Partial Symmetries and Symmetry Levels of Graphs – A Census

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Abstract

The majority of graphs are well-known to be asymmetric, i.e., having no non-trivial automorphisms. Moreover, removing just a single vertex from a vertex transitive graph may result in a graph with a trivial automorphism group; while removing a vertex from a graph belonging to the family of minimal asymmetric graphs (introduced by Nešetřil) always leads to a graph with a non-trivial automorphism group. Moreover, every two-vertex induced subgraph of any given graph Γ possesses a non-trivial automorphism. These observations call for the use of the concept of a partial (graph) automorphism which is an isomorphism between two induced subgraphs of a given graph.

The set of all partial automorphisms together with the operation of partial composition forms an inverse monoid. Based on the study of inverse monoids of partial automorphisms, we propose to use a graph parameter we call the *symmetry level of a graph*, defined to be the ratio between the maximal rank of a nontrivial partial automorphism and the order of the graph, as a measure of the graph's asymmetricity. In our paper, we present some basic observations concerning the symmetry levels of graphs, and present some computational results concerning the symmetry levels of small asymmetric graphs.

Keywords

symmetry of graphs, asymmetric graphs, partial automorphism, experiments to determine the level of symmetry

1. Introduction

While the order of the automorphism group of a finite graph is considered to be one of the more important parameters of a given graph Γ, in view of the well-known 1963 result of Erdős and Rényi [1] asserting that almost all graphs are *asymmetric*, i.e., having no non-trivial automorphisms, this parameter turns out to be generally rather irrelevant. This fact has already been acknowledged in [1], where the authors proposed to study the *symmetrization* of a graph Γ achieved via removing r and adding s edges, and thereby producing a graph possessing at least one non-trivial automorphism. The *degree of asymmetry*, $A(\Gamma)$, is then defined to be the minimum of the sum $r + s$ taken over all possible symmetrizations of Γ. They also noted that the asymmetry of a graph of order n can not exceed $\frac{n-1}{2}$, and showed that this estimate is asymptotically best possible, which led them to the concept of the *relative asymmetry* of Γ, $a(\Gamma) = \frac{A(\Gamma)}{\frac{n-1}{2}}$. Clearly, $0 \leq a(\Gamma) \leq 1$, for all finite graphs, and $a(\Gamma) = 0$ for graphs possessing at least one non-trivial automorphism.

In 1988, Nešetřil proposed a different approach to studying the asymmetry level of finite graphs by suggesting to study the order of asymmetric graphs with all induced subgraphs symmetric. Specifically, he proposed the concept of a *minimal asymmetric graph* which is a graph with no induced asymmetric subgraphs of order at least 2. He conjectured the existence of only a finite number of such graphs, and was proven to be right by Schweitzer and Schweitzer in 2017 who have shown that the complete list of such graphs consists of exactly 18 graphs [7].

In our paper, we propose to study a different measure of asymmetricity of a graph Γ from both of the above. Based on our research in the inverse monoids of partial automorphisms of graphs, we realized that inverse monoids are better suited for investigation of asymmetric graphs, as the inverse monoid of partial automorphisms of a graph Γ determines Γ uniquely [3] (regardless of whether Γ is asymmetric or not). A *partial automorphism* of a graph $\Gamma = (V, E)$ is an isomorphism between two *induced* subgraphs of Γ (with an automorphism of Γ being a partial automorphism from Γ onto itself). The set of all partial automorphisms of Γ together with the operation of partial composition form an inverse monoid we shall denote $PAut(\Gamma)$. The order of an inverse monoid is its cardinality, and the *rank* of a partial automorphism is the size of its domain. Any Γ of order at least 2 possesses at least one non-trivial (non-identity) partial automorphism of rank 2, namely a partial automorphism mapping a pair of adjacent vertices to any other such pair (in case of the complement of K_n, one can take pairs of non-adjacent vertices, and in a graph containing exactly one edge, one can take the partial automorphism swapping its end-points). On the other hand, the largest rank of a non-

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trivial partial automorphism of a non-asymmetric Γ is the order of Γ, while the largest rank of a non-trivial partial automorphism of a minimal asymmetric Γ of order n is $n-1$. Since it is not hard to see that the largest rank of a non-trivial partial automorphism of a graph Γ is related to the order of Γ, the measure of asymmetry of a graph Γ we propose to study is defined as the ratio between the largest rank of a non-trivial partial automorphism of a graph Γ and its order $|V(\Gamma)|$. As this ratio is equal to 1 if and only if Γ admits a non-trivial automorphism, we will call this ratio the *level of symmetry* of Γ, denote it by $S(\Gamma)$, and note that $S(\Gamma) < 1$ for almost all graphs Γ [1]. We find it important to emphasize, that the level of symmetry of Γ might be greater than the order of a smallest non-asymmetric induced subgraph of Γ; which may happen if Γ contains two distinct but isomorphic induced asymmetric subgraphs of orders larger than the order of a smallest non-asymmetric induced subgraph of Γ.

