On Graphs that are not Star-k-PCGs (short paper)

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Abstract
A graph G is a star-k-PCG if there exists a non-negative edge weighted star tree S and k mutually exclusive intervals $I_1, I_2, \ldots, I_k$ of non-negative reals such that each vertex of G corresponds to a leaf of S and there is an edge between two vertices in G if the distance between their corresponding leaves in S lies in $I_1 \cup I_2 \cup \ldots \cup I_k$. These graphs are related to different well-studied classes of graphs such as PCGs and multithreshold graphs. In this paper, we investigate the smallest value of n such that there exists an n vertex graph that is not a star-k-PCG, for small values of k.

Keywords
Pairwise compatibility graph, Multithreshold graph, Graph theory

1. Introduction

A graph G is a k-PCG (known also as multi-interval PCG) if there exists a non-negative edge weighted tree T and k mutually exclusive intervals $I_1, I_2, \ldots, I_k$ of non-negative reals such that each vertex of G corresponds to a leaf of T and there is an edge between two vertices in G if the distance between their corresponding leaves in T lies in $I_1 \cup I_2 \cup \ldots \cup I_k$ (see *e.g.* [1]). Such tree T is called the k-*witness* tree of G. The concept of 1-PCGs, also known as PCGs, originated from the problem of reconstructing phylogenetic trees [2]. The process of sampling leaves in a phylogenetic tree while considering distance constraints is closely connected to sampling cliques in a PCG [2]. Additionally, PCGs have proven valuable in describing and analyzing infrequent evolutionary scenarios, including those involving horizontal gene transfer [3]. These relationships highlight the significance of PCGs in understanding evolutionary processes.

In this paper we focus on k-PCGs for which the witness tree is a star. These graphs are known as star-k-PCGs [4]. Figure 1 depicts an example of a graph that is a star-1-PCG. The class of star-k-PCGs is equivalent to the class of $2k$-threshold graphs, which has gained considerable interest within the research community since its introduction in [5], as evidenced by the following studies [5, 6, 7, 8].

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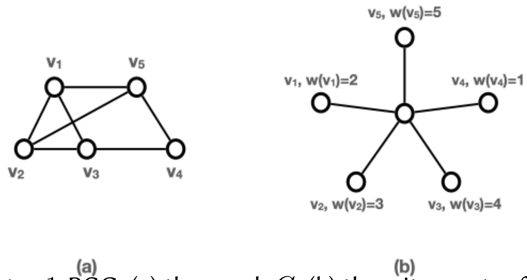

Figure 1: An example of a star-1-PCG: (a) the graph G, (b) the witness star for which G is a star-1-PCG for $I_1 = [5, 8]$.

Thus, the class of star-k-PCGs is particularly interesting as it serves as link between two significant graph classes: PCGs and multithreshold graphs, both of which currently lack a complete characterization. Indeed, the computational complexity of determining the minimum value of k for a graph to be a k-PCG remains an open question, and it is unknown whether this problem can be solved in polynomial time, even for the case of $k = 1$. Nevertheless, recent advancements have been made towards the recognition of star-k-PCGs. Recently, Xiao and Nagamochi [9] introduced the first polynomial-time algorithm for identifying graphs that are star-1-PCGs. Next, Kobayashi *et al.* in [10] improved upon this results by introducing a new characterization of star-1-PCGs that led a linear time algorithm for their recognition.

It is already established that every graph G is a star-k-PCG for some positive integer $k \leq |E(G)|$ [1]. Additionally, for each positive integer k, there exist graphs that are not star-k-PCGs but are star-$(k+1)$-PCGs [8]. A natural question is: for any given value of k which is the smallest value of n such that there exists an n vertex graph that is not a star-k-PCGs. This question has been already investigated for related graphs classes. Indeed, it is known that the smallest graph that is not a 1-PCG has 8 vertices [11, 12] and the smallest graph that is not a 2-PCG must have at least 9 vertices [13].

In this paper we ask a similar question for star-k-PCGs. We show that the smallest graph that is not a star-1-PCG has exactly 5 vertices. Moreover, we fully determine the membership to the star-k-PCG class for each graph with at most 5 vertices. We conclude with some open questions.

2. Preliminaries

For a graph $G = (V, E)$ and a vertex $u \in V$, the set $N(u) = \{v : \{u, v\} \in E\}$ is called the *neighborhood* of u.

Let S be an edge weighted star tree for each leaf v_i of S we denote by $w(v_i) = w_i$ the weight of the edge incident to v_i. For a graph G, *the weighted star tree of G* is a star whose leaves are the vertices of G.

It is already known that every graph G is a star-k-PCG for some positive integer k [1]. Thus, we introduce the following notation.

Definition 1. *Given a graph G, we define* the star number, *$\gamma(G)$, to be the smallest positive integer k, such that G is a star-k-PCG.*

From [1] it holds that for every graph G, $\gamma(G) \leq |E(G)|$.

