcal{I}_1, \mathcal{I}_2$ above. Let $\mathcal{B}$ be some minimal set of mosaics defined by $\mathcal{I}_1, \mathcal{I}_2$ such that: (i) all nominal generated mosaics are in $\mathcal{B}$; (ii) for every type t realized in $\mathcal{I}_i$ there exists $(T_1, T_2) \in \mathcal{B}$ with $t \in T_i$; (iii) $(T_1(d_1), T_2(d_1)) \in \mathcal{B}$. Observe that the size of $\mathcal{B}$ is at most exponential in the size of $\mathcal{O}, C_1, C_2$. Now select, for any mosaic $p = (T_1, T_2)$ defined by $\mathcal{I}_1, \mathcal{I}_2$ and any $\exists s.C \in t \in T_i$ such that there exists $r \in \Sigma$ with $\mathcal{O} \models s \sqsubseteq r$, a mosaic $q = (T_1', T_2')$ such that $(t, t') \in R^{s,i}_{p,q}$ and $C \in t'$, and denote the resulting set by $\mathcal{S}(p)$. Form the set $\mathcal{T}$ of sequences $\sigma = p_0 \cdots p_j = (T_1^0, T_2^0) \cdots (T_1^j, T_2^j)$, with $j \leq k$, $p_0 \in \mathcal{B}$ and $p_{i+1} \in \mathcal{S}(p_i)$ for $i < j$. Let $\text{tail}(\sigma) = p_j$ and $\text{tail}_i(\sigma) = T_i^j$. We next define the domain of $\mathcal{J}_1$ and $\mathcal{J}_2$ as

$$\Delta^{\mathcal{J}_i} = \{(t, p) \mid t \in \text{tail}_i(p), p \in \mathcal{B}\} \cup \{(t, \sigma) \mid \sigma \in \mathcal{T}, t \in \text{tail}_i(\sigma), |\sigma| > 1, t \text{ has no nominal}\}.$$

We define interpretations $\mathcal{J}_1, \mathcal{J}_2$ in the expected way. It can be shown that they are as required.

- For any individual name a and $(T_1, T_2) \in \mathcal{B}$ with $\{a\} \in t \in T_i$, we set $a^{\mathcal{J}_i} = (t, (T_1, T_2))$.
- For any concept name A, $(t, \sigma) \in A^{\mathcal{J}_i}$ iff $A \in t$.
- Let r be a role name. Then, we let for $\sigma p \in \mathcal{T}$:
  - $((t, \sigma), (t', \sigma p)) \in r^{\mathcal{J}_i}$ if $(t, t') \in R^{r,i}_{\text{tail}(\sigma),p}$ and t' contains no nominal;
  - $((t, \sigma), (t', p)) \in r^{\mathcal{J}_i}$ if $(t, t') \in R^{r,i}_{\text{tail}(\sigma),p}$ and t' contains a nominal.

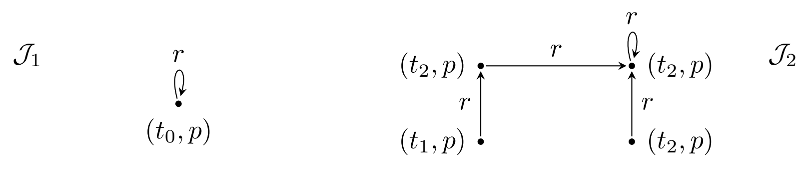
  Next assume that $\text{tail}(\sigma) = (T_1, T_2)$ and σ has length k. If $\text{tail}(\sigma') = (T_1, T_2)$ for some $|\sigma'| < k$, then choose as r-successors of any node of the form (t, σ) exactly the r-successors of (t, σ') defined above. If no such σ' exists, then all nodes of the form $(t, \text{tail}(\sigma))$ have distance exactly k from the roots (since no nominal occurs in any type in any mosaic in σ) and no successors are added.