After the next section, where we collect several basic results concerning the level of symmetry of graphs in general, we present some computational results concerning graphs of orders for which the complete lists of non-isomorphic graphs have already been determined.

2. Basic Results Concerning Symmetry Levels of Graphs

As the concepts of partial automorphism, the inverse monoid of partial automorphisms of a graph Γ, and the level of symmetry of Γ are relatively new, in this section, we present some basic facts.

The first result is a complete analogue of a well-known result about automorphism groups of graphs.

Lemma 1. *Let Γ be a graph, and let $\tilde{\Gamma}$ denote its complement. A partial permutation of the vertices of Γ is a partial automorphism of Γ if and only if, it is a partial automorphism of $\tilde{\Gamma}$, and thus*

$$PAut(\Gamma) = PAut(\tilde{\Gamma}), \text{ and } S(\Gamma) = S(\tilde{\Gamma}).$$

Proof 1. *The claim follows from the well-known fact that $Aut(\Gamma) = Aut(\tilde{\Gamma})$ and the observation that a subgraph of Γ induced by a subset $R \subseteq V(\Gamma)$ is the complement of the subgraph of $\tilde{\Gamma}$ induced by R.*

The next result will provide us with a lower bound on the level of symmetry of forests.

Lemma 2. *Let Γ be a forest of order n. Then, $S(\Gamma) \geq \frac{n-1}{n}$.*

Proof 2. *We proceed by considering all possible orders of connected components of Γ. As observed above, the level of symmetry of Γ containing no edges is equal to 1. If Γ contains a connected component consisting of a single edge uv or two isolated vertices u, v, it admits a non-trivial automorphism swapping u and v and leaving all other vertices fixed, hence, $S(\Gamma) = 1$ again. If Γ contains a component of order 3, it is necessarily a path u, w, v and Γ admits a non-trivial automorphism swapping u and v and leaving all other vertices fixed; $S(\Gamma) = 1$. Finally, suppose that Γ contains a component (a tree) $\mathcal{T}$ of order ≥ 4. Then $\mathcal{T}$ either contains two leaves u, v attached to the same w, or it contains a path w, u, v of length 3 in which u is of degree 2 and v is of degree 1 (a leaf). In either case, Γ admits a non-trivial* partial *automorphism of rank $n-1$ swapping u and v and fixing all the other vertices of Γ but the vertex w, which is left out of the domain of this partial automorphism. Therefore, $S(\Gamma) \geq \frac{n-1}{n}$.*

The next series of results are all based on the idea of constructing partial automorphisms by 'ignoring' neighbors of a specified pair of vertices.

Lemma 3. *Let Γ be a graph of order n, and let u and v be two vertices of Γ of degrees d_u, d_v sharing r common neighbors. Then there exists a non-trivial partial automorphism of Γ of rank at least $n - d_u - d_v + r$. In addition, if u and v are adjacent in Γ, there exists a partial automorphism of Γ of rank at least $n - d_u - d_v + r + 2$.*

Proof 3. *The domain of the desired partial automorphism is the set of vertices of Γ minus the neighbors of u and v that fixes all the vertices in its domain and swaps u and v. The rank (i.e., the cardinality of its domain) of such partial automorphism can be easily seen to be equal to the values stated in the statement of the lemma.*

Corollary 1. *Let Γ be a graph of order n, and let m be the maximum of the values $n - d_u - d_v + r$ or $n - d_u - d_v + r + 2$, if u and v are adjacent, over all pairs of distinct vertices $u, v \in V(\Gamma)$. Then $S(\Gamma) \geq \frac{n-m}{n}$. In particular, if u and v are vertices of minimum degrees d_u, d_v among all vertices of Γ, $S(\Gamma) \geq \frac{n-d_u-d_v}{n}$.*

The above corollary yields a relatively high level of symmetry for all graphs containing two vertices of small degree. Even though this may seem like a rather strong requirement, two vertices of high degree are likely to share some common neighbors, and moreover, Lemma 1 allows us to extend the result to the opposite case of graphs in which all the vertices have high degrees. The interesting cases lie therefore among the graphs in which the majority of vertices are of degree roughly $\frac{n}{2}$; with n being the order of the graph. However, instead of proceeding further and improving the above lower bounds, we choose to state a series of open questions inspired by our results on possible levels of symmetry which will be addressed in the last section of our paper where we present computational evidence toward answering the questions listed here.

