Group Pursuit Problem for Fraction Differential Systems with Pure Delay

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Abstract

In this paper, we consider for the first time the group problem of pursuit for linear fractional differential systems with several pursuers and one evader and with pure delay. We have developer an outline of the Method of Resolving Functions and the First Direct Method of Pontryagin for such conflict-controlled processes using the latest representation of the Cauchy formula. We compare the game end times guaranteed by these two methods. Sufficient conditions for ending the group pursuit game and the practical finding method of resolving functions are formulated.

Keywords

Conflict-controlled process, differential games, fractional differential equations, pursuit games, group games

1. Introduction

Mathematical models with fractional differential equations with a delay have become widely used today, in particular, in the theory of decision-making in conflict situations. In the theory of conflict-controlled processes, the problems of group pursuit are relevant, especially in military practice. The task is to find pursuer strategies (not necessarily optimal) that guarantee the solvability of the group pursuit problem under any admissible controls of the evader. The theory of conflict-controlled processes (differential games) enumerates several fundamental methods of researching various natural processes that function in conditions of conflict and uncertainty. Attempts to construct optimal behavior for opposing sides in dynamic game problems inevitably lead to the use of dynamic programming ideology, which is closely related to the Hamilton-Jacobi–Bellman-Isaacs equation, the main equation in the theory of differential games [1]. The desire to find optimal solutions for counter-acting parties in the game problems of dynamics encounters great difficulties of mathematical nature. Therefore, several effective mathematical methods have been created for deciding dynamic games, which provide a guaranteed result and give sufficient conditions for goal achievement without worrying about optimality, which is quite justified from the practical point of view. These are the First Direct Method of Pontryagin [2], the Rule of Extremal Aiming of N.N. Krasovskii [3], and the Method of Resolving Functions [4]. In this case, the range of group pursuit problems, that can be solved using this approach is much wider. This approach follows the rule of parallel pursuit, well known to design engineers.

In this paper, we use the Method of Resolving Functions and the First Direct Method of Pontryagin. Inverse Minkowski functionals [5, 6] play a key role in our approach. Resolving

XXII International Scientific and Practical Conference "Information Technologies and Security (ITS-2022)", November 16, 2022, Kyiv, Ukraine
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functions are typically determined from specific quadratic equations, making this technique a convenient and universal means of solving certain problems.

Delay differential equations are widely used in various fields, including control theory, computer engineering, signal analysis, and damage theory. Generally, these mathematical models have a peculiarity, which is that the rate of change of these processes is determined by their history. In 2003, Khusainov, Shuklin, and Pospíšil [7-9] represented the solutions of linear differential equations, proposing to consider the so-called exponential matrix function with a delay. Based on the presented analogs of the Cauchy formula for conflict-controlled processes described by differential-difference systems, sufficient conditions for solving pursuit games have been found in research papers [10, 11]. The work [12] presents a modification of the Method of Resolving Functions for differential-difference pursuit games for a group of pursuers and one evader.

Representations of solutions of fractional differential equations with a linear delay are increasingly considered. Li and Wang considered a representation of the solution of the linear homogeneous fractional differential systems with the pure delay with order $\alpha \in (0, 1)$ using delayed matrix Mittag-Leffler functions [13]. The basic studies by Chikry–Eidelman [14-19] contain sufficient conditions for solving the pursuit problem for systems with fractional derivatives of arbitrary order $\alpha \in (0, 1)$.

In 2018, Liang et al [20] presented the solution of the linear homogeneous fractional differential system with the pure delay with a term that includes Caputo double derivatives with order $\alpha \in (0, 1]$. In 2021, Liu et al [21] obtained exact solutions for a nonhomogeneous fractional oscillation equation with pure delay by constructing two functions derived from the extension of the Mittag-Leffler function.

In 2021, Elshenhab and Wang [22] introduced a new delay matrix of the Mittag–Leffler type with two Liu delay matrices. This study is based on the latest achievements in the presentation of analogs of the Cauchy formula.

2. Preliminaries

In this section, we present some necessary definitions and lemmas for linear fractional systems with pure delay used in our subsequent discussions.

