On Geometric Siamese Color Graphs

Nina Hronkovičová[1], Martin Mačaj[1]

[1]Department of Algebra and Geometry, Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská Dolina 824 48 Bratislava, Slovakia

Abstract

A Siamese color graph is an edge decomposition of a complete graph into strongly regular subgraphs sharing a spread. Using a computer aided exhaustive search we completely classify so called geometric Siamese color graphs on 40 vertices. We also independently confirm the classifications of Siamese color graphs on 15 vertices originally obtained by M. Klin, S. Reichard, and A. Woldar.

Keywords

Siamese color graphs, graph decompositions, generalized quadrangles, strongly regular graphs

1. Introduction

Siamese color graphs were initially introduced by Kharaghani and Torabi in [1] using algebraic methods and later studied by Klin, Reichard, and Woldar in [2, 3] from the geometric point of view. Kharaghani and Torabi provided an infinite class of Siamese color graphs arising from an infinite class of balanced generalized weighing matrices supplied by Gibbons and Mathon in [4]. Klin, Reichard, and Woldar presented a complete list of Siamese color graphs on 15 vertices and some geometric Siamese color graphs on 40 vertices [2, 3]. Most results obtained so far concern Siamese color graph with strongly regular graphs with parameters $(1+q+q^2+q^3, q^2+q, -1+q, 1+q, q)$. Such graphs are pseudo-geometric with respect to generalised quadrangles of order q and are known to exist for all prime powers.

2. Preliminaries

2.1. Partial geometries and strongly regular graphs

A *partial geometry* is an incidence structure with parameters (K, R, T) such that each block (or line) contains K points, each point lays on R lines, each pair of distinct points lay on at most one line, and for each line l and point p not on l, there exist exactly T lines through p that intersect l.

By double-counting, it is easy to see that any such structure has $K((K-1)(R-1)/T+1)$ points and $R((K-1)(R-1)/T+1)$ lines.

One of the objects related to an incidence structure is its *point graph* (or collinearity graph). It is a graph whose vertices are the points of the incidence structure and two vertices are connected by an edge if and only if they lay on the same line.

A *strongly regular graph* with parameters (v, k, λ, μ) is a regular graph with order v and valency $0 < k < v-1$ such that every pair of adjacent vertices in Γ have λ common neighbours, and every pair of non-adjacent vertices have μ common neighbours.

It is easy to show, that the point graph of any partial geometry is strongly regular with parameters

$$\begin{aligned}
v &= K\left({(K-1)(R-1)}/{T}+1\right),\\
k &= (K-1)R,\\
\lambda &= (K-2)+(R-1)(T-1),\\
\mu &= RT.
\end{aligned}$$

On the other hand, if a strongly regular graph is the point graph of a suitable partial geometry, it is said to be *geometric* and if its parameter set coincides with that of a geometric strongly regular graph, it is called *pseudo-geometric*.

A *spread* of a partial geometry is a set of pairwise disjoint lines that together contain all the points of the geometry.

Since a spread divides $K\left({(K-1)(R-1)}/{T}+1\right)$ points of the partial geometry into mutually disjoint sets of K points, there are ${(K-1)(R-1)}/{T}+1$ lines in one spread.

Any two points on the same line in the spread must be adjacent in the point graph, therefore all points belonging to the same line in the spread form a clique in the point graph. If we have a spread of a partial geometry, it spans the set of points by $\frac{(K-1)(R-1)}{T}+1$ lines, therefore, in the point graph, there is a set of $\frac{(K-1)(R-1)}{T}+1=\frac{v}{K}$ cliques of size K. In accordance, if we have any graph Γ with disjoint set of same-size cliques that span the whole Γ, we shall call it a *spread* in Γ.

