Fuzzy closure systems over Heyting algebras as fixed points of a fuzzy Galois connection

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Abstract

Galois connections are useful to model solutions for both pure and application-oriented problems. Throughout the paper, the general framework is a complete fuzzy lattice over a Heyting algebra. We have established a fuzzy Galois connection between the fuzzy powerset lattice and the set of functions in A. Furthermore, the fixed points, or *formal concepts*, of this fuzzy Galois connection are exactly the fuzzy closure systems and fuzzy closure operators on A. The extension of this fuzzy Galois connection to the general framework is discussed but the study of the fixed points is still an open problem.

Keywords

Closure system, Galois connection, Fuzzy lattice

1. Introduction

Galois connections occur in a great variety of mathematical theories and in several instances in the theory of relations [1]. Thus, it makes sense to study their properties separately. Birkhoff [2] pointed out that any binary relation defines a Galois connection between the subsets of two sets and it is easily seen that conversely every Galois connection can be constructed in this manner. This is deeply related to Formal Concept Analysis [3], thus the research on Galois connections complements that on FCA. An interesting source to find the classical notions from FCA, the interrelationship among fixed points, closure operators, systems, and complete lattices can be found in [4].

The fuzzy extension of the notion of Galois connection was introduced by Bělohlávek [5] and the so-called Galois condition, which is an "if and only if" condition in the crisp case, is substituted by the equality of the fuzzy orders. This definition is used for research concerning Fuzzy Formal Concept Analysis.

The other main concept used in this paper are closure structures. Closure operators, introduced by E.H. Moore in 1910 [6], play a major role in computer science and both pure and applied mathematics [7]. The extension to the fuzzy framework, the so-called fuzzy closure

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operators [8, 9], appear in several areas of fuzzy logic and its applications. Its counterpart, the so-called closure systems, have been extended to the fuzzy framework as well. Despite the general agreement upon the definition of fuzzy closure operator, this is not the case for fuzzy closure systems. Thus, we can find several distinct definitions of fuzzy closure system in the literature [8, 10, 11, 12, 13, 14]. The definition of fuzzy closure system used in this paper will be the one of [13], which, even though it is equivalent to the one in [14], is defined directly on Heyting algebras, the framework of this paper.

In [13], the framework of the paper was a complete $\mathbb{L}$-fuzzy lattice (A, ρ), where the infimum and the supremum are denoted by $\sqcap$ and $\sqcup$, respectively and $\mathbb{L}$ is a complete Heyting algebra. The main discussion was searching for a suitable definition of fuzzy closure system. In that, there are two mappings which transform fuzzy closure operators into fuzzy closure systems and vice versa. These mappings are defined as follows, let $\Phi \in L^A$ be a fuzzy closure system and $\mathrm{c}\colon A \to A$ be a fuzzy closure operator. Then, $\mathrm{c}_\Phi\colon A \to A$ is defined as $\mathrm{c}_\Phi(a) = \bigsqcap(a^\rho \cap \Phi)$ and $\Phi_\mathrm{c}(a) = \rho(\mathrm{c}(a), a)$.

In this paper, we elaborate on these mappings since their domain and codomain does not have to be the set of fuzzy closure systems and the set of fuzzy closure operators, respectively. Actually, these mappings are well-defined for all isotone functions $f\colon A \to A$ and all fuzzy sets $X \in L^A$. The main goal of the paper is studying whether these mappings defined in the most general domains and codomains form a fuzzy Galois connection. As a matter of fact, the fuzzy set Φ_f can be defined for any function f on A, but the fuzzy Galois connection found in the results of the paper requires the use of isotone mappings only. This restriction does not affect the main result of the paper since we are interested in including fuzzy closure operators, which are isotone mappings. Then, it is proved that the fixed points of this fuzzy Galois connection satisfy some additional properties, such as extensionality of inflationarity.