Consider the system

$$\left({}^{C}D_{0^+}^{\alpha} z\right)(t) = -Az(t-h) + f(t), \text{ for } t \geq 0, h > 0, \tag{1}$$

where ${}^{C}D_{0^+}^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha \in (1, 2)$ with the lower limit zero, $z = (z_1,\ z_2, \dots, z_n)^T: [-h, \infty) \to \mathbb{R}^n$ is a solution satisfying (1) for every $t \geq 0$, A is an $n \times n$ constant real nonzero matrix, $f: [0, \infty) \to \mathbb{R}^n$ be a given function, with the initial conditions

$$z(t) \equiv \varphi(t), \qquad z(t) \equiv \dot{\varphi}(t), \ \ -h \leq t \leq 0,$$

where $\varphi = (\varphi_1,\ \varphi_2, \dots, \varphi_n)^T: [-h, \infty) \to \mathbb{R}^n$ is an arbitrary differentiable function.

Definition 1. [22]. The two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}, \quad \alpha, \gamma > 0, z \in \mathbb{C}.$$

Especially, if $\gamma = 1$, then

$$E_{\alpha,1}(z) = E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0.$$

Definition 2. [22]. The delayed Mittag-Leffler type matrix functions $H_{h,\alpha}(At^{\alpha})$, $M_{h,\alpha}(At^{\alpha})$, and $S_{h,\alpha}(At^{\alpha})$ are defined as follows:

$$H_{h,\alpha}(At^{\alpha}) = \begin{cases} \Theta, \quad -\infty < t < -h, \\ E, \quad -h \le t < 0, \\ E - A\dfrac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad 0 \le t < h, \\ \cdots \\ E - A\dfrac{t^{\alpha}}{\Gamma(1+\alpha)} + A^2\dfrac{(t-h)^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots + \\ +(-1)^k A^k \dfrac{(t-(k-1)h)^{k\alpha}}{\Gamma(1+k\alpha)}, \quad (k-1)h \le t < kh; \end{cases}$$

$$M_{h,\alpha}(At^{\alpha}) = \begin{cases} \Theta, \quad -\infty < t < -h, \\ E(t+h), \quad -h \le t < 0, \\ E(t+h) - A\dfrac{t^{\alpha+1}}{\Gamma(2+\alpha)}, \quad 0 \le t < h, \\ \cdots \\ E(t+h) - A\dfrac{t^{\alpha+1}}{\Gamma(2+\alpha)} + A^2\dfrac{(t-h)^{2\alpha+1}}{\Gamma(2+2\alpha)} + \cdots + \\ +(-1)^k A^k \dfrac{(t-(k-1)h)^{k\alpha+1}}{\Gamma(2+k\alpha)}, \quad (k-1)h \le t < kh; \end{cases}$$

$$S_{h,\alpha}(At^{\alpha}) = \begin{cases} \Theta, \quad -\infty < t < -h, \\ E\dfrac{(t+h)^{\alpha-1}}{\Gamma(\alpha)}, \quad -h \le t < 0, \\ E\dfrac{(t+h)^{\alpha-1}}{\Gamma(\alpha)} - A\dfrac{t^{2\alpha-1}}{\Gamma(2\alpha)}, \quad 0 \le t < h, \\ \cdots \\ E\dfrac{(t+h)^{\alpha-1}}{\Gamma(\alpha)} - A\dfrac{t^{2\alpha-1}}{\Gamma(2\alpha)} + A^2\dfrac{(t-h)^{3\alpha-1}}{\Gamma(3\alpha)} + \cdots + \\ +(-1)^k A^k \dfrac{(t-(k-1)h)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))}, \quad (k-1)h \le t < kh; \end{cases}$$

where Γ is a gamma function, the notations Θ and E are the $n \times n$ zero and identity matrices, respectively, $k = 0, 1, \ldots$.