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nina.hronkovicova@fmph.uniba.sk (N. Hronkovičová); martin.macaj@fmph.uniba.sk (M. Mačaj)

Let there be a spread S in a partial geometry with parameters (K, R, T), let l, m be distinct lines of S, for any point p on m, there are exactly T lines through p that intersect l through T different points. Similarly, for any point q on l, there are exactly T lines through q that intersect m through T different points. Therefore there are exactly KT different lines that intersect both l and m, and they do so in pair-wise different pairs of points on l and m.

Lemma 1. *Let (K, R, T) be a partial geometry with a spread, then for any two lines l, m in the spread there are exactly KT other lines that intersect both of them. Each point on l is contained in exactly T of these lines.*

A (finite) generalized quadrangle with parameters (s, t) is an incidence structure W satisfying the following axioms:

1. Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.
2. Each line is incident with $s + 1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.
3. If x is a point and l is a line not incident with x, then there exists exactly one line through x that intersects l.

The pair (s, t) is called the order of W. Hereinafter, we will refer to a generalized quadrangle of order (s, t) as $GQ(s, t)$.

It is straightforward to show that generalised quadrangles are a particular case of partial geometries. In particular, the generalised quadrangles of orders (s, t) are exactly the partial geometries with parameters $(s + 1, t + 1, 1)$.

Hence, the point graph of a generalised quadrangle of order (s, t) is a strongly regular graph with parameters (v, k, λ, μ), where

$$\begin{aligned} v &= (s + 1)(st + 1), \\ k &= s(t + 1), \\ \lambda &= s - 1, \\ \mu &= t + 1. \end{aligned}$$

Every line in $GQ(s, t)$ gives rise to a clique of size $1 + s$ in the point graph. On the other hand, there are no other cliques as the third condition in the definition of generalized quadrangles tells us that every three points that induce a K_3 in the point graph must belong to the same line.

Incidentally, there is a one-to-one correspondence between spreads in $GQ(s, t)$ and those spreads in its point graph which consist of $(1 + st)$ cliques of size $1 + s$.

Based on Lemma 1, we can observe that any two cliques in the spread are connected by exactly $1 + s$ edges, that is, a perfect matching. This can be expressed as follows.

Lemma 2. *If we arrange the vertices of the point graph of $GQ(s, t)$ with a spread S according to their corresponding cliques, the resulting adjacency matrix can be represented by $(1+st) \times (1+st)$ blocks with each block of size of $(1+s) \times (1+s)$. The diagonal blocks of the matrix correspond to the adjacency matrices of the cliques, i.e. $J - I$, while the off-diagonal blocks correspond to permutation matrices, that is, the incidence matrices of 1-factors.*

Given a Γ with diameter d, Γ is a *distance-regular graph* if and only if there is an array of integers $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ such that for all $1 \leq j \leq d$, and any pair of vertices u and v at a distance j in Γ, b_j gives the number of neighbours of u at distance $j + 1$ from v and c_j gives the number of neighbours of u at a distance $j - 1$ from v. The array of integers characterising a distance-regular graph is known as its *intersection array.*

It was shown by Brouwer in [5] that the removal of a spread S from a pseudo-geometric or geometric strongly regular graph Γ with a spread and parameters coinciding to parameters of the point graph of $GQ(s, t)$ gives us a distance-regular graph of diameter 3 with antipodal system S, that is, the relation of being at distance 3 in distance-regular graph $\Gamma - S$ is an equivalence relation and its blocks are exactly the cliques of S.

If the strongly regular graph Γ is geometric, we shall call the distance-regular graph $\Gamma - S$ *geometric* as well.

2.2. Siamese color graphs

A *color graph* Γ is a pair $(V, \mathcal{R})$ where V is a set of vertices and $\mathcal{R}$ is a partition of V^2, i.e., elements of $\mathcal{R}$ are pairwise disjoint and $\bigcup_{R \in \mathcal{R}} R = V^2$. We refer to the relations in $\mathcal{R}$ as the colors of Γ and to the number $|\mathcal{R}|$ of its colors as the rank of Γ.