Question 1. *Does there exist a graph Γ of order n and level of symmetry equal to $\frac{n-k}{n}$ for arbitrarily large $k \geq 2$?*

The following two results do not resolve this question, but suggest a possible relation between the parameters n and k.

Lemma 4. *Let $k < n$ be positive integers, and suppose that the number of asymmetric graphs of order k is smaller than $\binom{n}{k}$. Then, the level of symmetry of any graph Γ of order n is greater than or equal to $\frac{k}{n}$.*

Proof 4. *If Γ is of order n, and $\binom{n}{k}$ is greater than the number of asymmetric graphs of order k, the list of all k-vertex induced graphs of Γ necessarily contains an induced non-asymmetric subgraph of order k or contains two* isomorphic *asymmetric induced subgraphs of order k. In either case, Γ admits a non-trivial partial automorphism whose domain is a k-vertex induced subgraph of Γ, and whose rank is therefore k.*

Corollary 2. *Let n be a positive integer and let k be the smallest positive integer satisfying the inequality*

$$n(n-1)(n-2)\cdots(n-k+1) \geq 2^{\binom{k}{2}}. \qquad (1)$$

The level of symmetry of any graph Γ of order n is greater than or equal to $\frac{k}{n}$.

Proof 5. *Our proof is based on a rather rough estimate. As it is well-known, the number of non-labeled non-isomorphic graphs of order k is at most $\frac{2^{\binom{k}{2}}}{k!}$, and so the same must be true for the number of non-isomorphic* asymmetric *graphs of order k. Thus, applying Lemma 4 yields the desired result for all graphs of order n satisfying the inequality*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \geq \frac{2^{\binom{k}{2}}}{k!}$$

which can be simplified into (1).

Even though the above corollary does not resolve Question 1, it does yield the result that the minimal rank of a partial automorphism of a graph increases with the order of the graph. More precisely, solving the equation $n(n-1)(n-2)\cdots(n-k+1) \approx 2^{\binom{k}{2}}$ yields $k \approx \log_{\sqrt{2}} n$ which gives the approximate lower bound $S(\Gamma) \geq \frac{\log_{\sqrt{2}} n}{n}$, for all graphs Γ of order n. This result does not, however, take into the account the structural properties of graphs. In fact, preliminary results of the first of the authors suggest the improved bound $S(\Gamma) \geq \frac{n}{3}$. This motivates us to state the following refinement of Question 1:

Question 2. *What is the minimal level of symmetry of a graph Γ of order n as a function of n?*

Finally, we pose one more question the answer to which might prove useful in determination of $PAut(\Gamma)$.

Question 3. *When given two graphs of the same order, does higher symmetry level of one of them necessarily mean that it will also have a larger monoid of partial automorphisms?*

3. Census of Symmetry Levels of Small Graphs and Their Inverse Monoids of Partial Automorphisms

The results contained in this section all come from a Diploma Thesis [2] and are based on an application and an algorithm for finding partial symmetries of graphs developed therein. A quick summary of the obtained results includes the following:

- An exhaustive search of all graphs of order $n \leq 10$ determined the symmetry levels of all of them and showed that all of them have level of symmetry at least $\frac{n-2}{n}$.
- A recursive construction of graphs of order $n = 11$ from asymmetric graphs of order 10 yielded a complete list of graphs of order 11 whose level of symmetry is $\frac{11-3}{11}$; it also determined that there are no graphs of order 11 with the level of symmetry $\frac{11-k}{11}$, $4 \leq k \leq n$.
- Constructions of graphs with smaller level of symmetry include a record graph of symmetry level $\frac{n-4}{n}$, for $n = 14$.
- Randomized constructions of graphs with smaller level of symmetry yield graphs of symmetry level $\frac{n-5}{n}$, for $n \geq 15$.

3.1. Minimal asymmetric graphs

As stated already in the Introduction, all minimal asymmetric graphs are of the symmetry level $\frac{n-1}{n}$, with n being the order of the graph. Figure 1 shows the smallest (with respect to the order and number of edges) asymmetric graph. Under inspection we see, that the graph does not have any non-trivial symmetries, but its every induced subgraph on at least two vertices has non-trivial automorphisms.