Definition 3. [22]. Let z: $[-h, \infty) \to \mathbb{R}^n$ be a function of order $\alpha \in (1, 2)$. Then the Caputo fractional derivative for z is given by

$$\left({}^{C}D_{0^+}^{\alpha} z \right)(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\ddot{y}(s)}{(t-s)^{\alpha-1}} ds, \ t > 0.$$

Lemma 1. [22]. The solution $z(t)$ of (1) is given by

$$z(t) = \begin{cases} \varphi(t), \quad -h \le t < 0, \\ H_{h,\alpha}(A(t-h)^{\alpha})\varphi(0) + M_{h,\alpha}(A(t-h)^{\alpha})\dot{\varphi}(0) - \\ -A \displaystyle\int_{-h}^{0} S_{h,\alpha}(A(t-2h-s)^{\alpha})\varphi(s) ds + \\ + \displaystyle\int_0^t S_{h,\alpha}(A(t-h-s)^{\alpha}) f(u(s), v(s)) ds, \ \ t \ge 0. \end{cases}$$

3. Problem statement

Let the motion of an object $z = (z_1, z_2, \dots, z_\nu)^T, z_i \in \mathbb{R}^{n_i}$, evolve in a finite-dimensional space $\mathbb{R}^n, n = n_1 + n_2 + \dots + n_\nu$ subject to the equations

$$\left({}^C D_{0^+}^\alpha z_i\right)(t) = -A_i z_i(t - h_i) + f_i\big(u_i(t), v(t)\big), i = 1, 2, \dots, \nu, \text{ for } t \ge 0, h_i > 0, \quad (2)$$

where A_i are square matrices of order n_i; $f_i(u_i, v)$, $f_i: U_i \times V \to \mathbb{R}^{n_i}$, are jointly continuous vector functions of its variables; U_i and V are nonempty compacts.

Let $z(t)$ be a solution of the system (2) under the initial condition

$$z_i(t) \equiv \varphi_i(t),\ \dot{z}_i(t) \equiv \dot{\varphi}_i(t),\ -h_i \le t \le 0,\ i = 1, 2, \dots, \nu. \quad (3)$$

The terminal set M^* consists of sets $M_1^*, \dots, M_\nu^*, M_i^* \subset \mathbb{R}^{n_i}$, each having the form

$$M_i^* = M_i^o + M_i, \quad (4)$$

where M_i^o is a linear subspace in $\mathbb{R}^{n_i}$ and M_i is a convex compact belonging to the orthogonal complement L_i to subspace M_i^o in the space $\mathbb{R}^{n_i}$.

The valid controls u_i and v are the measurable Lebesgue functions, $u_i \in U_i$ and $v \in V$. Denote

$$\Omega_{U_i} = \{u_i(s): u_i(s) \in U_i, s \in [0, +\infty)\},$$
$$\Omega_V = \{v(s): v(s) \in V, s \in [0, +\infty)\}.$$

The function $v(\cdot) \in \Omega_V$ is chosen by the evader based on knowledge of the initial condition [4].

The function

$$v_t = \{v(s): v(\cdot) \in \Omega_V, s \in [0, t]\},$$

is the prehistory of the control of evader [4].

We define quasistrategies [23] of the pursuers as the mappings $U_i\big(t, \varphi_i(\cdot), v_t(\cdot)\big)$. To each moment $t \ge 0$, v_t and condition (3) it assigns [4] a Lebesgue measurable function $u_i(t) = U_i\big(t, \varphi_i(\cdot), v_t(\cdot)\big), t \ge 0$, for all $i = 1, 2, \dots, \nu$.

The group pursuit game (2), (3) terminates when $z_i \in M_i^*$ for some *i*.

4. Scheme of the Method of the Resolving Functions

Denote by π_i the operator of orthogonal projection from $\mathbb{R}^n$ onto L_i. Consider the set-valued maps

$$W_i(t, v) = \pi_i S_{h_i,\alpha}(A_i (t)^\alpha) f_i(U_i v),$$
$$W_i(t) = \bigcap_{v \in V} W_i(t, v).$$

Condition 1. $W_i(t) \ne \emptyset$ for all $i = 1, 2, \dots, \nu, t \ge 0$.