In other words, a color graph is any edge-coloring of a complete digraph with a loop at each vertex. We define an *adjacency matrix* of a color graph to be a $v \times v$ matrix $A = (a_{i,j})$ such that $a_{i,j} = t$ if $(x_i, x_j) \in R_t$ for $R_t \in \mathcal{R}$.

Throughout this paper we will only consider color graphs such that all their colors are symmetric relations and one of them is an identity relation, i.e., ones that can be restricted to a simple graph, not a digraph.

Let Γ and Γ' be color graphs. An isomorphism $\phi : \Gamma \to \Gamma'$ is a bijection of V onto V' which induces a bijection $\psi : \mathcal{R} \to \mathcal{R}'$ of colors. A weak (or color) automorphism of Γ is an isomorphism $\phi : \Gamma \to \Gamma$. If, in addition, the induced map ψ is the identity on $\mathcal{R}$, we call ϕ a (strong) automorphism of Γ.

In 2003 in [1] Kharaghani and Torabi introduced the concept of a Siamese color graph, i.e., the decomposition of a complete graph into strongly regular graphs sharing a spread. This notion is formalised in the following definition.

Definition 1. Let $W = (V, \{Id_V, S, R_1, R_2, \ldots, R_n\})$ be a color graph for which

1. (V, S) is a partition of V into cliques of equal size.
2. For all i, graph (V, R_i) is an imprimitive distance-regular graph of diameter 3 with antipodal system S.
3. For all i, graph $(V, R_i \cup S)$ is a strongly regular graph with the same parameters.

Then W is a *Siamese color graph.* We call S the spread of Γ and n – the number of distance-regular graphs – the Siamese rank of W.

We shall denote W by $SCG(v, k, \lambda, \mu, \sigma)$ where (v, k, λ, μ) are common parameters of all $srg(V, R_i \cup S)$ and σ is the valency of the spread S. Kharaghani and Torabi used the term *Siamese* here to indicate that all these strongly regular graphs share a common spread.

Kharaghani and Torabi [1] further proved the existence of an infinite family of Siamese color graphs with special parameters.

Theorem 1. *For any prime power q, there exists a $SCG(1 + q + q^2 + q^3, q + q^2, -1 + q, 1 + q, q)$, that is a SCG on $1 + q + q^2 + q^3$ vertices consisting of $1 + q$ strongly regular graphs sharing $1 + q^2$ disjoint cliques of size $1 + q$.*

Parameters of strongly regular graphs mentioned above are interesting because these are the parameters of a point graph of generalised quadrangle $GQ(q, q)$. In the following we will refer to Siamese color graphs with these parameters as Siamese color graphs of order q and denote them $SCG(q)$. By the Theorem of Brouwer [5] mentioned after the definition of distance-regular graphs, for this class of Siamese color graphs, we do not have to check the second condition in Definition 1 if the remaining two are fulfilled.

We shall call a Siamese color graph $SCG(q)$ *geometric* if all its strongly regular graphs $(V, R_i \cup S)$ are geometric.

3. Some known results on geometric Siamese color graphs

Geometric Siamese color graphs were studied by Reichard in his thesis [6] and further by Klin, Reichard, and Woldar in a series of articles [2, 3]. In these papers, the authors constructed an infinite family of geometric Siamese color graphs which is conjectured to be isomorphic to the family of Kharabhani and Torabi and proved the following result.

Theorem 2. *Let W be a geometric Siamese color graph of order q. For each point graph $(V, R_i \cup S)$, construct the corresponding generalised quadrangle. Let B denote the union of all lines in all resulting generalized quadrangles. Then the incidence structure*

$$\mathcal{S} = (V, B)$$

is a Steiner design

$$\mathcal{S} = S\left(2, q + 1, \frac{q^4 - 1}{q - 1}\right).$$

Using Theorem 2, Klin, Reichard, and Woldar completely classified Siamese color graphs of order 2 and found hundreds of geometric Siamese color graphs of order 3. The classification of Siamese color graphs of order 2 was expressed in the following theorem.

