Setting the Path to the Combinatorial Characterization of Prime Double Square Polyominoes

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Abstract

Moving from the seminal work of Beauquier and Nivat (1991) about the characterization of polyominoes that tile the plane by translation, Blondin Massé et al. (2013) found that their boundary words, encoded by the Freeman chain coding on a four letters alphabet, have interesting combinatorial properties. In particular, they considered the specific class of double square polyominoes, and they defined two operators that allow to generate them starting from the basic class of the so called prime double squares. However, the proposed algorithm suffers few drawbacks due to repetitions and outliers generation. Here a different combinatorial approach to the double square characterization is proposed. In particular we provide a series of properties for the boundary words of prime double square tiles, that lead to detect some factors of them where a specific letter of the alphabet never occurs. The possibility of extending this property to the whole boundary word of a prime double square, as it seems, would naturally provide a valuable characterization and a tool for their generation and enumeration.

Keywords

Discrete Geometry, Combinatorics on words, Tiling of the plane, Exact tile

1. Introduction

A polyomino is defined as a connected finite subset of points in the integer lattice, commonly represented as a set of cells on a squared surface, each square being associated to an integer point. In [2], Beauquier and Nivat characterized the polyominoes that tile the plane by translation through properties of the Freeman chain coding on a four letters alphabet of their boundary. In particular, the boundary word P of an exact polyomino, say tile, can be factorized according to the equation $P = X_1 X_2 X_3 \widehat{X}_1 \widehat{X}_2 \widehat{X}_3$, where $\widehat{X}$ refers to the word X considered as a path and travelled in the opposite direction. We will refer to such decomposition as *BN-factorization*. According to [2], at most one among X_1, X_2 and X_3 can be empty, and we refer to pseudo squares in this case, pseudo hexagons otherwise.

The names pseudo square and pseudo hexagon refer to the behavior of a tile of being surrounded, in a tiling, with four or six copies of itself, respectively, see Fig. 1(a) and (b). It is easy to verify that some tiles also show these two behaviours at the same time in a tiling, as witnessed in

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Fig. 1(c).

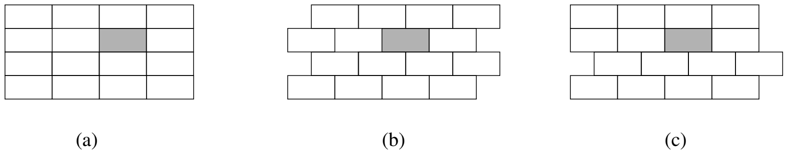

Figure 1: Three tilings of the plane with the two cell domino showing different behaviours. In (a) the domino acts as a pseudo square, since it is surrounded by four copies of itself. In (b) the domino acts as a pseudo hexagon, since six copies surround each domino. In (c) both the behaviours are present. Here the dark domino is surrounded by five copies of itself, but this is not the case for each element of the tiling.

Focusing on pseudo square tiles, in [3] it was proved that an exact polyomino tiles the plane as a pseudo square in at most two distinct ways. Furthermore, if two different pseudo square factorizations of P exist, then no decomposition as a pseudo hexagon does. The authors refer to these exact polyominoes as *double squares*.
Double squares have been studied under different aspects, some of them leading to the possibility of their exhaustive generation. To hit this target, in [5] two operators have been defined and recursively applied to a basic subclass, say *prime double squares*, defined throughout the notion of *homologous morphism*. Unfortunately, those operators suffer from some drawbacks: in particular, as observed by the authors, their iterative application may not generate a double square polyomino, or may generate more copies of the same double square (see Fig. 12 in [5]). Also from a combinatorial point of view, these drawbacks are undesirable in view of the characterization and the enumeration of the whole class.
Focusing on prime double squares, still in [5] the following conjecture, recently proved in [1], was proposed

Conjecture 35. Let w be the boundary word of a prime double square tile in a four letters alphabet Σ. Then, for any letter $\alpha \in \Sigma$, $\alpha\alpha$ is not a factor of w.

