On the Spectra and Spectral Radii of Token Graphs[⋆]

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Abstract

Let G be a graph on n vertices. The k-token graph (or symmetric k-th power) of G, denoted by $F_k(G)$ has as vertices the $\binom{n}{k}$ k-subsets of vertices from G, and two vertices are adjacent when their symmetric difference is a pair of adjacent vertices in G. In particular, $F_k(K_n)$ is the Johnson graph $J(n,k)$, which is a distance-regular graph used in coding theory. In this paper, we present some results concerning the (adjacency and Laplacian) spectrum of $F_k(G)$ in terms of the spectrum of G. For instance, when G is walk-regular, an exact value for the spectral radius ρ (or maximum eigenvalue) of $F_k(G)$ is obtained. When G is distance-regular, other eigenvalues of its 2-token graph are derived using the theory of equitable partitions. A generalization of Aldous' spectral gap conjecture (which is now a theorem) is proposed.

Keywords

Token graph, Adjacency spectrum, Local spectrum, Laplacian spectrum, Algebraic connectivity, Binomial matrix, Spectral radius, Walk-regular graph.

1. Introduction

For a (simple and connected) graph $G=(V,E)$ with adjacency matrix $\boldsymbol{A}$, its local spectrum at vertex u plays a role similar to the (standard adjacency) spectrum when the graph is 'seen' from vertex u. For instance, the local spectra of G, for every $u\in V$, were used by Fiol and Garriga [14] to prove the so-called 'spectral excess theorem', which gives a quasi spectral characterization of distance-regular graphs. In turn, this result was the crucial tool for the discovery, by van Dam and Koolen [10], of the first known family of non-vertex-transitive distance-regular graphs with unbounded diameter. Besides, Fiol, Garriga, and Yebra [16] used the local spectra to define the local predistance polynomials, which were used to characterize a general kind of local distance-regularity (intended for not necessarily regular graphs).

One of the most important parameters in spectral graph theory is the *index* or *spectral radius* of a graph, which corresponds to the largest eigenvalue of its adjacency matrix. This parameter has special relevance in the study of many integer-valued graph invariants, such as the diameter, the radius, the domination number, the matching number, the clique number, the independence number, the chromatic number, or the sequence of vertex degrees. In turn, this leads to studying the structure of graphs having an extremal spectral radius and fixed values of some of such parameters. See Brualdi, Carmona, Van den Driessche, Kirkland, and Stevanović [4, Cap. 3].

In this work, we use some information given by the local spectra to obtain new results about the spectral radius of an ample family of graphs, which are known as token graphs or symmetric k-th powers, defined as follows. For a given integer k, with $1\leq k\leq n$, the *k-token graph* $F_k(G)$ of G is the graph whose vertex set $V(F_k(G))$ consists of the $\binom{n}{k}$ k-subsets of vertices of G, and two vertices A and B of $F_k(G)$ are adjacent whenever their symmetric difference $A\bigtriangleup B$ is a pair $\{a,b\}$ such that $a\in A$, $b\in B$, and $\{a,b\}\in E(G)$. In Figure 1, we show the 2-token graph of the cycle C_9 on 9 vertices. In particular, if $k=1$, then $F_1(G)\cong G$; and if G is the complete graph K_n, then $F_k(K_n)\cong J(n,k)$, where $J(n,k)$ denotes the Johnson graph, see Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia, and Wood [12].

The name 'token graph' also comes from the observation in [12], that vertices of $F_k(G)$ correspond to configurations of k indistinguishable tokens placed at distinct vertices of G, where two configurations are adjacent whenever one configuration can be reached from the other by moving one token along an edge from its current position to an unoccupied vertex. Such graphs are also called *symmetric k-th power of a graph* in Audenaert, Godsil, Royle, and Rudolph [1]. They have applications in physics; a connection between symmetric powers of graphs and the exchange of Hamiltonian operators in quantum mechanics is given in [1]. Our interest is in relation to the graph isomorphism problem. It is well known that there are cospectral non-isomorphic graphs, where often the spectrum of the adjacency matrix of a

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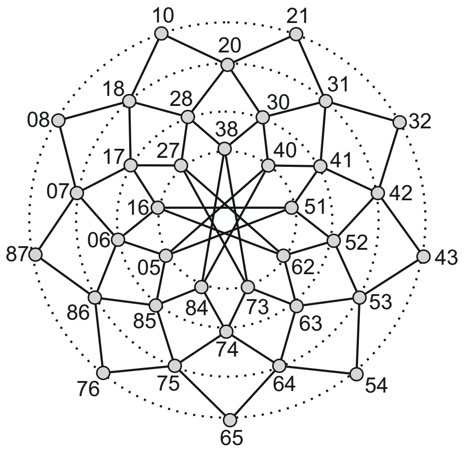