Theorem 3. *Every Siamese color graph on 15 vertices is necessarily geometric. There are exactly two non-isomorphic Siamese color graphs on 15 vertices. Their corresponding Steiner triple systems are $STS(15)\#1$ and $STS(15)\#7$ in the notation of [7].*

4. Computer-aided search

Our primary emphasis in the search was on Siamese color graphs of order 3, although we also examined the case for order 2, which represents the smallest non-trivial scenario. To begin with, we shall summarize the established information regarding the Siamese color graphs of order 2 and 3. Corresponding strongly- and distance-regular graphs possess the following properties.

Siamese color graphs or order 2

- The spread consists of five K_3
- The strongly regulars graphs have parameters $(15, 6, 1, 3)$ – there is only one such strongly regular graph, it is a point graph of $GQ(2, 2)$ and it has only one spread up to isomorphism
- The distance-regular graphs have intersection arrays $\{4, 2, 1; 1; 1; 4\}$ – there is only one such distance-regular graph and it is the line graph of the Petersen graph

Siamese color graphs of order 3

- The spread consists of ten K_4

- The strongly regular graphs have parameters $(40, 12, 2, 4)$ – there are 29 such strongly regular graphs [8], but only two of them have a spread
    - one is geometric and it has only one spread up to isomorphism
    - one is not geometric and it has two non-isomorphic spreads
- The distance-regular graphs have intersection arrays $\{9, 6, 1; 1; 2; 9\}$. There are three of them and only one is geometric

4.1. Computer-aided search for geometric Siamese color graphs of orders 2 and 3

Our goal is to obtain the set $\mathcal{C}(q)$ of all mutually non-isomorphic geometric Siamese color graphs of order q for $q \in \{2, 3\}$. It follows that for $q \leq 3$ and for a fixed spread S all geometric distance regular graphs with antipodal system S form a single orbit of $Aut(S)$. Therefore, the following four-step strategy is sufficient to obtain $\mathcal{C}(q)$.

1. For a fixed spread S, choose a geometric distance regular graph Γ_1 with the antipodal system S (i.e., $\Gamma_1 + S$ is the point graph of $GQ(3)$ with the spread S).
2. Apply all automorphisms of S to obtain all geometric distance-regular graphs which have S as the antipodal system and find the set A of all such distance-regular graphs which have no common edges with Γ_1.
3. In A, find all triples $\Gamma_2, \Gamma_3, \Gamma_4$ of mutually edge disjoint distance-regular graphs.
4. Check the resulting system of Siamese color graphs for isomorphism.

We implemented our strategy using Python [9], GAP [10], and GAP packages GRAPE and DESIGN [11, 12]. For $q = 2$ the literal implementations of the strategy was sufficient to obtain the set $\mathcal{C}(2)$. For $q = 3$ we implemented various improvements to speed up the computiations. In what follows, we present the most important modifications in each step.

Step 1:
We fixed the spread S with cliques $\{1, \ldots, 4\}$, $\{5, \ldots, 8\}, \ldots, \{37, \ldots, 40\}$. This choice of S enabled us to represent our graphs during the computation by any of the following

- their adjacency matrices in the block form, where every permutation matrix is represented by number in $\{1, 2, \ldots, 24\}$ and $J - I$ by 0 – Steps 1, 2 and 4
- binary numbers generated by concatenation of parts of rows of the full adjacency matrix that belong to the blocks above the diagonal blocks – Steps 2 and 3

Furthermore, we have chosen the graph Γ_1 with lexicographically maximal adjacency matrix M_1. As a consequence, all blocks in the first row of the block form of M_1 are equal to the identity matrix.

Step 2:
As S is the antipodal system of Γ_1 we have $H = Aut(\Gamma_1) \leq Aut(S)$ and it suffices to apply representatives of the cosets of H in $Aut(S)$ to Γ_1. Moreover, we used the action of $Aut(S) = S_4 \wr S_{10} = (S_4^{10} \rtimes S_{10})$ on cliques of S to implement an intelligent backtrack on each coset of S_4^{10}.