In this article we move from the results in [1] and we show that the boundary word of a prime double square has a peculiar form that involves factors that are repeated in different parts of both the BN-factorizations. We also prove that some of these factors are characterized by the absence of specific letters of the coding alphabet. This result opens a promising way both to the enumeration and to the generation of prime double squares (lately, of all the double square polyominoes), since it allows to generate parts of their boundary words whose path does not self intersect. So, in the next section we recall some basic definitions on combinatorics on words and few preliminary results to approach the study of prime double square polyominoes. Then Section 3 provides a series of properties of the boundary word of a prime double square polyomino that lead to state that in some of its parts one specific letter never occurs (consequently, since horizontal and vertical steps always alternate, as shown in [1], no self intersections of the related paths are possible).

2. Basic notions and preliminaries

Let us consider the lattice grid $\mathbb{Z}^2$. A *polyomino* is defined as a 4-connected finite subset of points of $\mathbb{Z}^2$. Each polyomino can be represented as a finite set of cells on a squared surface, each cell representing a lattice point (see Fig. 2). It is commonly required that polyominoes have no holes, i.e., the boundary of their representation as a set of cells is considered as a continuous, closed and non-intersecting path. We adopt this assumption here. Relying on that, we naturally code the boundary of a polyomino through a word, say *boundary word*, on an alphabet of four letters, $\Sigma = \{0, \overline{0}, 1, \overline{1}\}$. Each letter of the word represents a step in one of the four directions of the discrete grid, $\{\rightarrow, \leftarrow, \uparrow, \downarrow\}$ respectively. Due to the correspondence between letters of Σ and directions, we indicate 0 and $\overline{0}$, resp. 1 and $\overline{1}$, as *opposite*. Figure 2 shows an example of the coding.

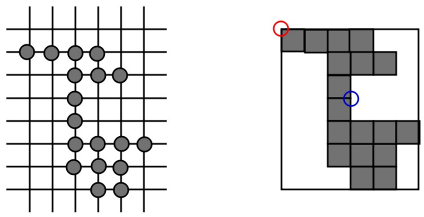

Figure 2: A connected set of points (polyomino) P on the left and its cell representation on the right. Moving clockwise and starting from the red circle, the polyomino is represented by the boundary word $P = 0000\overline{1}0\overline{10011}000\overline{101100}1\overline{0}11111\overline{00}1$, while starting from the blue circle the boundary word is $P' = \overline{1}000\overline{101100}1\overline{0}11111\overline{00}10000\overline{1}0\overline{1001}$. An easy check reveals that $P \equiv P'$.

Obviously, choosing different starting points and moving along the border clockwise or counter-clockwise lead to different boundary words for the same polyomino. So we need to introduce the following definitions to overcome these ambiguities. Using the standard notation, we indicate with Σ^* the free-monoid on Σ, i.e., the set of all words defined on Σ, where ε is the empty one, and with Σ^+ the set $\Sigma^* \setminus \{\varepsilon\}$. Given $w \in \Sigma^*$, $|w|$ denotes its length, while $|w|_\alpha$ is the number of occurrences of the letter α in the word w. The notation w^n indicates the concatenation of n copies of the word w. Finally, v is a *factor* of w if there exist $x, y \in \Sigma^*$ such that $w = xvy$. If $x = \varepsilon$ [resp. $y = \varepsilon$], then v is a *prefix* [resp. *suffix*] of w, while if $|x| = |y|$ then v is the *center* of w. The notation $v \notin w$ points out that v is not a factor of w.
Two words v and w are conjugate, say $w \equiv v$, if there exist two words x and y such that $v = xy$ and $w = yx$. The conjugacy is an equivalence relation, and the conjugacy class of a word w contains all its cyclic shifts, i.e., all the possible coding of a polyomino when fixing a travelling direction and moving all over the possible starting points. We decide to describe the boundary of a polyomino by travelling it clockwise, in order to identify it with any word of the class (see again Fig. 2 for an example). The *unit square* turns out to be $U = 10\overline{10}$. Moreover, if P is the boundary word of a polyomino, the following conditions hold: for all $\alpha \in \Sigma$ we have $|P|_\alpha = |P|_{\overline{\alpha}}$ (this ensures the closeness of the boundary), and there exists $\alpha \in \Sigma$ such that $|Q|_\alpha \neq |Q|_{\overline{\alpha}}$ for any Q proper factor of P (this ensures that the polyomino is 4-connected).
We define three operators on a word $w = w_1 w_2 \dots w_n \in \Sigma^*$:

1. the *opposite of* w, indicated with $\overline{w}$, is the word obtained by replacing each letter of w with its opposite;
2. the *reversal of* w, indicated with $\tilde{w}$, is defined as $\tilde{w} = w_n w_{n-1} \dots w_1$. A *palindrome* is a word s.t. $w = \tilde{w}$;
3. the *hat of w*, indicated with $\widehat{w}$, is the composition of the previous operations, $\widehat{w} = \tilde{\overline{w}}$.

We now introduce a particular subclass of polyominoes, the so called *prime double squares*, in which we are interested. A polyomino is called *exact* if it tiles the plane by translation. Beauquier and Nivat characterized exact polyominoes in relation to their boundary word, providing the following

Theorem 1 ([2]). *A polyomino P is exact if and only if there exist $X_1, X_2, X_3 \in \Sigma^*$ such that*

$$P = X_1 X_2 X_3 \widehat{X}_1 \widehat{X}_2 \widehat{X}_3,$$

where at most one of the words is empty. This factorization may be not unique.

We will refer to this decomposition as a *BN-factorization*, and call *BN-factors* the words X_i and $\widehat{X}_i$ provided by the decomposition. Starting from this result, exact polyominoes can be further divided in classes; we will focus on *pseudo squares*, that are the exact polyominoes where one of the BN-factors is empty. Among them, we specify the *double square polyominoes*, that admit two different (in terms of BN-factors) BN-factorizations as a square, $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$. Due to the presence of two BN-factorizations, double squares' boundary words can be written in the general form obtained from Corollary 6 in [6],

$$P = w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8, \tag{1}$$

where $A = w_1 w_2$, $B = w_3 w_4$, $\widehat{A} = w_5 w_6$, $\widehat{B} = w_7 w_8$ and $X = w_2 w_3$, $Y = w_4 w_5$, $\widehat{X} = w_6 w_7$, $\widehat{Y} = w_8 w_1$, with $w_1, \dots, w_8$ non empty.

We introduce the notion of *homologous morphism.* A morphism, in our framework, is a function $\varphi : \Sigma^* \to \Sigma^*$ s.t. $\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$ with $\alpha, \beta \in \Sigma$, i.e., it preserves concatenation, and it is said to be *homologous* if, for all $A \in \Sigma^*$, $\varphi(\widehat{A}) = \widehat{\varphi(A)}$, i.e., it preserves the hat operation. From now on, we will refer to homologous morphisms only. For each exact polyomino $P = AB\widehat{A}\widehat{B}$, we can define the trivial morphism that maps the unit square in P as $\varphi_P(1) = A$, $\varphi_P(0) = B$, $\varphi_P(\overline{1}) = \widehat{A}$ and $\varphi_P(\overline{0}) = \widehat{B}$. In general, the boundary word of an exact polyomino can also be obtained starting from the unit square through the composition of two or more morphisms (see Example 1). In [5] the authors defined the class of *prime double squares*, briefly pds, as the double squares whose boundary word P is such that, for any homologous morphism φ, the equality $P = \varphi(Q)$ implies that either $Q = P$ or Q is the boundary word of the unit square. This property can be rephrased saying that a double square is prime if its trivial morphism can not be obtained by composing two or more different morphisms. This last class, that constitutes the basis for the generation of double squares through homologous morphisms, will be the focus of our work. In particular, we will provide some properties of their boundary words setting the path for a suitable characterization to generate and then enumerate them.

Example 1. *The double square* $P = 11 \vdots 01011 \vdots 010 \vdots \overline{11}010 | \overline{11} \vdots \overline{01011} \vdots \overline{010} \vdots 11\overline{010}$ *is not prime, since it can be obtained applying to the unit square, in this order, the morphisms* $\varphi(0) = 0\overline{1}0$, $\varphi(1) = 101$ *and* $\psi(0) = 010$, $\psi(1) = 11$.