Table 1 shows the minimal asymmetric graphs, their numbers of vertices, edges, pairwise non-isomorphic induced subgraphs (number of isomorphism classes), and number of partial symmetries. We know from Lemma 1 that the partial automorphism monoids for a graph Γ

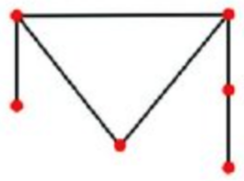

Figure 1: The smallest (with respect to the order and number of edges) asymmetric graph; graph X1 in the list of asymmetric graphs in [7].

and its complement $\tilde{\Gamma}$ are equal. Similarly, the number of induced subgraphs and the number of partial symmetries is the same for any graph and its complement. Hence, the list contains only 9 of the 18 minimal asymmetric graphs (which come in complementary pairs).

We also found the partial automorphism monoid for each of the minimal asymmetric graphs. [2] contains the complete list of these monoids.

3.2. Symmetry Levels of Small Graphs

The second author created an application to provide an interface for easy work with graphs, graph symmetries, and partial symmetries. As a result, a simple script allowed us to answer the question whether there exist graphs of order $n \leq 10$ of symmetry levels $\frac{n-1}{n}$. The script relied on the list of (non-isomorphic) unlabelled simple graphs generated by McKay and published in GAP format [4]. Erdős and Rényi already established that there are no asymmetric graphs with $2 \leq n \leq 5$ vertices [1]. Since there are no asymmetric graphs with 5 vertices, it follows that none of the 8 asymmetric graphs of order 6, of which all are minimal, have asymmetric subgraphs with 5 vertices [6][7]. Using the above mentioned script, it was then shown that of all the 1044 of order 7, exactly 152 are asymmetric [5]. None of these graphs have symmetry level $\frac{7-2}{7}$. Of the 3696 asymmetric unlabelled graphs with 8 vertices, there are 8 graphs with symmetry level $\frac{8-2}{8}$. Of all asymmetric graphs of orders 9 and 10, 2608 are of symmetry level $\frac{9-2}{9}$ and more than a million of symmetry level $\frac{10-2}{10}$.

Based on the findings from [1], we know that almost all finite graphs are asymmetric. Thus, as n grows, the number of asymmetric graphs with n vertices gets closer to the total number of all graphs of order n. It is also reasonable to expect that the number of asymmetric graphs of order n and symmetry level $\frac{n-2}{n}$ should be increasing with the growth of n. This pattern seems to be supported by the obtained data, as only 8 graphs of order 8 are of symmetry level $\frac{8-2}{8}$, 2608 graphs of order 9 have symmetry level $\frac{9-2}{9}$, and more than a million have symmetry level $\frac{10-2}{10}$ (of almost 8 million asymmetric graphs).

Checking all previously found graphs determined that no graphs of order $\leq$ 10 are of symmetry level $\frac{n-3}{n}$. We avoided using the developed program to check all unlabelled graphs with 11 vertices, since there are more than a billion such graphs. Instead, a different approach was used. Based on the list of all asymmetric graphs of order 10, a program created all possible 11-vertex graphs by adding a new vertex. In total, there are 2^{10} possible ways of adding a new vertex to a 10-vertex graph. Using this approach, we found 11-vertex graphs with symmetry level $\frac{11-3}{11}$.

Utilizing the previous findings and our extensive library, we implemented a function $find\_sym\_d$, which takes a graph as an argument and returns its symmetry level. Then we extended the approach described previously by taking any n-vertex asymmetric graph and creating an $n+1$-vertex graph by adding a new vertex. In total, there are 2^n different ways of doing this, since the new vertex is either isolated or it is added as a neighbor to any k-vertex subset of the set of vertices, $1 \leq k \leq n$. The function $has\_symmetry\_level$ is then used to verify whether the new graph has a desired symmetry level. Using this approach recursively, graphs of order 14 vertices and symmetry level $\frac{14-4}{14}$ were constructed. Unfortunately, this approach is computationally difficult and would not be appropriate for finding all 14 vertex graphs with a given level of symmetry.