Step 3:
As blocks in the first row of Γ_1 are all identity matrices, there are only nine permutation matrices disjoint with any of them and there are only four combinations of any three of these matrices and identity matrix such that they are disjoint and their sum is an all-ones matrix. We distributed the computations in such a way that in each instance we restricted the candidates for Γ_2, Γ_3, and Γ_4 to graphs with prescribed first three blocks of the first row of the block form of the adjacency matrix.

Clearly, in each Siamese color graph of order 3, Γ_4 is uniquely determined by S, Γ_1, Γ_2 and Γ_3. It turns out that for given edge-disjoint Γ_2 and Γ_3 it is faster to first check whether $K_{40} - (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ is a $srg(40, 12, 2, 4)$ and only then to verify whether it belongs to the set A.

Step 4:
Instead of testing all obtained Siamese color graphs for isomorphisms, it suffices to check whether for given Γ_1, Γ_2, Γ_3, and Γ_4 the quadruple of their adjacency matrices is maximal in the action of $Aut(S)$. In fact we implemented this modification already in Step 3, e.g., we considered only Γ_2 whose adjacency matrices were maximal in the action of $H = Aut(\Gamma_1)$.

5. Results

For $q = 2$ we confirmed the results of Klin, Reichard, and Woldar that there are only two Siamese color graphs of order 2.

For $q = 3$ we obtained the following result.

Theorem 4. *There are exactly* 399 *non-isomorphic geometric Siamese color graphs of order* 3.

We also found that in 357 of the geometric Siamese color graphs of order 3 the graph Γ_2 is the element with the largest adjacency matrix in the set A above. We will further refer to this subset as $\mathcal{C}'$.

For each of the 399 geometric Siamese color graphs of order 3, we computed its automorphism group and its orbit on vertices. Further, in accordance with Theorem 2, we computed the corresponding Steiner system, its automorphism group as well as its orbits on the points and the blocks. The results are compiled in the table below. The last column tells us, how many out of all of these non-isomorphic Siamese color graphs come from $\mathcal{C}'$.

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Table 1: Geometric Siamese structures for $q = 3$.