Since $dom W_i(t) = [0, +\infty)$ the set-valued map $W_i(t)$ is upper semicontinuous and each of them has a Borelian selection $g_i(t)$ [24]. Let us denote $g(\cdot) = (g_1(\cdot), \dots, g_\nu(\cdot))$ and $G_\nu = \{g(\cdot): g_i(t) \in W_i(t), t \ge 0, i = 1, \dots, \nu\}$. For fixed $g(\cdot) \in G_\nu$ we put

$$\xi_i\big(t, \varphi_i(\cdot), g_i(\cdot)\big) = \pi_i\big[H_{h_i,\alpha}(A_i (t - h_i)^\alpha)\varphi_i(0) + M_{h_i,\alpha}(A_i (t - h_i)^\alpha)\dot{\varphi}_i(0)\big] -$$
$$-\pi_i A_i \int_{-h_i}^0 S_{h_i,\alpha}(A_i (t - 2h_i - s)^\alpha)\varphi_i(s) ds + \int_0^t g_i(s)\, ds. \quad (5)$$

Consider the resolving function

$$\alpha_i(t, s, v) = \alpha_i\big(t, s, \varphi_i(\cdot), v, g_i(\cdot)\big) =$$
$$sup\{\rho \ge 0: [W_i(t - h_i - s, v) - g_i(t - h_i - s] \cap \rho\big[M_i - \xi_i\big(t, \varphi_i(\cdot), g_i(\cdot)\big)\big] \ne \emptyset\}. \quad (6)$$

We also observe that function $\alpha_i(t, s, \varphi_i(\cdot), v, g_i(\cdot)) = +\infty$ for all $s \in [0, t], v \in V$, if and only if $\xi_i(t, \varphi_i(\cdot), g_i(\cdot)) \in M_i$. If for some $t \geq 0$ $\xi_i(t, \varphi_i(\cdot), g_i(\cdot)) \notin M_i$, then the function (6) assumes finite values.

Denote

$$T(\varphi(\cdot), g(\cdot)) = \inf\left\{t \geq 0 : \inf_{v\in V} \max_{i} \int_0^t \inf_{v\in V} \alpha_i(t, s, v) ds \geq 1\right\}, \quad i = 1, \ldots, \nu,$$

$$g(\cdot) = (g_1(\cdot), \ldots, g_\nu(\cdot)), \varphi(\cdot) = (\varphi_1(\cdot), \ldots, \varphi_\nu(\cdot)).$$

If the inequality in the curly brackets does not hold for all $t \geq 0$, we put $T(\varphi(\cdot), g(\cdot)) = +\infty$.

Theorem. Let the conflict-controlled process (2) and (4) with the initial condition (3) satisfy Condition 1. Let $= \mathrm{coM_i}$, $i = 1, \ldots, \nu$, and let for given $\varphi(\cdot) = (\varphi_1(\cdot), \ldots, \varphi_\nu(\cdot))$ and some Borelian selection $g^0(\cdot) \in G_\nu$ hold the inequality: $T = T(\varphi(\cdot), g^0(\cdot)) < +\infty$.

Then for at least one *i* a trajectory of the corresponding process (2) can be brought from $\varphi(\cdot)$ to the set M_i^* at the moment $T(\varphi(\cdot), g^0(\cdot))$ under any controls *v*.

Proof. Let $v(\cdot) \in \Omega_V$. First consider the case $\xi_i\left(T, \varphi_i(\cdot), g_i^0(\cdot)\right) \notin M_i$ for all $i = 1, \ldots, \nu$. We introduce the controlling function

$$h(t) = h(T, t, \varphi(\cdot), v(\cdot), g^0(\cdot)) =$$
$$= 1 - \min_{i=1,\ldots,\nu} \int_0^t \alpha_i\left(T, s, \varphi(\cdot), v(s), g_i^0(\cdot)\right) ds, t \geq 0.$$

From the definition of *T*, there is a switching time $t_* = t_*(v(\cdot)), 0 < t_* \leq T$, such that the controlling function becomes zero:

$$h(t_*) = 0. \tag{7}$$

Consider the set-valued maps

$$U_i^1(s, v) = \{u_i \in U_i\colon \pi_i S_{h_i,\alpha}(A_i(T - h_i - s)^\alpha) f_i(u_i, v) - g_i^0(T - h_i - s) \in \alpha_i(T, s, v)[M_i - \xi_i\left(T, \varphi_i(\cdot), g_i^0(\cdot)\right)]\}. \tag{8}$$

The selection

$$u_i^1(s, v) = \mathrm{lexmin} U_i^1(s, v), i = 1, \ldots, \nu,$$

appear as a jointly Borelian function in their variables [25]. The control of each of the pursuers on the interval $[0, t_*]$ equals to

$$u_i(s) = u_i^1(s, v(s)), i = 1, \ldots, \nu, \tag{9}$$

and it is a Borelian function [25].