  It remains to consider existential restrictions $\exists r.C$ for the role names r not entailing any role name in Σ. If $\sigma \in \mathcal{T}$, $\exists r.C \in t \in T_i$ with $\text{tail}_i(\sigma) = T_i$ and $\mathcal{O} \not\models r \sqsubseteq s$ for any $s \in \Sigma$, we add $((t, \sigma), (t', p))$ to $r^{\mathcal{J}_i}$ (and all $s^{\mathcal{J}_i}$ with $\mathcal{O} \models r \sqsubseteq s$) for some $p = (T_1', T_2') \in \mathcal{B}$ and $t' \in T_i'$ with $C \in t'$ such that there are e, e' realizing t, t' in $\mathcal{I}_i$ and $(e, e') \in r^{\mathcal{I}_i}$.

A similar construction can be used for the case with inverse roles, but without nominals. ❑

The following example illustrates the construction of $\mathcal{J}_1, \mathcal{J}_2$ from the proof above, in the case with nominals and without inverse roles, using the interpretations $\mathcal{I}_1, \mathcal{I}_2$ from Example 5.

Example 8. *Let* $t_0 = \text{tp}_\Xi(\mathcal{I}_1, a^{\mathcal{I}_1})$, $t_1 = \text{tp}_\Xi(\mathcal{I}_2, b^{\mathcal{I}_2})$, $t_2 = \text{tp}_\Xi(\mathcal{I}_2, d)$. *We ignore the types realized by* $b^{\mathcal{I}_1}$ *in* $\mathcal{I}_1$ *and by* $a^{\mathcal{I}_2}$ *in* $\mathcal{I}_2$ *as not relevant for understanding the construction. Then only the mosaic* $p = (T_1, T_2)$, *with* $T_1 = \{t_0\}$, $T_2 = \{t_1, t_2\}$, *remains.* $\mathcal{J}_1$, $\mathcal{J}_2$ *are depicted below.*



For the lower bound, we show that it is NExpTime-hard to decide joint consistency of $\mathcal{L}$-concepts $C \sqcap C_0$ and $C \sqcap \neg C_0$ (under an ontology containing RIs) modulo $\mathcal{L}(\Sigma)$-bisimulations and then employ Theorem 7. The proof is via an encoding of an *(exponential torus) tiling problem*, known to be NExpTime-complete.

6. The Computation Problem

Unfortunately, the algorithms for *deciding* the existence of interpolants do not immediately give rise to a way of *computing* interpolants in case they exist. Intuitively, this is due to the fact that compactness is used in the proof of the model-theoretic characterization in Theorem 6. In this section, we address the computation problem for DLs that do not contain nominals.

Theorem 10. *Let $\mathcal{L}$ be a DL in $\mathsf{DL}_{\mathsf{nr}}$ that does not contain nominals, and let $\mathcal{O}$ be an $\mathcal{L}$-ontology, C_1, C_2 be $\mathcal{L}$-concepts, and Σ be a signature. Then, if there is an $\mathcal{L}(\Sigma)$-interpolant for $C_1 \sqsubseteq C_2$ under $\mathcal{O}$, we can compute the DAG representation of an $\mathcal{L}(\Sigma)$-interpolant in time $2^{2^{p(n)}}$ where p is a polynomial and $n = ||\mathcal{O}|| + ||C_1|| + ||C_2||$.*

Note that this implies that the DAG representation is also of double exponential size, and that a formula representation of the interpolant can be computed in triple exponential time. Moreover, this also allows us to compute explicit definitions since, given $\mathcal{O}$, C, and Σ, any $\mathcal{L}(\Sigma)$-interpolant for $C_\Sigma \sqsubseteq C$ under $\mathcal{O} \cup \mathcal{O}_\Sigma$ is an explicit $\mathcal{L}(\Sigma)$-definition of C under $\mathcal{O}$, where $\mathcal{O}_\Sigma$ and C_Σ are obtained from $\mathcal{O}$ and C by replacing all symbols not in Σ by fresh symbols.