Figure 1: The 2-token graph $F_2(C_9)$ of the cycle graph with vertex set $V(C_9) = \mathbb{Z}_9$. The vertices inducing a circumference (in dashed line) of radius r_ℓ, with $\ell = 1, 2, 3, 4$ and $r_1 > r_2 > r_3 > r_4$ are ij with $\mathrm{dist}(i, j) = \ell$ in C_9.

graph is used. For instance, Rudolph [23] showed that there are cospectral non-isomorphic graphs that can be distinguished by the adjacency spectra of their 2-token graphs, and he also gave an example for the Laplacian spectrum. Audenaert, Godsil, Royle, and Rudolph [1] also proved that 2-token graphs of strongly regular graphs with the same parameters are cospectral and also derived bounds on the (adjacency and Laplacian) eigenvalues of $F_2(G)$ for general graphs. For more information, see again [1] or [12].

What can be said about the spectrum of $F_k(G)$? The three main results that we want to recall are the following.

Theorem 1.1. (Audenaert, Godsil, Royle, and Rudolph [1]). *All the strongly regular graphs with the same parameters have cospectral symmetric squares (or 2-token graphs).*

Theorem 1.2. (Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete, Zaragoza Martínez [8]). *For any graph G on n vertices, the Laplacian spectrum of its h-token graph is contained in the Laplacian spectrum of its k-token graph for every $1 \leq h \leq k \leq n/2$.*

Theorem 1.3 (Lew [20]). *Let G have Laplacian eigenvalues $\lambda_1(=0) < \lambda_2 \leq \cdots \leq \lambda_n$. Let λ be an eigenvalue of $F_k(G)$ not in $F_{k-1}(G)$. Then,*

$$k(\lambda_2 - k + 1) \leq \lambda \leq k\lambda_n.$$

In this paper, we mainly derive new results about the spectral radius of token graphs, and it is organized as follows. The next section begins with some basic concepts, definitions, and results. More precisely, we recall some known results about the local spectra and derive the basic tools for computing the spectral radius. In Section 3, we introduce the new concepts of k-algebraic connectivity and k-spectral radius. There, we study some of their properties and propose and conjecture a generalization of Aldou's spectral gap conjecture, already a theorem (see Caputo, Liggett, and Richthammer [6]). In Section 4, we give both lower and upper bounds for the spectral radius of a token graph, which are shown to be asymptotically tight. In the same section, we present some infinite families in which the exact values of the spectral radius are obtained. Finally, in the last section, we deal with the case of distance-regular and strongly regular graphs, where two results are presented in the form of Audenaert, Godsil, Royle, and Rudolph's result [1], and Lew's result [20].

2. Preliminaries

2.1. Graphs and their spectra

Let G be a (simple and connected) graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. Let G have adjacency matrix $\boldsymbol{A}$, and spectrum

$$\mathrm{sp}\, G \equiv \mathrm{sp}\, \boldsymbol{A} = \{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\},$$

where $\theta_0 > \theta_1 > \cdots > \theta_d$. Thus, by the Perron-Frobenius theorem, G has spectral radius $\rho(G) = \theta_0$.

Let $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$ be the Laplacian matrix of G, with eigenvalues

$$\lambda_1(= 0) < \lambda_2 \leq \cdots \leq \lambda_n.$$

Recall that λ_2 is the algebraic connectivity, and D is the diagonal matrix whose diagonal entries are the vertex degrees of G.

2.2. The local spectra of a graph

Let G have different eigenvalues $\theta_0 > \cdots > \theta_d$, with respective multiplicities $m_0, \ldots, m_d$. If $\boldsymbol{U}_i$ is the $n \times m_i$ matrix whose columns are the orthonormal eigenvectors of θ_i, the matrix $\boldsymbol{E}_i = \boldsymbol{U}_i \boldsymbol{U}_i^\top$, for $i = 0, 1, \ldots, d$, is the (*principal*) *idempotent* of $\boldsymbol{A}$ and represents the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathrm{Ker}(\boldsymbol{A} - \theta_i \boldsymbol{I})$. The $(u\text{-})$*local multiplicities* of the eigenvalue θ_i are defined as

$$m_u(\theta_i) = \|\boldsymbol{E}_i \boldsymbol{e}_u\|^2 = \langle \boldsymbol{E}_i \boldsymbol{e}_u, \boldsymbol{e}_u \rangle = (\boldsymbol{E}_i)_{uu}$$