From (7) it follows that there exists a number i_*, such that

$$1 - \int_0^{t_*} \alpha_{i_*}\left(T, s, \varphi_{i_*}(\cdot), v(s), g_{i_*}^0(\cdot)\right) ds = 0. \tag{10}$$

Then the maps

$$U_i^2(s, v) = \{u_i \in U_i\colon \pi_i S_{h_i,\alpha}(A_i(T - h_i - s)^\alpha) f_i(u_i, v) - g_i^0(T - h_i - s) = 0\}, \tag{11}$$

$s \in [t_*, T], v \in V$, are Borelian functions in *s* and *v*, and so are the selections

$$u_i^2(s, v) = \mathrm{lexmin} U_i^2(s, v), i = 1, \ldots, \nu.$$

We set $\alpha_{i_*}\left(T, s, \varphi_{i_*}(\cdot), v(s), g_{i_*}^0(\cdot)\right) \equiv 0$ on the interval $(t_*, T]$, and the control of the i_*-th pursuer equal to

$$u_{i_*}(s) = u_{i_*}^2(s, v(s)). \tag{12}$$

At the same time, we can assign arbitrary control to all other pursuers.

If $\xi_i\left(T, \varphi_i(\cdot), g_i^0(\cdot)\right) \in M_i$ for some $i = 1, \ldots, \nu$, then we set

$$u_i(s) = u_i^2(s, v(s)), s \in [0, T].$$

At the same time, we can assign arbitrary control to all other pursuers.

Let us examine the case when $\xi_i\left(T,\varphi_i(\cdot),g_i^0(\cdot)\right)\notin M_i$ for all $i=1,\dots,\nu$. We shall follow the i_* pursuer. Since $\alpha_{i_*}\left(T,s,\varphi_{i_*}(\cdot),v(s),g_{i_*}^0(\cdot)\right)\equiv 0, s\in(t_*,T], v\in V$, equation (10) implies that

$$\int_0^T \alpha_{i_*}\left(T,s,\varphi_{i_*}(\cdot),v(s),g_{i_*}^0(\cdot)\right)ds=1. \tag{13}$$

From the Cauchy formula (Lemma 1) for the i_* pursuer we have

$$\pi_{i_*}z_{i_*}(T)=\pi_{i_*}\left[H_{h_{i_*},\alpha}\left(A_{i_*}(T-h_{i_*})^\alpha\right)\varphi_{i_*}(0)+M_{h_{i_*},\alpha}\left(A_{i_*}(T-h_{i_*})^\alpha\right)\dot\varphi_{i_*}(0)\right]-$$
$$-\pi_{i_*}A_{i_*}\int_{-h_{i_*}}^{0}S_{h_{i_*},\alpha}\left(A_{i_*}(T-2h_{i_*}-s)^\alpha\right)\varphi_{i_*}(s)ds+$$
$$+\pi_{i_*}\int_0^T S_{h_{i_*},\alpha}\left(A_{i_*}(T-h_{i_*}-s)^\alpha\right)f_{i_*}\left(u_{i_*}(s),v(s)\right)ds. \tag{14}$$

If we add and subtract from the right-hand side of (14) the value $\int_0^T g_{i_*}^0(T-s)ds$, we obtain the equality:

$$\pi_{i_*}z_{i_*}(T)=\Big[\pi_{i_*}H_{h_{i_*},\alpha}\left(A_{i_*}(T-h_{i_*})^\alpha\right)\varphi_{i_*}(0)+\pi_{i_*}M_{h_{i_*},\alpha}\left(A_{i_*}(T-h_{i_*})^\alpha\right)\dot\varphi_{i_*}(0)-$$
$$-\pi_{i_*}A_{i_*}\int_{-h_{i_*}}^{0}S_{h_{i_*},\alpha}\left(A_{i_*}(T-2h_{i_*}-s)^\alpha\right)\varphi_{i_*}(s)\,ds+\int_0^T g_{i_*}^0(T-h_{i_*}-s)ds\Big]+$$
$$+\int_0^T\left[\pi_{i_*}S_{h_{i_*},\alpha}\left(A_{i_*}(P-h_{i_*}-s)^\alpha\right)f_{i_*}\left(u_{i_*}(s),v(s)\right)-g_{i_*}^0(T-h_{i_*}-s)\right]ds.$$