The main result of the paper is proving that fixed points of a specific fuzzy Galois connection are the ones formed by a fuzzy closure system and the fuzzy closure operator induced by it. Contrary to the main results in Galois connections and FCA [2, 4] and the first approaches to the fuzzy framework [15], which state that the composition of the mappings in a (fuzzy) Galois connection is a closure operator, in this paper we study one fuzzy Galois connection in particular such that closure operators are the fixed points, i.e., we show that there is a certain fuzzy Galois connection $\langle f, g\rangle$ such that if $\mathrm{c}\colon A \to A$ is a closure operator in A^A, then $fg(\mathrm{c}) = \mathrm{c}$. Last, there is a section of conclusions and further work where the results are discussed and some hints of future research lines are given.

2. Preliminaries

For the reader's convenience, we give the main notions and theorems that are used along this paper.

An algebra $\mathbb{L} = (L, \leq, 0, 1, \to)$ is a complete Heyting algebra, [16], if $(L, \leq)$ is a complete lattice, 0 and 1 are the minimum and maximum elements, respectively, and the following condition holds, for all $p, q, r \in L$

$$p \wedge q \leq r \qquad \text{if and only if} \qquad p \leq q \to r \tag{1}$$

In addition, we have the following properties:

$$(p \to q) \wedge (r \to s) \le (p \wedge r) \to (q \wedge s) \tag{2}$$

$$p \wedge (p \to q) \le p \wedge q \tag{3}$$

$$p \le q \text{ implies } p \to r \le p \to (r \wedge q) \tag{4}$$

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \tag{5}$$

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r) \tag{6}$$

for all $p, q, r \in L$.

An $\mathbb{L}$-*set* or fuzzy set is a mapping $X \colon U \to L$ from the universe set U to the membership values set L where $X(u)$ means the degree in which u belongs to X. An $\mathbb{L}$-set A is said to be a fuzzy subset of X if $A(u) \le X(u)$, for all $u \in U$. As usual, the *core* of X is the set of elements such that $X(x) = 1$ and is denoted by $Core(X)$. An $\mathbb{L}$-set $X \in L^A$ is said to be extensional if $X(a) \wedge (a \approx b) \le X(b)$, for all $a, b \in A$. Finally, a crisp set is considered to be a particular case of $\mathbb{L}$-set by using its characteristic mapping $X \colon U \to \{0, 1\}$ with $X(u) = 1$ iff $u \in X$. Operations with $\mathbb{L}$-sets are defined pointwise as usual.

Let $S \colon L^U \times L^U \to L$ be a mapping defined by $S(A, B) = \bigwedge_{x \in U} \big(A(x) \to B(x)\big)$. This mapping is called the subsethood degree relation. Whenever A is a fuzzy subset of B, we have $S(A, B) = 1$, and this is denoted by $A \subseteq B$.

The concepts of fuzzy binary relation, its properties, fuzzy poset and fuzzy lattice are used as in the standard definitions, see [17, 18, 19]. Throughout this section (A, ρ) will be a fuzzy poset.

The lower and upper cone of a fuzzy set X are defined as

$$\begin{aligned} X_\rho(a) &= \bigwedge_{x \in A} \big(X(x) \to \rho(a, x)\big) && \text{and} \\ X^\rho(a) &= \bigwedge_{x \in A} \big(X(x) \to \rho(x, a)\big) && \text{for all } a \in A. \end{aligned}$$

Similarly, the following concepts were introduced in [20],

Definition 1. *An element $a \in A$ is an* infimum *(resp.* supremum*) of a fuzzy set $X \in L^A$ if:*

i. $X_\rho(a) = 1$ *(resp. $X^\rho(a) = 1$).*
ii. $X_\rho \subseteq a_\rho$ *(resp. $X^\rho \subseteq a^\rho$).*

As a consequence, if $a \in A$ is infimum (resp. supremum) of $X \in L^A$, then $X \subseteq a^\rho$ (resp. $X \subseteq a_\rho$).