for $u \in V$ and $i = 0, 1, \ldots, d$, where the vector $\boldsymbol{e}_u$ is an n-dimensional vector with a 1 in the u-th entry and zeros elsewhere. In particular, $m_u(\theta_0) = v_u^2 > 0$, where v is the corresponding normalized Perron eigenvector. Although the local multiplicities are, of course, not necessarily integers, they have nice properties when the graph is studied from a vertex, so justifying their name. Thus, they satisfy $\sum_{i=0}^{d} m_u(\theta_i) = 1$ and $\sum_{u\in V} m_u(\theta_i) = m_i$, for $i = 0, 1, \ldots, d$. The number $a_{uu}^{(\ell)}$ of closed walks of length ℓ rooted at vertex u can be computed as

$$a_{uu}^{(\ell)} = \sum_{i=0}^{d} m_u(\theta_i)\theta_i^{\ell} \tag{1}$$

(see Fiol and Garriga [14]). By picking up the eigenvalues θ_i with non-null local multiplicities, $\mu_0 (= \theta_0) > \mu_1 > \cdots > \mu_{d_u}$, we define the *(u-)local spectrum* of G as

$$\mathrm{sp}_u\, G := \{\mu_0^{m_u(\mu_0)}, \mu_1^{m_u(\mu_1)}, \ldots, \mu_{d_u}^{m_u(\mu_{d_u})}\},$$

with *(u-)local mesh*, or set of distinct eigenvalues, $\mathrm{ev}_u\, G := \{\mu_0 > \mu_1 > \cdots > \mu_{d_u}\}$. The eccentricity of a vertex u satisfies an upper bound similar to that satisfied by the diameter of G in terms of its distinct eigenvalues. More precisely,

$$\mathrm{ecc}(u) \le d_u = |\,\mathrm{ev}_u\, G| - 1. \tag{2}$$

In coding theory, d_u corresponds to the so-called 'dual degree' of the trivial code $\{u\}$. For more information, see Fiol, Garriga, and Yebra [16].

We use the following lemma to prove the results of Section 4. Notice that this is just a reformulation of the power method in terms of the number of walks given by (1).

Lemma 2.1. *Let G be a finite graph with different eigenvalues $\theta_0 > \cdots > \theta_d$. Let $w_u^{(\ell)}$ be the number of ℓ-walks starting from (any fixed) vertex u, and let $w_{uu}^{(\ell)}$ be the number of closed ℓ-walks rooted at u. Then,*

$$\rho(G) = \lim_{\ell\to\infty} \sqrt[\ell]{w_u^{(\ell)}} = \lim_{\ell\to\infty} \sup \sqrt[\ell]{w_{uu}^{(\ell)}},$$

where '$\sup$' denotes the supremum.

2.3. Regular partitions and their spectra

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$, adjacency matrix $\boldsymbol{A}$, and Laplacian matrix $\boldsymbol{L}$. A partition π of its vertex set V into r cells $C_1, C_2, \ldots, C_r$ is called *regular* (or *equitable*) whenever, for any $i, j = 1, \ldots, r$, the *intersection numbers* $b_{ij}(u) = |G(u) \cap C_j|$, where $u \in C_i$, do not depend on the vertex u but only on the cells C_i and C_j. In this case, such numbers are simply written as b_{ij}, and the $r \times r$ matrices $\boldsymbol{Q}_A = \boldsymbol{A}(G/\pi)$ and $\boldsymbol{Q}_L = \boldsymbol{L}\,(G/\pi)$ with entries $(\boldsymbol{Q}_A)_{ij} = b_{ij}$ and

$$(\boldsymbol{Q}_L)_{ij} = \begin{cases} -b_{ij} & \text{if } i \neq j,\\ \displaystyle\sum_{j=1}^{r} b_{ij} - b_{ii} & \text{if } i = j, \end{cases}$$