Finally, in order to construct graphs of order n and symmetry level $\frac{n-5}{n}$, random graphs of orders between 15 and 30 were generated and their symmetry levels were determined. The search did yield some graphs of symmetry level $\frac{n-5}{n}$. Due to the combinatorial explosion occurring when working with graphs, their subgraphs, and symmetries, exhaustive searches become very quickly

Graph code	Vertices	Edges	# of non-isomorphic induced subgraphs	# of partial symmetries
X1	6	6	20	768
X2	6	7	22	704
X3	6	7	20	714
X4	6	7	21	680
X9	7	6	28	3373
X10	7	7	30	2793
X11	7	8	29	2553
X15	8	9	45	9728
X16	8	10	45	8560

Table 1
The number of non-isomorphic induced subgraphs and partial symmetries of minimal asymmetric graphs (graph codes taken from [7]).

infeasible.

3.3. Number of partial automorphisms

Due to the fast rise of the number of partial symmetries with respect to the rise of the orders of the considered graphs, we wanted to know if one can predict the number of partial symmetries of a graph by looking at its structure. For this reason, the numbers of partial symmetries for all unlabelled graphs of orders n, $3 \leq n \leq 9$, were calculated.

To find the number of partial automorphisms of a given graph Γ, it suffices to find all isomorphism classes of induced subgraphs of Γ, the corresponding numbers of induced subgraphs belonging to these classes, and the orders of the automorphism groups of representatives in these classes. The number of partial symmetries within a specific isomorphism class I can then be calculated using the formula $|I|^2 \times |Aut(rep_I)|$, where rep_I is a representative of I.

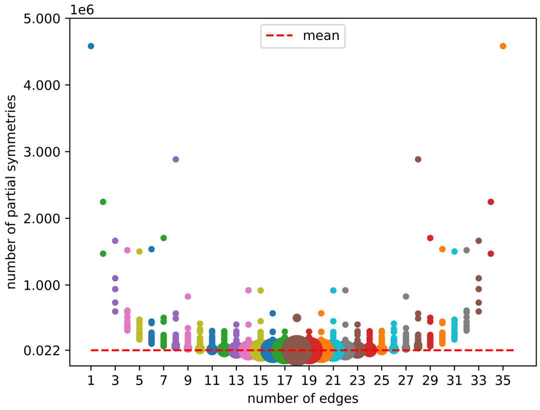

Figure 2: The number of partial symmetries for all 9-vertex graphs, dot size denotes the number of graphs.

Figure 2 shows the number of partial symmetries for graphs with 9 vertices, where we excluded K_9 and its complement. The size of the dot scales proportionally to the number of graphs having that number of partial symmetries. One can quickly notice how symmetrical this image is. It is to be expected, since we already know that a graph and its complement share the same partial automorphism monoid structure and therefore have the same number of partial symmetries.

By studying the results obtained for all graphs with 9 or fewer vertices, we made the following observations. Obviously, the number of partial symmetries of an n-vertex graph Γ is equal to the number of partial permutations of an n-element set, if and only if, Γ is isomorphic to the complete graph K_n or its complement. Removal of just one edge from K_n significantly reduces the number of partial symmetries. For $n = 9$, the number drops from $17,572,114$ partial symmetries in case of K_9 to $4,582,270$ partial symmetries for $K_9 \setminus \{e\}$.

Next, we noticed that for graphs with k edges, $1 \leq k \leq 9$, there is always one graph with a number of partial symmetries higher than all the other k-edge graphs. These special graphs consist of a k-vertex star with all other vertices being isolated.

Despite K_9 having more than 17 million partial symmetries, the mean number of partial symmetries for graphs of order 9 is only 22154, a decrease of 99.5%. We also calculated the mean values for graphs of 3 to 8 vertices, and based on the data, we predicted the mean of partial symmetries for graphs with fewer than 17 vertices. The prediction can be seen in Fig. 3.

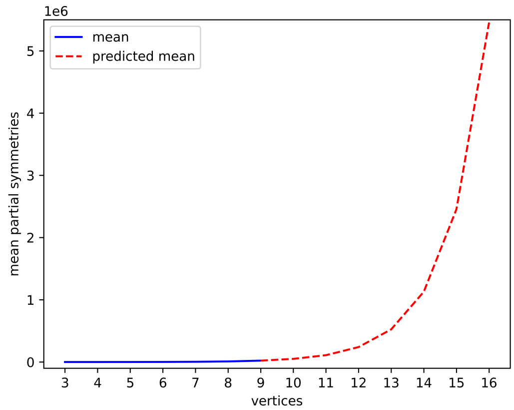

Figure 3: Mean number of partial symmetries for graphs with 3-9 vertices with predictions for graphs with 10-16 vertices.

4. Acknowledgments

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