As in the classical case, if the infimum (resp. supremum) of a set exists then it is unique and is denoted by $\bigsqcap X$ (resp. $\bigsqcup X$). In addition, an element $m \in A$ is a *minimum* (resp. *maximum*) of X if and only if m is an infimum (resp. supremum) of X and $X(m) = 1$.

Theorem 1. *Let $X \in L^A$. An element $a \in A$ is the infimum (resp. supremum) of X if and only if $X_\rho = a_\rho$ (resp. $X^\rho = a^\rho$).*

A complete fuzzy lattice (A, ρ) is a fuzzy poset such that every fuzzy subset $X \in L^A$ has supremum and infimum. We denote by $\perp$ and $\top$ the bottom and top elements of the complete fuzzy lattice, respectively. The pair (L^U, S) is an example of complete fuzzy lattice [21], which is called the *L-powerset lattice of U*. A set $B \subseteq A$ is said to be a complete fuzzy sublattice of A if, for all fuzzy set $X \in L^B$, the elements $\bigsqcap X$ and $\bigsqcup X$ are in B.

The focus of this paper is the characterization of closure structures as fixed points of a fuzzy Galois connection. The definition of closure operator is the following.

Definition 2. *Given a fuzzy preposet $\mathbb{A} = (A, \rho)$, a mapping $\mathrm{c}\colon A \to A$ is said to be a* closure operator *on $\mathbb{A}$ if the following conditions hold:*

i. $\rho(a, b) \leq \rho(\mathrm{c}(a), \mathrm{c}(b))$, *for all $a, b \in A$*
ii. $\rho(a, \mathrm{c}(a)) = 1$, *for all $a \in A$*
iii. $\rho(\mathrm{c}(\mathrm{c}(a)), \mathrm{c}(a)) = 1$, *for all $a \in A$.*

Conditions i and ii are well-known and are called isotony and inflationarity, respectively. Observe that condition iii could be replaced by $(\mathrm{c}(\mathrm{c}(a)) \approx \mathrm{c}(a)) = 1$, and, thus, if $\mathbb{A}$ is a fuzzy poset, a closure operator is idempotent in a classical sense, i.e., $\mathrm{c}(\mathrm{c}(a)) = \mathrm{c}(a)$ for all $a \in A$.

On the other hand, there is not a unique extension of the notion of closure system to a fuzzy setting. The definition we will use in this paper is the following one, introduced in [13].

Definition 3. *Let (A, ρ) be a complete fuzzy lattice. We say that an $\mathbb{L}$-set $\Phi \in L^A$ is a* fuzzy closure system *if it is extensional and $\bigsqcap X \in Core(\Phi)$ for all $X \subseteq \Phi$.*

Fuzzy closure systems and fuzzy closure operators are related concepts. In fact, similarly to the classical case, there is a one-to-one relation among them.

Theorem 2. *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. The following assertions hold:*

i. *If Φ is a fuzzy closure system, the mapping $\mathrm{c}_\Phi\colon A \to A$ defined as $\mathrm{c}_\Phi(a) = \bigsqcap(a^\rho \cap \Phi)$ is a closure operator.*
ii. *If c is a closure operator, the fuzzy set Φ_c defined as $\Phi_\mathrm{c}(a) = \rho(\mathrm{c}(a), a)$ is a fuzzy closure system.*
iii. *If Φ is a fuzzy closure system, then $\Phi = \Phi_{\mathrm{c}_\Phi}$.*
iv. *If $\mathrm{c}\colon A \to A$ is a closure operator, then $\mathrm{c}_{\Phi_\mathrm{c}} = \mathrm{c}$.*

The following result is a characterization of fuzzy closure systems which is useful in the proofs.