The notation $\vdots$ *separates the factors* w_i *in* P, *while* $|$ *denotes half of the word.*
On the other hand, the cross polyomino in the intermediate step is clearly prime.

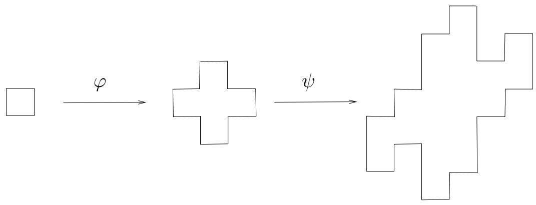

Figure 3: The visual representation of the actions of the two homologous morphisms φ and ψ to reach a double square from the unit square. The intermediate step is the cross, a prime double square polyomino.

We conclude this section with some useful results from [1, 5]:

Property 1 ([5]). *Let* P *be a double square, and* $AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ *its BN-factorizations. If* P *is prime, then the factors* A, B, X, Y *are palindrome.*

Property 2 ([5]). *Given* $P = w_1 w_2 \dots w_8$ *the boundary word of a double square as in* (1), *for all* $i = 1, \dots, 8$ *there exist* $u_i, v_i \in \Sigma^*$ *and* $n_i \geq 0$ *such that*

$$\begin{cases} w_i = (u_i v_i)^{n_i} u_i, \\ \widehat{w}_{i-3} w_{i-1} = u_i v_i. \end{cases}$$

Theorem 2 ([1]). *If* P *is the boundary word of a* pds *(prime double square), then it fits in one of the two following forms:*

a) $P = (u_1 k\tilde{u}_1 p)^{n_1} u_1 \vdots k\tilde{u}_1 \vdots (\overline{pu}_1 k\tilde{u}_1)^{n_3} u_3 \vdots \widehat{u}_1 \overline{p} | (\overline{u}_1 \overline{k} \widehat{u}_1 \overline{p})^{n_1} \overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots (pu_1 \overline{k}\widehat{u}_1)^{n_3} \overline{u}_3 \vdots \tilde{u}_1 p$, *with* k *and* $\overline{p}$ *palindrome and* $n_1, n_3 \geq 0$,

b) $P = u_1 \vdots (\tilde{u}_3 u_1)^{n_2} k\tilde{u}_1 \vdots u_3 \vdots (\widehat{u}_1 u_3)^{n_4} \widehat{u}_1 \overline{p} | \overline{u}_1 \vdots (\widehat{u}_3 \overline{u}_1)^{n_2} \overline{k}\widehat{u}_1 \vdots \overline{u}_3 \vdots (\tilde{u}_1 \overline{u}_3)^{n_4} \tilde{u}_1 p$, *with* $n_2, n_4 \geq 0$,

where the factors u_i *are those ones provided by Property* 2 *and under the assumption that* $|u_1| \leq |u_2|, |u_3|, |u_4|$, *so that* $u_2 = k\tilde{u}_1$ *and* $u_4 = \widehat{u}_1 \overline{p}$.

Finally, the following recent result proves Conjecture 35 in [5]

Theorem 3 ([1]). *Given a* pds, *its boundary word is* couple-free, *i.e., no two consecutive occurrences of a same letter of* Σ *are present.*

The two possible forms of the boundary word of a pds provided in Theorem 2 can be merged into a single one according to the following

Proposition 1. *Let P be the boundary word of a pds having form **b)** of Theorem 2. Then, it is always possible to rephrase P in the form **a)** of Theorem 2.*

From Proposition 1, it follows that the boundary word of a pds has a unique form according to the choices of $u_1, u_3, k, \overline{p}$ and the values $n_1, n_3 \geq 0$.