Let $\mathcal{L}$, $\mathcal{O}$, C_1, C_2, and Σ be as in Theorem 10. The computation of the $\mathcal{L}(\Sigma)$-interpolant (if it exists) is based on a *mosaic elimination procedure* for deciding joint consistency, which is a simplified variant of a procedure that was presented in [15] and which decides a slightly more general variant of joint consistency. As in Section 5, a *mosaic* is a pair (T_1, T_2) with T_1, T_2 sets of Ξ-types, where $\Xi = \mathrm{sub}(\mathcal{O}, C_1, C_2)$. We denote with $\mathrm{Tp}(\Xi)$ the set of all Ξ-types. The aim of the mosaic elimination procedure is to determine all pairs $(T_1, T_2) \in 2^{\mathrm{Tp}(\Xi)} \times 2^{\mathrm{Tp}(\Xi)}$ such that all $t \in T_1 \cup T_2$ can be realized in mutually $\mathcal{L}(\Sigma)$-bisimilar elements of models of $\mathcal{O}$. In order to formulate the elimination conditions, we need some preliminary notions. Throughout the rest of the section, we treat the universal role u as a role name contained in Σ, in case $\mathcal{L}$ allows the universal role. Note that u^- is equivalent to u, and that $\mathcal{O} \models r \sqsubseteq u$, for every role r.

Let t_1, t_2 be Ξ-types. We call t_1, t_2 *u-equivalent* if for every $\exists u.C \in \Xi$, we have $\exists u.C \in t_1$ iff $\exists u.C \in t_2$. This condition is trivial if $\mathcal{L}$ does not use allow the universal role. For a role r, we call t_1, t_2 *r-coherent for $\mathcal{O}$*, in symbols $t_1 \rightsquigarrow_{r,\mathcal{O}} t_2$, if t_1, t_2 are u-equivalent and the following conditions hold for all roles s with $\mathcal{O} \models r \sqsubseteq s$: (1) if $\neg\exists s.C \in t_1$, then $C \notin t_2$ and (2) if $\neg\exists s^-.C \in t_2$, then $C \notin t_1$. Note that $t \rightsquigarrow_{r,\mathcal{O}} t'$ iff $t' \rightsquigarrow_{r^-,\mathcal{O}} t$. We lift the definition of r-coherence from types to mosaics $(T_1, T_2), (T_1', T_2')$. We call (T_1, T_2), (T_1', T_2') *r-coherent*, in symbols $(T_1, T_2) \rightsquigarrow_r (T_1', T_2')$, if for $i = 1, 2$, *(i)* for every $t \in T_i$ there exists a $t' \in T_i'$ such that $t \rightsquigarrow_{r,\mathcal{O}} t'$, and *(ii)* if $\mathcal{L}$ allows for inverse roles, then for every $t' \in T_i'$, there is a $t \in T_i$ such that $t \rightsquigarrow_{r,\mathcal{O}} t'$. Note that $(T_1, T_2) \rightsquigarrow_r (T_1', T_2')$ iff $(T_1', T_2') \rightsquigarrow_{r^-} (T_1, T_2)$ if $\mathcal{L}$ allows for inverses.

Let $\mathcal{S} \subseteq 2^{\mathrm{Tp}(\Xi)} \times 2^{\mathrm{Tp}(\Xi)}$. We call $(T_1, T_2) \in \mathcal{S}$ *bad* if it violates one of the following conditions.

1. *Σ-concept name coherence*: $A \in t$ iff $A \in t'$, for every concept name $A \in \Sigma$ and any $t, t' \in T_1 \cup T_2$;

2. *Existential saturation*: for $i = 1, 2$ and $\exists r.C \in t \in T_i$, there exists $(T_1', T_2') \in \mathcal{S}$ such that (1) there exists $t' \in T_i'$ with $C \in t'$ and $t \rightsquigarrow_{r,\mathcal{O}} t'$ and (2) if $\mathcal{O} \models r \sqsubseteq s$ for a Σ-role s, then $(T_1, T_2) \rightsquigarrow_s (T_1', T_2')$.

The mosaic elimination procedure is now as follows. We start with the set $\mathcal{S}_0$ of all mosaics $(T_1, T_2) \in 2^{\text{Tp}(\Xi)} \times 2^{\text{Tp}(\Xi)}$ such that, for $i = 1, 2$, T_i contains only types that are realizable in some model of $\mathcal{O}$. Then obtain, for $i \geq 0$, $\mathcal{S}_{i+1}$ from $\mathcal{S}_i$ by eliminating all mosaics (T_1, T_2) that are bad in $\mathcal{S}_i$. Let $\mathcal{S}^*$ be where the sequence stabilizes. This elimination procedure decides joint consistency (and thus interpolant existence via Theorem 6) since the following are equivalent:

(A) $C_1, \neg C_2$ are jointly consistent under $\mathcal{O}$ modulo $\mathcal{L}(\Sigma)$-bisimulations;

(B) there exists $(T_1, T_2) \in \mathcal{S}^*$ and Ξ-types $t_1 \in T_1, t_2 \in T_2$ with $C_1 \in t_1$ and $\neg C_2 \in t_2$.