|A(SCG)|	V/A(SCG)	|A(SS)|	V/A(SS)	B/A(SS)	# in $\mathcal{C}'$ / #
720	40^1	12130560	40^1	130^1	1/1
120	20^2	480	40^1	$10^1, 20^1, 40^1, 60^1$	1/1
72	$4^1, 36^1$	10368	$4^1, 36^1$	$1^1, 48^1, 81^1$	1/1
72	$4^1, 36^1$	1296	$4^1, 36^1$	$1^1, 24^2, 81^1$	0/1
72	$4^1, 36^1$	288	$4^1, 36^1$	$1^1, 9^1, 24^2, 36^2$	1/1
72	$4^1, 36^1$	144	$4^1, 36^1$	$1^1, 9^1, 24^2, 36^2$	0/1
48	$8^2, 24^1$	192	$16^1, 24^1$	$4^2, 6^1, 12^1, 24^1, 32^1, 48^1$	1/1
48	$8^2, 24^1$	96	$8^2, 24^1$	$2^2, 4^1, 6^1, 12^3, 16^2, 48^1$	1/1
36	$2^2, 18^2$	5184	$2^2, 36^1$	$1^1, 24^2, 81^1$	1/1
36	$2^2, 18^2$	1296	$4^1, 36^1$	$1^1, 24^2, 81^1$	0/1
36	$2^2, 18^2$	1296	$4^1, 36^1$	$1^1, 24^2, 81^1$	0/1
36	$2^2, 18^2$	1296	$4^1, 18^2$	$1^1, 24^2, 81^1$	0/1
36	$2^2, 18^2$	648	$4^1, 36^1$	$1^1, 12^4, 81^1$	0/1
36	$2^2, 18^2$	648	$4^1, 18^2$	$1^1, 12^4, 81^1$	0/1
36	$2^2, 18^2$	288	$4^1, 18^2$	$1^1, 9^1, 24^2, 72^1$	1/1
36	$2^2, 18^2$	144	$4^1, 18^2$	$1^1, 9^1, 24^2, 72^1$	0/1
24	$8^2, 12^2$	96	$12^2, 16^1$	$4^2, 6^1, 12^1, 16^2, 24^1, 48^1$	1/1
20	10^4	160	40^1	$10^1, 40^1, 80^1$	1/1
20	10^4	40	$10^2, 20^1$	$5^2, 20^6$	1/1
18	$1^4, 9^4$	2592	$4^1, 18^2$	$1^1, 24^2, 81^1$	0/1
18	$1^4, 9^4$	648	$4^1, 9^4$	$1^1, 12^4, 81^1$	0/1
18	$1^4, 9^4$	648	$1^2, 2^1, 36^1$	$1^1, 12^4, 81^1$	0/1
18	$1^4, 9^4$	324	$1^2, 2^1, 18^2$	$1^1, 6^8, 81^1$	0/1
18	$1^4, 9^4$	144	$4^1, 36^1$	$1^1, 9^1, 24^2, 72^1$	0/1
18	$1^4, 9^4$	72	$2^2, 18^2$	$1^1, 9^1, 12^4, 72^1$	0/1
16	8^5	256	$8^1, 32^1$	$2^1, 16^2, 32^1, 64^1$	1/1
16	8^5	32	$8^3, 16^1$	$2^3, 4^5, 8^3, 16^3, 32^1$	1/1
16	8^5	32	$8^1, 16^2$	$2^1, 4^2, 8^3, 16^2, 32^2$	1/2
12	$2^2, 6^2, 12^2$	48	$4^1, 12^1, 24^1$	$1^1, 3^1, 4^1, 6^1, 8^1, 12^3, 24^3$	2/2
12	$2^2, 6^2, 12^2$	24	$2^2, 6^2, 12^2$	$1^1, 2^2, 3^1, 4^2, 6^5, 12^5, 24^1$	9/9
8	$4^6, 8^2$	64	$4^2, 8^2, 16^1$	$2^1, 4^2, 8^3, 16^4, 32^1$	1/1
8	$4^6, 8^2$	32	$8^1, 16^2$	$2^1, 4^4, 8^2, 16^4, 32^1$	1/3
8	$4^6, 8^2$	32	$4^2, 8^2, 16^1$	$2^1, 4^6, 8^1, 16^6$	4/4
8	$4^6, 8^2$	16	$8^3, 16^1$	$2^3, 4^5, 8^5, 16^4$	1/1
8	$4^6, 8^2$	16	$8^3, 16^1$	$2^3, 4^5, 8^3, 16^5$	0/1
8	$4^6, 8^2$	16	$4^6, 16^1$	$2^9, 4^2, 8^9, 16^2$	1/1
8	$4^6, 8^2$	16	$4^2, 8^4$	$2^3, 4^7, 8^6, 16^3$	2/3
8	$4^2, 8^4$	64	$8^1, 32^1$	$2^1, 8^2, 16^1, 32^3$	1/1
8	$4^2, 8^4$	32	$8^1, 32^1$	$2^1, 8^2, 16^1, 32^3$	1/3
8	$4^2, 8^4$	32	$8^1, 16^2$	$2^1, 4^4, 8^2, 16^4, 32^1$	2/2
8	$4^2, 8^4$	32	$4^2, 16^2$	$1^2, 4^4, 8^2, 16^6$	1/3
8	$4^2, 8^4$	16	$8^3, 16^1$	$2^3, 4^5, 8^3, 16^5$	1/1
8	$4^2, 8^4$	16	$8^1, 16^2$	$2^1, 4^4, 8^2, 16^6$	0/2
8	$4^2, 8^4$	16	$8^1, 16^2$	$2^1, 4^2, 8^3, 16^6$	0/1
8	$4^2, 8^4$	16	$8^1, 16^2$	$2^1, 4^2, 8^3, 16^6$	0/1
8	$4^2, 8^4$	16	$4^2, 16^2$	$1^2, 4^2, 8^3, 16^6$	0/1
8	$4^2, 8^4$	16	$4^2, 8^2, 16^1$	$1^2, 2^2, 4^5, 8^7, 16^3$	2/2
6	$2^2, 6^6$	24	$4^1, 12^3$	$1^1, 3^1, 4^3, 6^1, 12^5, 24^2$	3/3
6	$2^2, 6^6$	12	$2^2, 6^6$	$1^1, 2^6, 3^3, 6^8, 12^5$	2/2
4	$2^4, 4^8$	64	$2^2, 4^1, 16^2$	$1^2, 8^2, 16^5, 32^1$	1/1