3. New properties of the factors of a pds boundary word

This section is dedicated to the study of the factor u_1 of a pds' boundary word in the form **a)** of Theorem 2, providing the main result of Theorem 4. In particular, we will show that the non-self intersection property of the boundary word of a pds implies that u_1 contains three letters only. To simplify the proofs, we will assume $n_1 = n_3 = 0$, so obtaining the boundary word of a pds in the form

$$P = u_1 \vdots k\tilde{u}_1 \vdots u_3 \vdots \widehat{u}_1\overline{p}|\overline{u}_1 \vdots \overline{k}\widehat{u}_1 \vdots \overline{u}_3 \vdots \tilde{u}_1 p, \tag{2}$$

with $k, \overline{p}$ non-empty palindromes. We underline that all the steps needed for the proof of Theorem 4 can be performed setting n_1 or n_3 different from 0. From now on, we will consider the BN-factors $A = u_1 k \tilde{u}_1$, $B = u_3 \widehat{u}_1 \overline{p}$, $X = k\tilde{u}_1 u_3$ and $Y = \widehat{u}_1 \overline{p}\, \overline{u}_1$.
Moreover, we point out that the same properties of u_1 that we will show in the sequel hold when the values of n_1 and n_3 are greater than zero, through similar arguments.

Lemma 1. *Let $P = AB\widehat{A}\widehat{B} \equiv XY\widehat{X}\widehat{Y}$ be (the boundary word of) a pds. For each factorization, the four BN-factors begin (and end) with a different letter of the alphabet $\Sigma = \{0, \overline{0}, 1, \overline{1}\}$.*

It directly follows by the palindromicity of the BN-factors and the fact that no two consecutive equal letters occur in P (see [1]).
Without loss of generality, we assume that A and B begin with the letters 1 and 0, respectively; as a consequence, X starts with 0 and Y with $\overline{1}$, since the polyomino is travelled clockwise in both factorizations.

Proposition 2. *Given a pds as in Eq. (2), the factor u_1 begins and ends with the letter 1, while u_3, k and $\overline{p}$ all begin and end with the letter 0.*

Proof. According to the choice that A and B start with 1 and 0 (respectively), we have that X and Y begin with 0 and $\overline{1}$, respectively, so that k starts with 0 (from X) and $\overline{p}$ ends with 0 (B is palindrome). Again by palindromicity, we obtain that the last letters of u_1 and u_3 are respectively 1 and 0. ☐
Hereafter we state our main result, whose proof will be obtained through the following lemmas.

Theorem 4. *Given a pds with boundary word P expressed as in Eq. (2), the factor u_1 contains only three letters of Σ, i.e., $u_1 \in \{0, \overline{0}, 1\}^+$.*

The proof of Theorem 4 relies on the following lemmas where, proceeding by contradiction, it is assumed that only one occurrence of $\overline{1}$ is present in u_1, i.e., $u_1 = 1v\overline{1}w1$ with $\overline{1} \notin v, w$ and $v, w \neq \varepsilon$. Similarly, a contradiction is obtained if we assume that more occurrences of $\overline{1}$ are present in u_1.

Proof of Theorem 4

We start this section with two technical lemmas.

Lemma 2 ([7]). *Assume that $w = xy = yz$, with $y \neq \varepsilon$. Then, for some palindromes $a, b \in \Sigma^+$ and some $i \geq 0$, we have $x = ab$, $y = (ab)^i a$ and $z = ba$.*

Lemma 3. *Let x_1, x_2, $x_3 \in \Sigma^+$ be three palindromes such that $x_1x_2x_3$ is palindrome too. Then, for some palindromes $a, b \in \Sigma^+$, x_1, x_2 and x_3 can be obtained as their alternate concatenations.*

The proof can be obtained from Lemma 1 in [4].

Lemma 4. *Let us assume $u_1 = 1v\overline{1}w1$ with $\overline{1} \notin v, w$. Then, both v and w have length $|v|, |w| > 1$.*

Lemma 5. *Let us assume that $u_1 = 1v\overline{1}w1$ is a factor of the boundary word of a pds P expressed as in Eq. (2). The position of $\overline{1}$ in u_1 is not the center of the BN-factor $X = k\tilde{u}_1u_3$.*

Proof. By contradiction, let $X = k1\tilde{w}\overline{\mathbf{1}}\tilde{v}1u_3$ be such that $w1k = \tilde{v}1u_3$, and let us consider $B = u_3\widehat{u}_1\overline{p} = u_3\overline{1}\widehat{w}\mathbf{1}\widehat{v}\overline{1}\overline{p}$. Recall that both X and B are palindrome since BN-factors of a pds. Let us suppose $\overline{1} \in u_3$ and, as a consequence, $\overline{1} \in k$ too. We make the first occurrence in u_3 explicit, $u_3 = x\overline{1}y$ with $\overline{1} \notin x$. From X palindrome, we get $w1k' = \tilde{v}1x$ for a suitable k' s.t. $k = k'\overline{1}y$; we also notice that $\overline{1} \notin k'$. We now analyze the length of the words v and w:

1. Case $|w| < |v|$. There exists a factor v' such that $v = v'\tilde{w}$ and $1k' = \tilde{v}'1x$. We now move to the other BN-factor, $B = u_3\widehat{u}_1\overline{p}$, to study its palindromicity.
   We have $B = x\overline{1}y\overline{1}\widehat{w}1\overline{w}\widehat{v}'\overline{1}\overline{p}$ palindrome and, since $\overline{1} \notin x$, $|x| \leq |\overline{p}|$. As a consequence, if the inequality is strict, we can write the last palindrome as $\overline{p} = x\overline{p}'\overline{1}\tilde{x}$. It follows from the palindromicity of B that $y\overline{1}\widehat{w}\mathbf{1}\overline{w}\widehat{v}'\overline{1}x\overline{p}'$ has to be palindrome too. Even in this case, we can deduce $|y| \leq |x\overline{p}'|$ from the property $\overline{1} \notin x$. If the inequality is strict, then $|y| < |\overline{p}'|$ and, for the same reason, the letter 1 in boldface is part of the factor x. We also remind that $1 \notin \overline{w}$, then the factor w is made of one letter only, $w = 0$ or $w = \overline{0}$, in contradiction with Lemma 4. Then $|y| = |x\overline{p}'|$, and we get that $\widehat{w}\mathbf{1}\overline{w}\widehat{v}'$ in B is palindrome. The letter in boldface is the only occurrence of 1, so the center, and then v' is the empty word. It follows that $w = \tilde{v}$, so u_1 is palindrome, and $\overline{p} = u_3$. Studying again X, we immediately argue that $u_3 = k$ too. We can now define a non-trivial morphism, $\varphi(0) = u_3$, $\varphi(1) = u_1$, that maps the cross in the pds P, reaching a contradiction.
   We finally have to study the case $(\overline{1} \notin)x = \overline{p}$, that gives in B the palindrome $y\overline{1}\widehat{w}\mathbf{1}\overline{w}\widehat{v}'$. Again, we can distinguish two cases, if the letter in boldface is the unique occurrence of 1 or not. If $v' = \overline{y}1$, then from $1k' = \tilde{v}'1x$ we get $k' = \widehat{y}1x$. Then k starts with k' and u_3

ends with y, that is impossible since they both start and finish with the same letter 0 (see Proposition 2). Then, there exists y' such that $y = \overline{v}'\widehat{w}\mathbf{1}y'$ and $y'\overline{1}\widehat{w}$ is palindrome. From this last condition, we argue that the second letter of y is $\overline{1}$, since $|w| = 1$ does not hold by Lemma 4. We now study the palindrome $y'\overline{1}\widehat{w}$:

i) $|y'| < |w|$. In this case, there exists w' such that $\widehat{w} = \widehat{w}'\widetilde{y'}$, i.e. $w = \overline{y}'w'$, and $1 \notin y'$. So, $X = (k'\overline{1}y)(1\tilde{w}\overline{1}\tilde{v}1)(a\overline{1}y) = k'\overline{1}\overline{v}'\widehat{w}'\widetilde{y'}1y'1\tilde{w}\overline{1}\tilde{v}1\overline{p}\overline{1}\overline{v}'\widehat{w}'\tilde{y}'1y'$. Since $\overline{1} \notin k'$ and $1 \notin y'$, it must be $k' = y' = 0$ (we remind that k starts with 0 by Proposition 2). Then, $k = 0\overline{1}\overline{v}'\widehat{w}'010$ is not palindrome, contradiction. If $|y'| = |w|$, then $y' = \overline{w}$, so that $1 \notin y'$ and the same contradiction is reached.