Proposition 1. *Let (A, ρ) be a complete fuzzy lattice. A fuzzy set $\Phi \in L^A$ is a fuzzy closure system if and only if Φ is extensional and $\min(a^\rho \cap \Phi)$ exists for all $a \in A$.*

Fuzzy Galois connections are a main concept in this paper as well. Let us recall the definition.

Definition 4 ([19]). *Let $\mathbb{A} = \langle A, \rho_A \rangle$ and $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy posets, $f\colon A \to B$ and $g\colon B \to A$ be two mappings. The pair (f, g) is called a* Galois connection between $\mathbb{A}$ and $\mathbb{B}$, *denoted by $(f, g) : \mathbb{A} \rightharpoonup \mathbb{B}$, if*

$$\rho_A(a, g(b)) = \rho_B(b, f(a)) \quad \text{for all } a \in A \text{ and } b \in B.$$

3. A fuzzy Galois connection between $\mathbb{L}$-sets and isotone mappings

Consider, on the one hand, the $\mathbb{L}$-*powerset lattice of* A, denoted by (L^A, S), and, on the other hand, the pair $(A^A, \tilde{\rho})$ consisting of the set of (crisp) mappings on A and the pointwise $\mathbb{L}$-order defined as

$$\tilde{\rho}(f_1, f_2) = \bigwedge_{x \in A} \rho(f_1(x), f_2(x)) \text{ for all } f_1, f_2 \in A^A.$$

Among the mappings in $(A^A, \tilde{\rho})$, the isotone (or "order preserving") ones play an important role because they reflect the idea of homomorphism betweeen $\mathbb{L}$-posets. We denote the set of isotone mappings on (A, ρ) as $(\mathrm{Isot}(A^A), \tilde{\rho})$.

Proposition 2. *The couple* $(A^A, \tilde{\rho})$ *is a complete fuzzy lattice and* $(\mathrm{Isot}(A^A), \tilde{\rho})$ *is a complete fuzzy sublattice.*

Consider now the functions $\mathrm{c}\colon (L^A, S) \to (\mathrm{Isot}(A^A), \tilde{\rho})$ and $\Psi\colon (\mathrm{Isot}(A^A), \tilde{\rho}) \to (L^A, S)$ defined as follows:

- c assigns to any $\Phi \in L^A$ the mapping $\mathrm{c}_\Phi\colon A \to A$ where

$$\mathrm{c}_\Phi(a) = \bigsqcap (a^\rho \cap \Phi) \text{ for all } a \in A.$$

- Ψ assigns to any isotone mapping $f\colon A \to A$ the $\mathbb{L}$-set $\Psi_f \in L^A$ where

$$\Psi_f(a) = \rho(f(a), a) \text{ for all } a \in A.$$

The following proposition ensures that the mapping c is well-defined.

Proposition 3. *For all* $\Phi \in L^A$, *the mapping* $\mathrm{c}(\Phi)$ *is isotone.*

Proof. For all $x, y \in A$, we have that

$$\begin{aligned}
\rho(x, y) &\le \bigwedge_{z \in A} (\rho(y, z) \to \rho(x, z)) && \text{by transitivity of } \rho \\
&\le \bigwedge_{z \in A} ((y^\rho \cap \Phi)(z) \to (x^\rho \cap \Phi)(z)) && \text{by (2.45) in [20]} \\
&\le \bigwedge_{z \in A} ((y^\rho \cap \Phi)(z) \to \rho(\mathrm{c}_\Phi(x), z)) && \text{by Def. 1 and (2.43) in [20]} \\
&= \rho(\mathrm{c}_\Phi(x), \mathrm{c}_\Phi(y)) && \text{by Theorem 1.}
\end{aligned}$$

□

Hence, both Ψ and c are well-defined. In addition, these mappings are related since they form a Galois connection.

Theorem 3. *The pair* (c, Ψ) *forms a fuzzy Galois connection between* (L^A, S) *and* $(\mathrm{Isot}(A^A), \tilde{\rho})$.