We will show how to read off interpolants from the run of the elimination procedure, but we need one more notion. For a set T of Ξ-types, an interpretation $\mathcal{I}$, and a family d_t, $t \in T$ of domain elements of $\mathcal{I}$, we say that $\mathcal{I}$ and d_t, $t \in T$ *jointly realize* T *modulo* $\mathcal{L}(\Sigma)$*-bisimulations* if, for all $t, t' \in T$, $\mathsf{tp}_\Xi(\mathcal{I}, d_t) = t$ and $\mathcal{I}, d_t \sim_{\mathcal{L},\Sigma} \mathcal{I}, d_{t'}$. The elimination procedure decides joint realizability since T is jointly realizable modulo $\mathcal{L}(\Sigma)$-bisimulations iff a mosaic $(T, \emptyset)$ survives elimination. In what follows, let Real denote the set of all sets of types T which are jointly realizable modulo $\mathcal{L}(\Sigma)$-bisimulations.

Lemma 11. *Let $T_1, T_2 \in \mathsf{Real}$. If (T_1, T_2) is eliminated in the elimination procedure, then we can compute an $\mathcal{L}(\Sigma)$-concept I_{T_1,T_2} such that:*

1. *for all models $\mathcal{I}$ of $\mathcal{O}$ and elements d_t, $t \in T_1$ that realize T_1 modulo $\mathcal{L}(\Sigma)$-bisimulations, $d_t \in I^{\mathcal{I}}_{T_1,T_2}$ for some (equivalently: all) $t \in T_1$;*
2. *for all models $\mathcal{I}$ of $\mathcal{O}$ and elements d_t, $t \in T_2$ that realize T_2 modulo $\mathcal{L}(\Sigma)$-bisimulations, $d_t \notin I^{\mathcal{I}}_{T_1,T_2}$ for some (equivalently: all) $t \in T_2$.*

The concepts I_{T_1,T_2} are computed inductively in the order in which the (T_1, T_2) got eliminated in the elimination procedure. We distinguish cases why (T_1, T_2) got eliminated.

Suppose first that (T_1, T_2) was eliminated because of (failing) Σ-concept name coherence. Since T_1, T_2 are both jointly realizable, there are two cases:

(a) There is an $A \in \Sigma$ with $A \in t$ for all $t \in T_1$, but $A \notin t$, for all $t \in T_2$. Then $I_{T_1,T_2} = A$.

(b) There is an $A \in \Sigma$ with $A \in t$ for all $t \in T_2$, but $A \notin t$, for all $t \in T_1$. Then $I_{T_1,T_2} = \neg A$.

Now, suppose that (T_1, T_2) was eliminated due to (failing) existential saturation from $\mathcal{S}_i$ during the elimination. Since T_1, T_2 are both jointly realizable, there are two cases:

(a) There exist $t \in T_1$, $\exists r.C \in t$, and a Σ-role s with $\mathcal{O} \models r \sqsubseteq s$, such that there is no $(T_1', T_2') \in \mathcal{S}_i$ such that *(i)* $(T_1, T_2) \rightsquigarrow_s (T_1', T_2')$ and *(ii)* there is $t' \in T_1'$ with $C \in t'$ and $t \rightsquigarrow_{r,\mathcal{O}} t'$. Then, take

$$I_{T_1,T_2} = \exists s.\Big(\bigsqcup_{\substack{T_1' \in \mathsf{Real}, \\ T_1 \rightsquigarrow_s T_1', t \rightsquigarrow_{r,\mathcal{O}} t', C \in t' \in T_1'}} \ \bigsqcap_{\substack{T_2' \in \mathsf{Real}, \\ T_2 \rightsquigarrow_s T_2'}} I_{T_1',T_2'} \Big)$$