Taking the pursuer's control choice laws (8), (9), (11), (12) into account we deduce the inclusion

$$\pi_{i_*}z_{i_*}(T)\in\xi_{i_*}\left(T,\varphi_{i_*}(\cdot),g_{i_*}^0(\cdot)\right)+\int_0^T\alpha_{i_*}(T,s,v)\left[M_{i_*}-\xi_{i_*}\left(T,\varphi_{i_*}(\cdot),g_{i_*}^0(\cdot)\right)\right]ds=$$
$$=\xi_{i_*}\left(T,\varphi_{i_*}(\cdot),g_{i_*}^0(\cdot)\right)+\int_0^T\alpha_{i_*}(T,s,v)M_{i_*}ds-$$
$$-\int_0^T\alpha_{i_*}(T,s,v)\xi_{i_*}\left(T,\varphi_{i_*}(\cdot),g_{i_*}^0(\cdot)\right)ds.$$

Then

$$\pi_{i_*}z_{i_*}(T)\in\xi_{i_*}\left(T,\varphi_{i_*}(\cdot),g_{i_*}^0(\cdot)\right)\left(1-\int_0^{t_*}\alpha_{i_*}(T,s,v)ds\right)+\int_0^{t_*}\alpha_{i_*}(T,s,v)M_{i_*}ds.$$

Since (13) and the set M_{i_*} is convex then $\pi_{i_*}z_{i_*}(T)\in M_{i_*}$.

If for some i $\xi_i\left(T,\varphi_i(\cdot),g_i^0(\cdot)\right)\in M_i$, then, because of the control choice law, in this case, formula (5), and Lemma 1, we infer the inclusion $\pi_i z_i(T)\in M_i$. The proof is therefore complete.

Remark. For the linear process (2)

$$W_i(t)=\pi_i S_{h_i,\alpha}\left(A_i(t)^\alpha\right)U_i \overset{*}{\_}\, \pi_i S_{h_i,\alpha}\left(A_i(t)^\alpha\right)V,$$

where $\overset{*}{\_}$ is a geometric subtraction of the sets (Minkowski's difference) [26-29].

Corollary. Let the conflict-controlled process (2), (4) be linear ($f(u_i, v) = u_i - v$), Condition 1 is fulfilled, and let there exist continuous positive functions $r_i(t)$, and nonnegative numbers l_i such that

$$\pi_i S_{h_i,\alpha}(A_i(t)^\alpha)U_i = r_i(t)C_i,$$
$$M_i = l_i C_i,$$

where C_i is a unit ball centered at zero in the subspace L_i.

Then when $\xi_i(T, \varphi_i(\cdot), g_i(\cdot)) \notin M_i$, the resolving functions $\alpha_i(t, s, \varphi_i(\cdot), v, g_i(\cdot))$ turn out to be the largest roots of the quadratic equations for ρ_i, $\rho_i \geq 0$,

$$\|\pi_i S_{h_i,\alpha}(A_i(t - h_i - s)^\alpha)v + g_i(t - h_i - s) - \rho_i \xi_i(t, \varphi_i(\cdot), g_i(\cdot))\| =$$
$$= r_i(t - h_i - s) + \rho_i l_i. \tag{15}$$

Proof. Under the assumption of Corollary, we conclude from expression (6) that the resolving functions $\alpha_i(t, s, \varphi_i(\cdot), v, g_i(\cdot))$ are the maximal numbers ρ_i such that

$$\{r_i(t - h_i - s)C_i - \pi_i S_{h_i,\alpha}(A_i(t - h_i - s)^\alpha)v - g_i(t - h_i - s)\} \cap$$
$$\cap\, \rho_i\{l_i C_i - \xi_i(t, \varphi_i(\cdot), g_i(\cdot))\} \neq \emptyset.$$

This expression is equivalent to include

$$\pi_i S_{h_i,\alpha}(A_i(t - h_i - s)^\alpha)v + g_i(t - h_i - s) - \rho_i \xi_i(t, \varphi_i(\cdot), g_i(\cdot)) \in$$
$$\in [r_i(t - h_i - s) + \rho_i l_i]C_i. \tag{16}$$