In the forthcoming proofs we will use the following results.

Lemma 1. *[4, 9]. Let G be a graph and let k be a positive integer. If for any weighted star S of G, there exist $x \in V(G)$, vertices $v_1, \dots, v_{k+1}$ in $N(x)$ and $u_1, \dots u_k$ not in $N(x) \cup \{x\}$, such that $w(v_1) \leq w(u_1) \leq \dots \leq w(u_k) \leq w(v_{k+1})$, then G is not a star-k-PCG.*

The next lemma follows trivially by the definition of star-k-PCG.

Lemma 2. *Let G be a star-k-PCG and let S be a weighted witness star for G. If there are two leaves u, v in S for which $w(u) = w(v)$ then $N(u) = N(v)$.*

3. Not all 5-vertex graphs are star-1-PCGs

There are 34 non isomorphic graphs with 5 vertices [14]. These graphs are depicted in Fig. 2 based on increasing number of edges (see also [15]). Let $\mathcal{G}_5 = \{G_1, G_2, \dots, G_{34}\}$ be the set of all non isomorphic graphs with 5 vertices. In this section we show that these graphs are star-1-PCGs or star-2-PCGs. For the sake of simplicity in the forthcoming constructions we will omit to present the star tree proving the membership to star-k-PCG. Instead, for each leaf vertex v_i in a witness star tree S, we will simply associate the weight $w(v_i)$ to the vertex v_i in the graph G. We will refer to this representation as the *witness graph.* In Fig. 2 we show for each graph $G \in \mathcal{G}_5$ its witness graph together with the corresponding interval(s) proving the membership to star-1-PCG or star-2-PCG. To fully determine the membership to star-1-PCG or star-2-PCG classes, we need the following lemmas.

Lemma 3. $\gamma(G_{20}) = 2$

Lemma 4. $\gamma(G_{25}) = \gamma(G_{27}) = 2.$

Proof. Due to space limits we will only detail the proof for the graph G_{25}. Let $V(G_{25}) = \{a, b, c, d, e\}$ as shown in Fig. 2. Assume on the contrary that G_{25} is a star-1-PCG and let S and $I = [m, M]$ be the witness star tree and the corresponding interval. Notice that from Lemma 2, all the vertices are associate to a different weight in S. Let $l_1 = \min\{w(b), w(c)\}$ and $l_2 = \min\{w(d), w(e)\}$. Due to the symmetry of the graph, we can assume without loss of generality that $l_1 = w(b), l_2 = w(d)$ and $w(b) < w(d)$. Now, we focus on the weight of vertex a relative to the weight of the vertices b and d. We need to consider the following three cases.

- We have $w(a) < w(b) < w(d)$. Then the following holds:

$$m \leq w(a) + w(e) < w(b) + w(e) < w(d) + w(e) \leq M.$$

  Where the first and last inequalities follow as the edges $\{a, e\}, \{d, e\}$ belong to $E(G_{25})$. We reach a contradiction as $w(b) + w(e) \in I$ but $b, e \notin E(G_{25})$.
- We have $w(b) < w(a) < w(d)$. Then the following holds:

$$m \leq w(a) + w(b) < w(d) + w(b) < w(d) + w(a) \leq M.$$

  Where the first and last inequalities follow as the edges $\{a, b\}, \{d, a\}$ belong to $E(G_{25})$. We reach a contradiction as $w(d) + w(b) \in I$ but $d, b \notin E(G_{25})$.

- We have $w(b) < w(d) < w(a)$. Then the following holds:

$$m \leq w(b) + w(c) < w(d) + w(c) < w(a) + w(c) \leq M.$$

  Where the first and last inequalities follow as the edges $\{b, c\}, \{a, c\}$ belong to $E(G_{25})$. We reach a contradiction as $w(d) + w(c) \in I$ but $d, c \notin E(G_{25})$.

We thus, showed that G_{25} is not a star-1-PCG. The result for the graph G_{27} follows in a case by case analysis. $\square$. $\square$

Theorem 1. *All graphs with at most 5 vertices are star-1-PCGs, except for the graphs $\{G_{15}, G_{20}, G_{25}, G_{27}\}$ which are star-2-PCGs.*

Proof. For graphs with exactly 5 vertices the proof follows directly by Lemma 3 and Lemma 4 and by noticing that for the graph G_{15}, a cycle on five vertices, $\gamma(G_{15}) = 2$ [4]. It is easy to see that the rest of the graphs in Fig. 2 are star-1-PCG by simply checking the witness graph together with the corresponding interval.