(b) There exist $t \in T_2$, $\exists r.C \in t$, and a Σ-role s with $\mathcal{O} \models r \sqsubseteq s$, such that there is no $(T_1', T_2') \in \mathcal{S}$ such that *(i)* $(T_1, T_2) \rightsquigarrow_s (T_1', T_2')$ and *(ii)* there is $t' \in T_2'$ with $C \in t'$ and $t \rightsquigarrow_{r,\mathcal{O}} t'$. Then, take

$$I_{T_1,T_2} = \forall s.(\bigsqcup_{\substack{T_1' \in \mathsf{Real},\\ T_1 \rightsquigarrow_s T_1'}} \bigsqcap_{\substack{T_2' \in \mathsf{Real},\\ T_2 \rightsquigarrow_s T_2', t \rightsquigarrow_{r,\mathcal{O}} t', C \in t' \in T_2'}} I_{T_1',T_2'})$$

We show in the long version that the I_{T_1,T_2} are as required. Observe that we can represent the I_{T_1,T_2} in DAG shape by using a single node for every I_{T_1,T_2} (plus some auxiliary nodes connecting them). Overall, we obtain double exponentially many nodes in the DAG and the DAG can be constructed in double exponential time (both in $p(||\mathcal{O}|| + ||C_1|| + ||C_2||)$).

Given Lemma 11 it is relatively straightforward to construct the desired interpolants.

Lemma 12. *Suppose the result $\mathcal{S}^*$ of the elimination procedure does not contain a pair $(T_1, T_2) \in$ $\mathsf{Real} \times \mathsf{Real}$ such that $C_1 \in t_1$ and $\neg C_2 \in t_2$ for some types $t_1 \in T_1$ and $t_2 \in T_2$. Then,*

$$C = \bigsqcup_{\substack{T_1 \in \mathsf{Real}:\\ \textit{there is } t_1 \in T_1 \textit{ with } C_1 \in t_1}} \bigsqcap_{\substack{T_2 \in \mathsf{Real}:\\ \textit{there is } t_2 \in T_2 \textit{ with } \neg C_2 \in t_2}} I_{T_1,T_2}$$

is an $\mathcal{L}(\Sigma)$-interpolant for $C_1 \sqsubseteq C_2$ under $\mathcal{O}$. Moreover, a DAG representation of C can be computed in time $2^{2^{p(n)}}$, for some polynomial p and $n = ||\mathcal{O}|| + ||C_1|| + ||C_2||$.

To conclude, we give an intuition as to why the proof of Theorem 10 cannot be easily adapted to logics from $\mathsf{DL}_{\mathsf{nr}}$ that allow for nominals. Observe that in any two interpretations $\mathcal{I}_1, \mathcal{I}_2$, every nominal a is realized (modulo bisimulation) in exactly one mosaic. Thus, for the mosaic elimination procedure to work (in the sense of the equivalence between (A) and (B) above) one has to “guess” for every a exactly one mosaic that describes a [15]. Then, there is an interpolant for $C_1 \sqsubseteq C_2$ under $\mathcal{O}$ iff the runs of the mosaic elimination procedure for all possible guesses of the nominal mosaics in $\mathcal{S}_0$ lead to an $\mathcal{S}^*$ which does not satisfy (B). It is, however, unclear how to combine these different runs in proving analogues of Lemmas 11 and 12.

7. Conclusion and Future Work

We have determined tight complexity bounds for the problem of deciding the existence of interpolants and explicit definitions in standard DLs with nominals and/or role inclusions, with and without ontologies. It would be of interest to investigate these decision problems also for DLs with additional constructors (such as number restrictions and transitive closure) and for acyclic ontologies.

We have also performed first steps in the analysis of the computation problem, but many interesting problems remain to be addressed. First, note that our analysis only applies to the case with ontologies and we expect interpolants in the ontology free case to be one exponential smaller (if the universal role is not present). Second, we have provided only upper bounds on the size of interpolants and it remains to see whether the construction is optimal (we conjecture it to be). Finally, it is of great interest to compute interpolants also in the presence of nominals. An alternative approach might be to derive them from a suitably constrained proof of $\mathcal{O} \models C_1 \sqsubseteq C_2$ in an appropriate proof system, see e.g. [23].

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