Since the left part of inclusion (16) is linear in ρ_i, the vector

$$\pi_i S_{h_i,\alpha}(A_i(t - h_i - s)^\alpha)v + g_i(t - h_i - s) - \rho_i \xi_i(t, \varphi_i(\cdot), g_i(\cdot)) \in$$

lies on the boundary of the sphere $[r_i(t - h_i - s) + \rho_i l_i]C_i$ for the maximal value of ρ_i for each $i = 1, \ldots, \nu$. In other words, the length of this vector is equal to the radius of this ball for each $i = 1, \ldots, \nu$, which is demonstrated by (15). The proof is complete.

5. Scheme of the First Direct Method of Pontryagin

Denote the Pontryagin function

$$P_i(\varphi_i(\cdot)) = min\{t \geq 0 : \pi_i[H_{h_i,\alpha}(A_i(t - h_i)^\alpha)\varphi_i(0) + M_{h_i,\alpha}(A_i(t - h_i)^\alpha)\dot{\varphi}_i(0)]$$
$$- \pi_i A_i \int_{-\tau}^{0} S_{h_i,\alpha}(A_i(t - 2h_i - s)^\alpha)\varphi_i(s)\, ds \in$$
$$M_i - \int_0^t W_i(t - h_i - s)ds\}, i = 1, \ldots, \nu. \tag{17}$$

Let us prove that the quantity (17) is the guaranteed moment when the *i*-th pursuer catches the evader, i.e., it is the end of the pursuit game by the First Direct Method of Pontryagin [23].

Theorem 2. Let the process (2), (4) with the initial condition (3) satisfy Condition 1, and for the given initial state $\varphi(\cdot) = (\varphi_1(\cdot), \ldots, \varphi_\nu(\cdot))$ the inequality holds: P=$P_i(\varphi_i(\cdot)) < +\infty$, where$P_i(\varphi_i(\cdot))$ is determined by the equality (17).

Then for at least one *i* a trajectory of the process (2)-(4) can be brought from $\varphi(\cdot)$ to the terminal set M_i^* at the moment *P*.

Proof. We shall follow the *i*-th pursuer. The following inclusion holds

$$\pi_i[H_{h_i,\alpha}(A_i(P - h_i)^\alpha)\varphi_i(0) + M_{h_i,\alpha}(A_i(P - h_i)^\alpha)\dot{\varphi}_i(0)]$$
$$-\pi_i A_i \int_{-\tau}^{0} S_{h_i,\alpha}(A_i(P - 2h_i - s)^\alpha)\varphi_i(s)\, ds \in$$

$$M_i - \int_0^P W_i(P - h_i - s)ds.$$

Hence, there exists a point $m_i \in M_i$ and a selector $g^0(\cdot) \in G_v$ such that

$$\pi_i\left[H_{h_i,\alpha}(A_i(P - h_i)^\alpha)\varphi_i(0) + M_{h_i,\alpha}(A_i(P - h_i)^\alpha)\dot{\varphi}_i(0)\right]$$

$$-\pi_i A_i \int_{-\tau}^{0} S_{h_i,\alpha}(A_i(P - 2h_i - s)^\alpha)\varphi_i(s)\, ds =$$

$$= m_i - \int_0^{\mathrm{P}} g^0(\mathrm{P} - h_i - s)ds.$$

Consider the set-valued maps

$$U_i(s, v) = \left\{u_i \in U_i: \pi_i S_{h_i,\alpha}(A_i(P - h_i - s)^\alpha)f_i(u_i, v) - g_i^0(P - h_i - s) = 0\right\}, \quad (18)$$

where $s \in [0; \mathrm{P}], v \in V$.

They are Borel measurable functions in s, v. The selections

$$u_i(s, v) = lexmin U_i(s, v)$$

are Borel measurable functions in s, v.