The table continues on the next page.

Table 1 – continuing from the previous page.

|A(SCG)|	V/A(SCG)	|A(SS)|	V/A(SS)	B/A(SS)	# in $\mathcal{C}'$ / #
4	$2^4, 4^8$	32	$8^1, 16^2$	$2^1, 4^2, 8^3, 16^2, 32^2$	1/1
4	$2^4, 4^8$	32	$2^2, 4^1, 16^2$	$1^2, 8^4, 16^4, 32^1$	3/7
4	$2^4, 4^8$	16	$8^3, 16^1$	$2^3, 4^5, 8^5, 16^4$	1/1
4	$2^4, 4^8$	16	$8^1, 16^2$	$2^1, 4^4, 8^4, 16^5$	1/1
4	$2^4, 4^8$	16	$4^2, 8^4$	$2^3, 4^7, 8^6, 16^3$	2/2
4	$2^4, 4^8$	16	$4^2, 8^2, 16^1$	$2^1, 4^6, 8^5, 16^4$	2/3
4	$2^4, 4^8$	16	$4^2, 8^2, 16^1$	$1^2, 2^2, 4^5, 8^7, 16^3$	0/1
4	$2^4, 4^8$	16	$2^4, 8^4$	$1^2, 4^8, 8^8, 16^2$	8/9
4	$2^4, 4^8$	16	$2^2, 4^1, 16^2$	$1^2, 8^4, 16^6$	0/1
4	$2^4, 4^8$	8	$4^6, 8^2$	$2^{11}, 4^7, 8^{10}$	4/4
4	$2^4, 4^8$	8	$4^2, 8^4$	$2^3, 4^7, 8^{12}$	0/1
4	$2^4, 4^8$	8	$2^4, 4^4, 8^2$	$1^2, 2^8, 4^{12}, 8^8$	27/29
4	$2^4, 4^8$	8	$2^2, 4^1, 8^4$	$1^2, 4^8, 8^{12}$	9/10
4	$2^4, 4^8$	4	$2^4, 4^8$	$1^2, 2^{16}, 4^{24}$	25/25
2	2^{20}	8	8^5	$2^5, 4^6, 8^{12}$	1/1
2	2^{20}	8	$4^8, 8^1$	$2^5, 4^{14}, 8^8$	9/9
2	2^{20}	8	$4^4, 8^3$	$1^2, 2^2, 4^{11}, 8^{10}$	13/13
2	2^{20}	4	4^{10}	$1^4, 2^9, 4^{27}$	3/3
2	2^{20}	4	4^{10}	$1^2, 2^4, 4^{30}$	11/11
2	2^{20}	4	$2^8, 4^6$	$1^4, 2^{17}, 4^{23}$	93/93
2	2^{20}	2	2^{20}	$1^{10}, 2^{60}$	95/95
2	$1^8, 2^{16}$	16	$4^2, 16^2$	$1^2, 8^6, 16^5$	0/1
2	$1^8, 2^{16}$	8	$2^4, 8^4$	$1^2, 4^4, 8^{14}$	0/1