Proof. Consider $\Phi \in L^A$ and $f \in (\mathrm{Isot}(A^A), \tilde{\rho})$. First, we have that

$$\begin{aligned}
\tilde{\rho}(f, \mathrm{c}(\Phi)) &= \bigwedge_{a \in A} \rho(f(a), \mathrm{c}_\Phi(a)) \\
&= \bigwedge_{a,x \in A} \big((a^\rho \cap \Phi)(x) \to \rho(f(a), x)\big) && \text{by Theorem 1} \\
&\leq \bigwedge_{a \in A} \big((a^\rho \cap \Phi)(a) \to \rho(f(a), a)\big) && \text{by considering } x = a \\
&= \bigwedge_{a \in A} \big(\Phi(a) \to \rho(f(a), a)\big) && \text{by reflexivity} \\
&= S(\Phi, \Psi(f))
\end{aligned}$$

Conversely,

$$\begin{aligned}
S(\Phi, \Psi(f)) &= \bigwedge_{x \in A} \big(\Phi(x) \to \rho(f(x), x)\big) \\
&\leq \bigwedge_{x,a \in A} \big((\rho(a, x) \wedge \Phi(x)) \to (\rho(a, x) \wedge \rho(f(x), x)\big) && \text{by (2.45) in [20]} \\
&\leq \bigwedge_{x,a \in A} \big((a^\rho \cap \Phi)(x) \to (\rho(f(a), f(x)) \wedge \rho(f(x), x)\big) && \text{by isotonicity} \\
&\leq \bigwedge_{x,a \in A} \big((a^\rho \cap \Phi)(x) \to \rho(f(a), x)\big) && \text{by transitivity} \\
&= \tilde{\rho}(f, \mathrm{c}(\Phi)) && \text{by Theorem 1.}
\end{aligned}$$

□

As usual, a fixed point, or a *formal concept*, of the fuzzy Galois conection is a couple (Φ, f) such that $\mathrm{c}(\Phi) = f$ and $\Psi(f) = \Phi$. The fuzzy Galois connection between these mappings is interesting due to the properties its *formal concepts* have. The next result proves that the image by c or Ψ satisfy interesting additional properties, such as inflationarity in mappings and extensionality in fuzzy sets.

Lemma 1.

i. *For all* $\Phi \in L^A$*, the mapping* $\mathrm{c}(\Phi)$ *is inflationary.*
ii. *For all* $f \in (\mathrm{Isot}(A^A), \tilde{\rho})$*, we have that* $\Psi(f)$ *is extensional.*

Proof. First, given $\Phi \in L^A$ and $a \in A$, by Theorem 1, we have that

$$\rho(a, \mathrm{c}_\Phi(a)) = \rho(a, \bigsqcap(a^\rho \cap \Phi)) = (a^\rho \cap \Phi)_\rho(a) = \bigwedge_{x \in A} \big((a^\rho \cap \Phi)(x) \to \rho(a, x)\big) = 1$$

On the other hand, given $f \in (\mathrm{Isot}(A^A), \tilde{\rho})$, for all $x \in A$, we have that

$$\begin{aligned}
\Psi_f(z) \wedge \rho(z, x) \wedge \rho(x, z) &= \rho(f(z), z) \wedge \rho(z, x) \wedge \rho(x, z) \\
&\leq \rho(f(z), x) \wedge \rho(x, z) && \text{by transitivity} \\
&\leq \rho(f(z), x) \wedge \rho(f(x), f(z)) && \text{by isotoncity of } f \\
&\leq \rho(f(x), x) = \Psi_f(x) && \text{by transitivity.}
\end{aligned}$$

□

The following theorem is the main result of the paper. It characterizes the *formal concepts* of the fuzzy Galois connection. Remarkably, every fixed point of this connection is formed exactly by a fuzzy closure system and a fuzzy closure operator.