are, respectively, referred to as the *quotient matrix* and *quotient Laplacian matrix* of G with respect to π. In turn, these matrices correspond to the *quotient (weighted) directed graph* G/π, whose vertices representing the r cells, and there is an arc with weight b_{ij} from vertex C_i to vertex C_j if and only if $b_{ij} \neq 0$. Of course, if $b_{ii} > 0$, for some $i = 1, \ldots, r$, the quotient graph (or digraph) G/π has loops. Given a partition π of V with r cells, let $\boldsymbol{S}$ be the *characteristic matrix* of π, that is, the $n \times r$ times matrix whose columns are the characteristic vectors of the cells of π. Then, π is a regular partition if and only if $\boldsymbol{AS} = \boldsymbol{SQ}_A$ or $\boldsymbol{LS} = \boldsymbol{SQ_L}$. Moreover, $\boldsymbol{Q}_A = (\boldsymbol{S}^\top \boldsymbol{S})^{-1}\boldsymbol{S}^\top \boldsymbol{AS}$, and $\boldsymbol{Q}_L = (\boldsymbol{S}^\top \boldsymbol{S})^{-1}\boldsymbol{S}^\top \boldsymbol{L}\,\boldsymbol{S}$.
Thus, there is a strong analogy with similar results satisfied by the Laplacian matrices of the h-token graph and k-token graph of G for $h \le k$.

2.4. Walk-regular graphs

Let $a_u^{(\ell)}$ denote the number of closed walks of length ℓ rooted at vertex u, that is, $a_u^{(\ell)} = a_{uu}^{(\ell)}$. If these numbers only depend on ℓ, for each $\ell \ge 0$, then G is called *walk-regular*, a concept introduced by Godsil and McKay in [18].

Notice that, as $a_u^{(2)} = \delta_u$, the degree of vertex u, a walk-regular graph is necessarily regular.

Moreover, a graph G is called *spectrally regular* when all vertices have the same local spectrum: $\mathrm{sp}_u\, G = \mathrm{sp}_v\, G$ for any $u, v \in V$. The following result (in Delorme and Tillich [11], Fiol and Garriga [15], and also Godsil and McKay [18]) provide some characterizations of such graphs.

Lemma 2.2 ([11],[15],[18]). *Let $G = (V, E)$ be a graph. The following statements are equivalent.*

- *(i) G is walk-regular.*
- *(ii) G is spectrally regular.*
- *(iii) The spectra of the vertex-deleted subgraphs are all equal: $\mathrm{sp}\,(G \setminus u) = \mathrm{sp}\,(G \setminus v)$ for any $u, v \in V$.*

3. The k-algebraic connectivity and k-spectral radius

In this section, we always consider the Laplacian spectrum. Let G be a graph on n vertices, and $F_k(G)$ its k-token graph for $k \in \{0, 1, \ldots, n\}$. Note that $F_k(G) \cong F_{n-k}(G)$ where, by convenience, $F_0(G) \cong F_n(G) = K_1$ (a singleton). Moreover, $F_1(G) \cong G$. From Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete, and Zaragoza Martínez [8], it is known that the Laplacian spectra of the token graphs of G satisfy

$$\{0\} = \mathrm{sp}\, F_0(G) \subset \mathrm{sp}\, F_1(G) \subset \mathrm{sp}\, F_2(G) \subset \cdots \subset \mathrm{sp}\, F_{\lfloor n/2 \rfloor}(G). \tag{3}$$

Let denote $\alpha(G)$ and $\rho(G)$ the algebraic connectivity (see Fiedler [17]) and the spectral radius of a graph G, respectively. Then, from (3), we have

$$\alpha(G) \geq \alpha(F_2(G)) \geq \cdots \geq \alpha(F_{\lfloor n/2 \rfloor}(G)), \quad (4)$$
$$\rho(G) \leq \rho(F_2(G)) \leq \cdots \leq \rho(F_{\lfloor n/2 \rfloor}(G)). \quad (5)$$

The concepts of algebraic connectivity and spectral radius, together with (3)–(5), suggest the following definitions.

Definition 3.1. *Given a graph G on n vertices and an integer k such that $1 \leq k \leq \lfloor n/2 \rfloor$, the* k-algebraic connectivity $\alpha_k = \alpha_k(G)$ *and the* k-spectral radius $\rho_k = \rho_k(G)$ *of G are, respectively, the minimum and maximum eigenvalues of the multiset* $\operatorname{sp} F_k(G) \setminus \operatorname{sp} F_{k-1}(G)$.

Notice that, since $\binom{n}{k} > \binom{n}{k-1}$ for $1 \leq k \leq \lfloor n/2 \rfloor$, the parameters α_k and ρ_k always exist.

For instance, with $G = P_6$, the path on 6 vertices, we have (approximately)

$$\alpha_1(P_6) = 0.267, \alpha_2(P_6) = 0.572, \alpha_3(P_6) = 0.930,$$

and

$$\rho_1(P_6) = 3.732, \rho_2(P_6) = 6.504, \rho_3(P_6) = 7.487,$$

whereas for $G = C_7$, the cycle on 7 vertices, we get

$$\alpha_1(C_7) = 0.753, \alpha_2(C_7) = 1.163, \alpha_3(C_7) = 1.269.$$

Moreover, from these definitions, the following facts hold.