We set the control of i-th pursuer equal to

$$u_i(s) = u_i\big(s, v(s)\big), \qquad s \in [0; P],$$

where $v(s),\ v(s) \in V,$ is a measurable function. Under (18) and (17), we obtain

$$\pi_i z_i(P) = \Big[\pi_i H_{h_i,\alpha}(A_i(P - h_i)^\alpha)\varphi_i(0) + \pi_i M_{h_i,\alpha}(A_i(P - h_i)^\alpha)\dot{\varphi}_i(0) -$$

$$\left.-\pi_i A_i \int_{-h_i}^{0} S_{h_i,\alpha}(A_i(P - 2h_i - s)^\alpha)\varphi_i(s)\, ds\right] +$$

$$+\int_0^P \pi_i S_{h_i,\alpha}(A_i(P - h_i - s)^\alpha)f_i\big(u_i(s), v(s)\big)\, ds = m_i \in M_i.$$

Finally, we have the inclusion $z_i(P) \in M_i^*$. The proof is complete.

Theorem 3. Let the conflict-controlled process (2), and (4) with the initial condition (3) satisfy Condition 1.

Then the inclusion

$$\pi_i\left[H_{h_i,\alpha}(A_i(t - h_i)^\alpha)\varphi_i(0) + M_{h_i,\alpha}(A_i(t - h_i)^\alpha)\dot{\varphi}_i(0)\right]$$

$$-\pi_i A_i \int_{-\tau}^{0} S_{h_i,\alpha}(A_i(t - 2h_i - s)^\alpha)\varphi_i(s)\, ds \in$$

$$M_i - \int_0^t W_i(t - h_i - s)ds \geq 0,$$

holds if and only if there exists a selector $\mathrm{g}^0(\cdot) \in \mathrm{G}_\mathrm{v}$ such that $\xi_i\left(t, \varphi_i(\cdot), g_i^0(\cdot)\right) \in M_i$.

<u>*Proof.*</u> Let

$$\pi_i\left[H_{h_i,\alpha}(A_i(t - h_i)^\alpha)\varphi_i(0) + M_{h_i,\alpha}(A_i(t - h_i)^\alpha)\dot{\varphi}_i(0)\right]$$

$$-\pi_i A_i \int_{-\tau}^{0} S_{h_i,\alpha}(A_i(t - 2h_i - s)^\alpha)\varphi_i(s)\, ds \in$$

$$M_i - \int_0^t W_i(t - h_i - s)ds.$$

Then there exists a point $m_i \in M_i$ and a selection $g^0(\cdot) \in G_\nu$ such that

$$\pi_i\left[H_{h_i,\alpha}(A_i(t-h_i)^\alpha)\varphi_i(0) + M_{h_i,\alpha}(A_i(t-h_i)^\alpha)\dot{\varphi}_i(0)\right]$$

$$-\pi_i A_i \int_{-\tau}^{0} S_{h_i,\alpha}(A_i(t-2h_i-s)^\alpha)\varphi_i(s)\,ds =$$

$$= m_i - \int_0^{P} g^0(t-h_i-s)ds.$$

It follows that $\xi_i\left(t,\varphi_i(\cdot),g_i^0(\cdot)\right) = m_i \in M_i$.

If we assume that for some $g^0(\cdot) \in G_\nu$ $\xi_i\left(t,\varphi_i(\cdot),g_i^0(\cdot)\right) \in M_i$, then reasoning in the reverse order, we will get the desired result.

Corollary. Let the conflict-controlled process (2)-(4) satisfy Condition 1.

Then for at least one *i* and initial state (3), there exists a selection $g^0(\cdot) \in G_\nu$ such that

$$T\big(\varphi(\cdot),g^0(\cdot)\big) \le P_i\big(\varphi_i(\cdot)\big).$$

Remark. In the case when the resolving function $\alpha\big(t,s,z^0(\cdot),v,g(\cdot)\big) = +\infty$ the Method of Resolving Functions coincides with the First Direct Method of Pontryagin [4].

6. Conclusions

This paper elaborates the results obtained in previous research [30] and focuses on group pursuit games, which are described by fractional differential systems with pure delay. We construct outlines of the Method of Resolving Functions and the First Direct Method of Pontryagin using an analog of the Cauchy formula for these systems, and formulate sufficient conditions for the ending of the game. The game end times guaranteed by these two methods are comparable. The method of practical implementation of the resolving functions is presented.

In the future, it is planned to develop outlines of Methods of Resolving Functions and the First Direct Method of Pontryagin for processes described by linear fractional systems with multiple delays given by commutative and noncommutative matrices.

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