Theorem 4. *The following statements are equivalent:*

i. *The couple* (Φ, f) *is a fixed point of* (c, Ψ).
ii. *The fuzzy set* Φ *is fuzzy closure system and* $f = \mathrm{c}(\Phi)$.
iii. *The isotone mapping* f *is a fuzzy closure operator and* $\Phi = \Psi(f)$.

Proof. First, items *ii* and *iii* being equivalent is a direct consequence of Theorem 2. Furthermore, the equivalence between items *ii* and *iii* proves directly that the couple (Φ, f) is a fixed point of the fuzzy Galois connection. To prove *i* implies *ii*, assume (Φ, f) is a fixed point. Then, we have to prove that Φ is a fuzzy closure system. Recall that fuzzy closure systems are extensional fuzzy sets Φ which satisfy $\bigsqcap X \in Core(\Phi)$, for all $X \in L^A$ such that $S(X, \Phi) = 1$. Let $X \in L^A$ such that $S(X, \Phi) = 1$ and let $m = \bigsqcap X$. Then, by Definition 1, we get $X \subseteq m^\rho$, intersecting Φ we get $X = X \cap \Phi \subseteq m^\rho \cap \Phi$ and taking infima we have $\rho(\bigsqcap(m^\rho \cap \Phi), \bigsqcap X) = 1$. Using the hypothesis this is $\rho(f(m), m) = \rho(\mathrm{c}_\Phi(m), m) = 1$. Thus, $\Phi(m) = \Psi_f(m) = 1$ and $\bigsqcap X \in Core(\Phi)$. By Lemma 1, Φ is extensional. Hence, Φ is a fuzzy closure system. □

Notice that the proof uses strongly the restriction of being in a Heyting algebra. However, in the general residuated lattice case, where, as explored in [14], the natural construction of c_Φ is $\bigsqcap(x^\rho \otimes \Phi)$, the analogous result does not hold. A counterexample of last theorem in the general residuated lattice case is shown below.

Example 1. Let $\mathbb{L} = (\{0, 0.5, 1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be the three-valued Łukasiewicz residuated lattice, and (A, ρ) be the fuzzy lattice with $A = \{\bot, a, b, c, d, e, \top\}$ and the fuzzy relation $\rho\colon A \times A \rightarrow L$ is described by the following table:

ρ	$\bot$	a	b	c	d	e	$\top$
$\bot$	1	1	1	1	1	1	1
a	0.5	1	0.5	1	1	1	1
b	0.5	0.5	1	1	1	1	1
c	0.5	0.5	0.5	1	1	1	1
d	0	0.5	0	0.5	1	0.5	1
e	0	0	0.5	0.5	0.5	1	1
$\top$	0	0	0	0.5	0.5	0.5	1

For the fuzzy set $\Phi = \{a/1, b/0.5\}$, the mapping $\mathrm{c}_\Phi(x) = \bigsqcap(x^\rho \otimes \Phi) = f(x)$, which is isotone and inflationary, is $f(\bot) = f(a) = a$; $f(b) = c$, $f(c) = f(d) = d$ and $f(e) = f(\top) = \top$. However, f is not a closure operator since it is not idempotent, $\rho(f(f(b)), f(b)) = \rho(d, c) = 0.5 \neq 1$.

4. Conclusions and further work

This paper continues the line of work which initiated in [13], where fuzzy closure systems were introduced in the Heyting algebra framework. The mappings that take fuzzy closure systems to closure operators and vice versa are studied in a more general setting and are proved to form a fuzzy Galois connection. In addition, the *formal concepts* of this fuzzy Galois connection are shown to be exactly the fuzzy closure systems and closure operators in A.

As a prospect of future work, since we already know the images of the mappings introduced in [14] are not closure operators and fuzzy closure systems in general, this analysis can be extended to the general framework and study whether these mappings form a fuzzy Galois connection as well and, in the case they do, study the nature of its fixed points.

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