(i) $\rho_k(G) \geq \alpha_k(G) \geq 0$.

(ii) $\alpha_1(G) = \alpha(G)$ (the standard algebraic connectivity of G) and $\rho_1(G) = \rho(G)$ (the standard spectral radius of G).

(iii) Since $F_k(K_n) \cong J(n,k)$ (the Johnson graph), we have

$$\alpha_k(K_n) = \rho_k(K_n) = k(n+1-k),$$

$k = 1, \ldots, \lfloor n/2 \rfloor$.
In particular, $\alpha_1(K_n) = \rho_1(K_n) = n$, $\alpha_2(K_n) = \rho_2(K_n) = 2(n-1)$, and so on.

The equalities in (iii) come from the fact that the Johnson graph $J(n,k)$ has different Laplacian eigenvalues $\lambda_j = j(n+1-k)$, with multiplicities $m_j = \binom{n}{j} - \binom{n}{j-1}$ for $j = 0, 1, \ldots, k$.

From what is known about token graphs, we can suggest some conjectures and state some results, as follows.

Conjecture 3.2. *For any graph G,*

$$\alpha_1(G) \leq \alpha_2(G) \leq \cdots \leq \alpha_{\lfloor n/2 \rfloor}(G).$$

Because of (4), if Conjecture 3.2 holds, also does the conjecture proposed in [8], that is, $\alpha(F_k(G)) = \alpha(G)$ for any $k \leq n/2$. In fact, the last equality follows from the proof of Aldous' spectral gap conjecture given by Caputo, Ligget, and Richthammer in [6]. By this result, what we can state is that $\min\{\alpha_2, \ldots, \alpha_{\lfloor n/2 \rfloor}\} \geq \alpha_1$.

Conjecture 3.3. *For any graph G,*

$$\rho_1(G) \leq \rho_2(G) \leq \cdots \leq \rho_{\lfloor n/2 \rfloor}(G).$$

Notice that, from (5), if this conjecture holds, then $\rho_k(G) = \rho(F_k(G))$ for any $k \leq n/2$.

Lemma 3.4. *For any graph G and its complementary graph $\overline{G}$, the k-algebraic connectivity and k-spectral radius of $\overline{G}$ satisfy*

$$\alpha_k(G) + \rho_k(\overline{G}) = k(n-k+1).$$

Moreover, $\alpha(G) = n - \rho(\overline{G})$, as it is well known.

Corollary 3.5. *For any graph G on n vertices,*

$$\alpha_k(G) \leq k(n-k+1), \qquad k = 1, \ldots, \lfloor n/2 \rfloor.$$
$$\rho_k(\overline{G}) \leq k(n-k+1), \qquad k = 1, \ldots, \lfloor n/2 \rfloor.$$

From Lemma 3.4 and Proposition 5.2, we get the following result, which will be proved in Section 5.

Corollary 3.6. *Let G be a bipartite distance-regular graph. Let $L(F_2/\pi)$ be the quotient matrix in* (15) *with spectral radius $\rho_L(F_2/\pi)$. Then,*

$$\alpha_2(\overline{G}) = \binom{n}{2} - \rho_L(F_2/\pi).$$

4. The spectral radius of token graphs

In contrast with the previous section, in this section, we always consider the spectral radius of the adjacency matrix of a (connected) graph. Consider a graph G with spectral radius $\rho(G)$ and vertex-connectivity κ (the minimum number of vertices whose suppression either disconnects the graph or results in a singleton). By taking the spectral radii of its U-deleted subgraphs, with $U \subset V$ and $|U| = k < \kappa$, we define the following two parameters:

$$\rho_M^k(G) = \max\{\rho(G \setminus U) : U \subset V, |U| = k\},$$
$$\rho_m^k(G) = \min\{\rho(G \setminus U) : U \subset V, |U| = k\}.$$

Notice that, if G is walk-regular, then $\rho_M^1(G) = \rho_m^1(G) = \rho(G \setminus u)$ for every vertex u. If G is distance-regular with degree δ, it is known that it has vertex-connectivity $\kappa(G) = \delta$ (see Brouwer and Koolen [3]).