Notice that if a graph is a star-k-PCG, removing a vertex from the graph will still result in a graph that belongs to the class of star-k-PCGs. A graph with 4 vertices can be viewed as a graph with 5 vertices with one isolated vertex. These graphs are depicted in Fig. 2 and are namely, $G_1 - G_8, G_{13}, G_{14}, G_{18}, G_{24}$, which are shown to be star-1-PCGs. The graphs with at most 3 vertices are obtained from the ones of 4 vertices by removing vertices and thus are clearly star-1-PCGs. $\square$ $\square$

4. Conclusion and open problems

In this paper we consider star-multi-interval pairwise compatibility graphs. We show that the smallest graph that is not a star-1-PCG has exactly 5 vertices. Moreover, we fully determine the membership to the star-k-PCG class for each graph with at most 5 vertices. Many problems remain open.

Problem 1: Determine the smallest graph that is not a star-2-PCG.

From the results in this paper we know that this number is at least 6. From the results in [8] we have that $3K_4$, the graph consisting of 3 disjoint cliques on four vertices is a star-3-PCG. We conjecture that the smallest graph that is not a star-2-PCG has indeed 12 nodes, and all the graphs with at most 11 nodes are star-2-PCGs.

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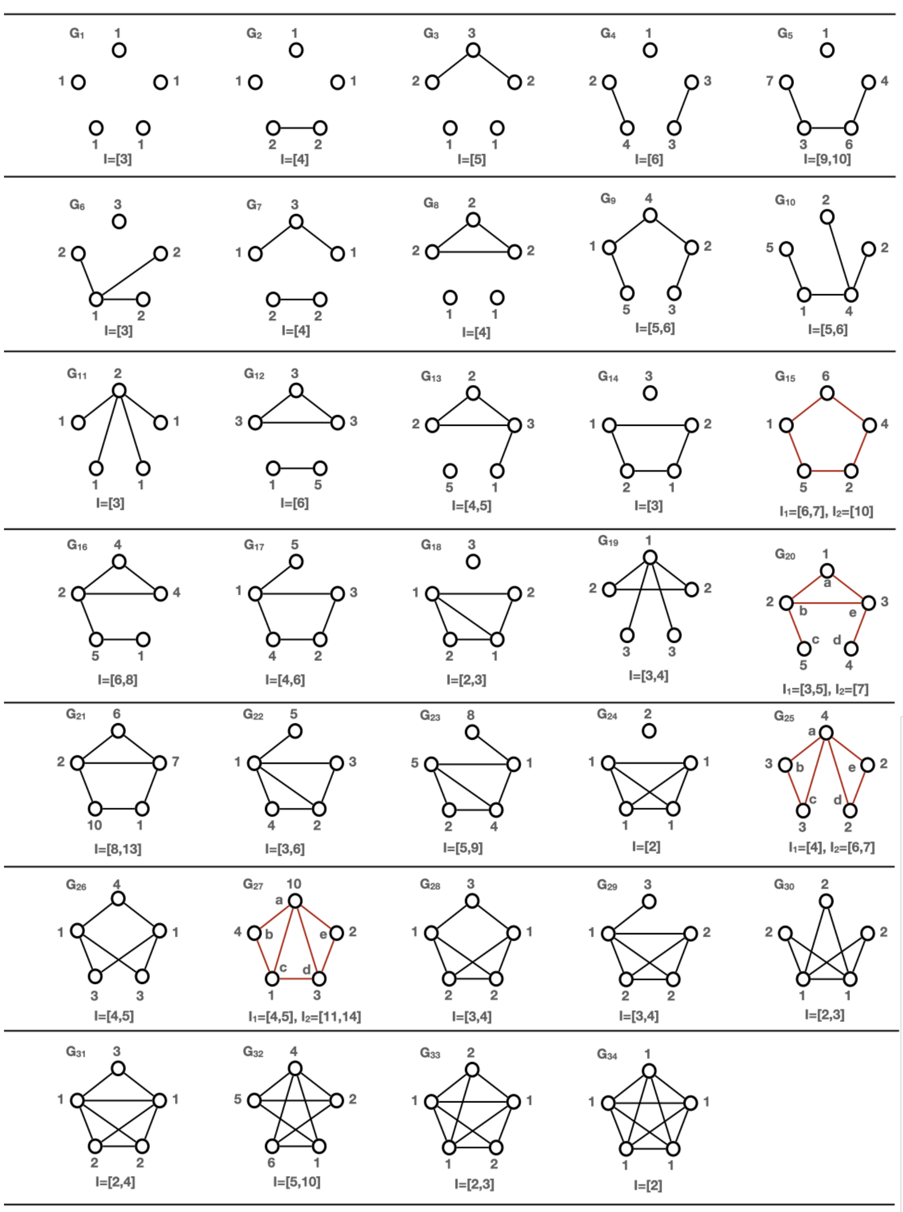

Figure 2: The list for all non isomorphic graphs with at most 5 vertices. The graphs with red edges, namely $G_{15}, G_{20}, G_{25}, G_{27}$ are star-2-PCGs. The rest of the graphs are all star-1-PCGs.

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