Moreover, Dalfó, Van Dam, and Fiol [7] showed that $\mathrm{sp}(G \setminus U)$ only depends on the distances in G between the vertices of U. Thus, for every $k \leq \delta - 1$, the computation of $\rho_M^k(G)$ and $\rho_m^k(G)$ can be drastically reduced by considering only the subsets U with different 'distance-pattern' between vertices. For instance, if G has diameter D,

$$\rho_M^2(G) = \max_{1\leq \ell \leq D} \{\rho(G \setminus \{u,v\}) : \mathrm{dist}_G(u,v) = \ell\},$$
$$\rho_m^2(G) = \min_{1\leq \ell \leq D} \{\rho(G \setminus \{u,v\}) : \mathrm{dist}_G(u,v) = \ell\}.$$

In general, by using interlacing (see Haemers [19] or Fiol [13]), we have the following result.

Lemma 4.1. *Let G be a graph with n vertices, vertex-connectivity κ, and eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then, for every $k = 1, \ldots, \kappa - 1$,*

$$\lambda_{k+1} \leq \rho_M^k(G) \leq \lambda_1, \tag{6}$$
$$\lambda_n \leq \rho_m^k(G) \leq \lambda_{n-k}. \tag{7}$$

From the above results, Lemma 2.1, and the bounds for the spectral radius of graph perturbations obtained in Dalfó, Garriga, and Fiol [9] and Nikiforov [21], we get the following main result.

Theorem 4.2. *Let G be a graph with spectral radius $\rho(G)$ and vertex-connectivity $\kappa > 1$. Given an integer k, with $1 \leq k < \kappa$, let $\rho_M^k(G)$ and $\rho_m^k(G)$ be the maximum and minimum of the spectral radii of the U-deleted subgraphs of G, where $|U| = k$.*

(i) *The spectral radius of the k-token graph $F_k(G)$ satisfies*

$$k\rho_m^{k-1}(G) \leq \rho(F_k(G)) \leq k\rho_M^{k-1}(G). \tag{8}$$

(ii) *If G is a graph of order n and diameter D, the spectral radius of the k-token graph $F_k(G)$ satisfies*

$$\rho(F_k(G)) \leq k\left(\rho(G) - \frac{1}{n\rho(G)^{2D}}\right). \tag{9}$$

(iii) *If G is walk-regular and $k = 2$ (that is, $F_2(G)$ is the 2-token graph of G), then*

$$\rho(F_2(G)) = 2\rho_m^1(G) = 2\rho_M^1(G). \tag{10}$$

As commented in [22], for large values of $\rho(G)$ and D, the right hand of (9) yields the correct order of magnitude of $\rho(H)$, with H a proper subgraph of G. Thus, we can say that, asymptotically, the spectral radius of $F_k(G)$ is k times the spectral radius of G. Moreover, in the case when G is regular, (ii) can be rewritten as

$$\rho(F_2(G)) \leq k\left(\rho(G) - \frac{1}{n(D+1)}\right), \tag{11}$$

(see again[22, Th. 4]).

Since the different eigenvalues of the path P_n on n vertices are $\theta_i = 2\cos\left(\frac{i\pi}{n+1}\right)$ for $i = 1, \ldots, n$, and the spectral radius of the complete bipartite graph is $\rho(K_{m,n}) = \sqrt{mn}$, we get the following results.

Corollary 4.3. *Let P_n and C_n be, respectively, the path and cycle graph on n vertices. Let P_∞ and C_∞ be, respectively, the infinite path and cycle graphs.*

(i) $\rho(F_2(P_n)) \leq 4\cos(\pi/n)$ *and* $\rho(F_2(P_\infty)) = 4$,
(ii) $\rho(F_2(C_n)) = 4\cos(\pi/n)$ *and* $\rho(F_2(C_\infty)) = 4$,
(iii) $\rho(F_2(K_{n,n})) = 2\sqrt{n(n-1)}$.

Notice a pair of examples:

- $F_2(C_3) = C_3 = K_3$ has spectrum $\{-1^{[2]}, 2\}$, whereas P_2 has $\{-1, 1\}$.
- $F_2(C_4) = K_{2,4}$ has spectrum $\{-2\sqrt{2}, 0^{[6]}, 2\sqrt{2}\}$, whereas P_3 has $\{-\sqrt{2}, 0, \sqrt{2}\}$.

5. The case of distance-regular graphs

In this section, we adopt the terminology of Brouwer, Cohen, and Neumaier [2] for distance-regular graphs. Furthermore, since we examine both the adjacency and Laplacian spectra, we will denote their respective spectral radii as ρ_A and ρ_L. In the following result, consider that G is a distance-regular graph with degree $\delta = b_0$, diameter d, intersection array

$$\iota(G) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}. \tag{12}$$

or intersection matrix

$$\boldsymbol{B} = \begin{pmatrix} 0 & c_1 & & & \\ b_0 & a_1 & c_2 & & \\ & b_1 & a_2 & \ddots & \\ & & \ddots & \ddots & c_d \\ & & & b_{d-1} & a_d \end{pmatrix}, \tag{13}$$

where $a_i = d - b_i - c_i$, for $i = 1, \ldots, d$.

Lemma 5.1. *Let $F_2(G)$ be the 2-token graph of a distance-regular graph G with degree $\delta = b_0$, diameter d, and intersection array $\iota(G)$ as in (12). Then, $F_2 = F_2(G)$ has a regular partition π with quotient matrix and quotient Laplacian matrix*

$$\boldsymbol{A}(F_2/\pi) = 2\begin{pmatrix} a_1 & b_1 & & & \\ c_2 & a_2 & b_2 & & \\ & c_3 & a_3 & \ddots & \\ & & \ddots & \ddots & b_{d-1} \\ & & & c_d & a_d \end{pmatrix}, \tag{14}$$

$$\boldsymbol{L}\,(F_2/\pi) = 2\begin{pmatrix} b_1 & -b_1 & & & \\ -c_2 & c_2+b_2 & -b_2 & & \\ & -c_3 & c_3+b_3 & \ddots & \\ & & \ddots & \ddots & -b_{d-1} \\ & & & -c_d & c_d \end{pmatrix}, \tag{15}$$

where $c_i + a_i + b_i = \delta$, for $i = 0, 1, \ldots, d$.

The following result shows that the quotient matrices of a regular partition can be used to find the spectral radius or Laplacian spectral radius of the 2-token graph of G.

Proposition 5.2. *Let G be a distance-regular graph with adjacency and Laplacian matrices $\boldsymbol{A}$ and $\boldsymbol{L}$. Let $F_2(G)$ be its 2-token graph with adjacency and Laplacian matrices $\boldsymbol{A}(F_2)$ and $\boldsymbol{L}(F_2)$ with respective spectral radii $\rho_A(F_2)$ and $\rho_L(F_2)$. Let $\boldsymbol{A}(F_2/\pi)$ and $\boldsymbol{L}(F_2/\pi)$ be the quotient matrices in (14) and (15) with respective spectral radii $\rho_A(F_2/\pi)$ and $\rho_L(F_2/\pi)$. Then, the following holds:*

- (a) $\rho_A(F_2) = \rho_A(F_2/\pi)$.
- (b) $\rho_L(F_2) \geq \rho_L(F_2/\pi)$, *with equality if G is bipartite.*

In fact, the eigenvalues of $\boldsymbol{A}(F_2/\pi)$ are 2 times the zeros of the so-called *conjugate polynomial* $\overline{p}_d$ of the distance polynomial p_d of G (with $p_d(\boldsymbol{A}) = \boldsymbol{A}_d$, where $\boldsymbol{A}_d$ is the d-distance matrix of G). The conjugate polynomials were introduced by Fiol and Garriga in [14], and are defined on the mesh $\{\theta_0, \theta_1, \ldots, \theta_d\}$ in terms of the distance polynomials $p_0, p_1, \ldots, p_d$ as

$$\overline{p}_i(\theta_i) = \frac{p_{d-i}(\theta_i)}{p_d(\theta_i)} \quad \text{for} \quad i = 0, 1, \ldots, d.$$

Thus, in particular $\overline{p}_d(\theta_i) = p_d(\theta_i)^{-1}$ and, up to a constant, $\overline{p}_d$(x) equals the characteristic polynomial of $\frac{1}{2}\boldsymbol{A}(F_2/\pi)$ (for more details, see Cámara, Fàbrega, Fiol, and Garriga [5]).

Let us show an example.

Example 5.3. *The Heawood graph H (which is the point/line incidence graph of the Fano plane) is a bipartite distance-regular graph with $n = 14$ vertices, diameter three, and intersection array $\{b_0, b_1, b_2; c_1, c_2, c_3\} = \{3, 2, 2; 1, 1, 3\}$. The Laplacian spectral radius of $\overline{H}$ is $\rho_L(H) = 6$, and the algebraic connectivity of $\overline{H}$ is $\alpha_1(\overline{H}) = n - \rho_L(H) = 8$. By Proposition 5.2, the 2-token graph $F_2 = F_2(H)$ has a regular partition π with quotient matrix*

$$\boldsymbol{A}(F_2/\pi) = 2\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix},$$

and quotient Laplacian matrix

$$\boldsymbol{L}\,(F_2/\pi) = 2\begin{pmatrix} 2 & -1 & 0 \\ -2 & 3 & -3 \\ 0 & -2 & 3 \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{A}(F_2/\pi)$ are $0, \pm 4\sqrt{2}$, whereas those of $\boldsymbol{L}(F_2/\pi)$ are $0, 8 \pm 2\sqrt{3}$. Thus, we conclude that $\rho_A(F_2(H)) = 4\sqrt{2}$ and $\rho_2(H) = \rho_L(F_2(H)) = 8 + 2\sqrt{3}$. From this and Lemma 3.4, we have that $\alpha_2(\overline{H}) = 2(n-1) - \rho_2(H) = 18 - 2\sqrt{3}$, which is greater than $\alpha_1(\overline{H}) = 8$. Consequently, the algebraic connectivity of $F_2(\overline{H})$ equals that of $\overline{H}$, as expected. Besides, the 3-distance polynomial of H is $p_3(x) = \frac{1}{3}(x^3 - 5)$, so that the conjugate polynomial $\overline{p}_3$ must satisfy $\overline{p}_3(\pm 3) = p_3(\pm 3)^{-1} = \pm\frac{1}{4}$, and $\overline{p}_3(\pm\sqrt{2}) = p_3(\pm\sqrt{2})^{-1} = \mp\frac{\sqrt{2}}{2}$. This results into $\overline{p}_3(x) = \frac{1}{12}x^3 - \frac{2}{3}x$, with roots $0, \pm 2\sqrt{2}$, which correspond to the eigenvalues of $\frac{1}{2}\boldsymbol{A}(F_2/\pi)$, as predicted.

Other consequences of Lemma 5.1 and Proposition 5.2 are the following. First, in the form of Audenaert, Godsil, Royle, and Rudolph's (see [1]), we get the following result.

Corollary 5.4. *All distance-regular graphs with diameter d and the same parameters have symmetric squares with the same, at least, d different (adjacency and Laplacian) eigenvalues. In particular, such graphs have the spectral radii ρ_A and ρ_L.*

Thus, the natural question is if, as in the case of strongly regular graphs, all distance-regular graphs with the same parameters are also cospectral.

Moreover, in the vein of Lew's result (see [20]) and, by using interlacing, we get the following consequence.

Corollary 5.5. *Let G be a distance-regular graph with (adjacency) eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Then, 2-token graph $F_2(G)$ has some eigenvalues $\mu_0 > \mu_1 > \cdots > \mu_{d-1}$ satisfying*

$$2\theta_{i+1} \leq \mu_i \leq 2\theta_i, \qquad i = 0, \ldots, d-1.$$

5.1. Strongly regular graphs

Let G be a (connected) strongly regular graph on n vertices, which is a distance-regular graph with diameter 2. Let G have parameters (n, d, a, c), that is, G is d-regular (with $b_0 = d$), $a_1 = a$, and $c_2 = c$. Then, its intersection matrix is

$$\boldsymbol{B} = \begin{pmatrix} 0 & 1 & 0 \\ d & a & c \\ 0 & d-a-1 & d-c \end{pmatrix}.$$

Then, the 2-token graph $F_2 = F_2(G)$ has a regular partition π with quotient matrix

$$\boldsymbol{A}(F_2/\pi) = \begin{pmatrix} 2a & 2c \\ 2d-2a-2 & 2d-2c \end{pmatrix},$$

and quotient Laplacian matrix

$$\boldsymbol{L}\,(F_2/\pi) = \begin{pmatrix} 2d-2a-2 & -2c \\ -2d+2a+2 & 2c \end{pmatrix}.$$

Such a regular partition was given by Audenaert, Godsil, Royle, and Rudolph in [1], and noted that the adjacency eigenvalues of $\boldsymbol{A}(F_2/\pi)$ are

$$\theta_{1,2} = d + (a-c) \pm \sqrt{[d-(a-c)]^2 - 4c}.$$

They also commented that the positive eigenvalue θ_1 has a positive eigenvector (Perron vector) and, so, it corresponds to the (adjacency) spectral radius $\rho_A(F_2(G))$. In contrast, the quotient Laplacian matrix $\boldsymbol{L}(F_2/\pi)$ has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2(d-1) - 2(a-c)$. Now, the eigenvector of λ_2 is orthogonal to $\mathbf{1}$. Then, we can only conclude that the Laplacian spectral radius of $F_2(G)$ satisfies

$$\rho_L(F_2(G)) \geq 2(d-1) - 2(a-c). \tag{16}$$

For instance, the cycle on five vertices C_5 is strongly regular with parameters $(5, 2, 0, 1)$. Its Laplacian spectral radius is (approximately) $\rho_L(F_2(G)) = 6.2361$, whereas the lower bound in (16) gives 4.

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