ii) $|w| < |y'|$. In this case, there exists a palindrome $y'' \neq \varepsilon$ such that $y' = \overline{w}\overline{1}y''$. We get $u_1 = 1v\overline{1}w1$, $u_3 = x\overline{1}y = \overline{p}\overline{1}\overline{v}1\overline{w}\overline{1}y'' = \overline{pu}_1y''$, $k = k'\overline{1}y = k'\overline{1}\overline{v}1\overline{w}\overline{1}y'' = k'\overline{u}_1y''$ and y'' palindrome, with $\overline{1} \notin \overline{p}, k', v, w$. The boundary word of the pds is now

$$P = u_1 \vdots k'\overline{u}_1y''\tilde{u}_1 \vdots \overline{pu}_1y'' \vdots \widehat{u}_1\overline{p}| \ldots,$$

and $A = u_1k\tilde{u_1}$ is palindrome if and only if $k'\overline{u}_1y'' = k'\overline{1}\overline{v}1\overline{w}1y''$ is palindrome. Since $\overline{1} \notin k'$ and $|w| > 1$, we argue that $\widetilde{k'}$ is a suffix of y'', $y'' = y'''\widetilde{k'}$ palindrome, and $\overline{u}_1y'''$ is palindrome too. Moving to $X = k'\overline{u}_1y'''\widetilde{k'}\tilde{u}_1\overline{pu}_1y'''\widetilde{k'}$, we deduce that $\tilde{k}'\tilde{u}_1\overline{p}$ is palindrome with one only occurrence of $\overline{1}$ (that one in u_1). So, $\overline{p} = k'$ and u_1 are palindrome. We get

$$P = u_1 \vdots \overline{pu}_1y'''\overline{pu}_1 \vdots \overline{pu}_1y'''\overline{p} \vdots \overline{u}_1\overline{p}| \ldots$$

with $\overline{u}_1y'''$ and $y'''\overline{p}$ palindromes (since center of the BN-factors A and Y, respectively). In particular, y''' ends with $\overline{1}$ and, since $\overline{1} \notin \overline{p}$, there exists a palindrome q such that $y''' = \overline{p}q$, with $\overline{p}q\overline{p}$ and $\overline{u}_1\overline{p}q$ both palindrome. By Lemma 3, there exist two palindromes z_1 and z_2 such that $\overline{u}_1$, q and $\overline{p}$ can be written as their concatenation. We underline that at least one among z_1 and z_2 has length greater than one, since $\overline{u}_1$ contains occurrences of both the letters 1 and $\overline{1}$.
We finally get that u_1, k, u_3 and $\overline{p}$ are concatenation of the palindromes z_1 and z_2 (or their opposite), and then it is possible to define a non-trivial morphism, $\varphi(0) = z_1$, $\varphi(1) = z_2$, that makes P non prime, contradiction.

2. Case $|v| < |w|$. There exists a word w' such that $w = \tilde{v}w'$ and $w'1k' = 1x$. The palindrome BN-factor is now $B = x\overline{1}y\overline{1}\widehat{w}'\overline{v}1\overline{\tilde{v}}\overline{1}\overline{p}$, and again $\overline{p} = x\overline{p}'\overline{1}\tilde{x}$ for a suitable $\overline{p}'$ since $\overline{1} \notin x$. Applying the same argument as in the previous case we reach again a contradiction.

The case $|v| = |w|$ immediately gives a contradiction through the definition of a morphism φ, as shown before. We then conclude that no occurrences of the letter $\overline{1}$ appear in the factor u_3. So, from the palindromicity of B, we argue that $|u_3| \leq |\overline{p}|$. If $u_3 = \overline{p}$ palindrome, then $\tilde{w} = v$ and u_1 is palindrome too. Going back to the BN-factor X, we get $k = u_3$ and a non-trivial morphism $\varphi(0) = u_3$, $\varphi(1) = u_1$ can be defined to map the cross in P, contradiction. Then, $|u_3| < |\overline{p}|$.

Being $B = u_3\overline{1}\widehat{w}1\widehat{v}\overline{1}\overline{p}$, there exists $\overline{p}'$ such that $\overline{p} = \overline{p}'\overline{1}\tilde{u}_3$ and $\widehat{w}1\widehat{v}\overline{1}\overline{p}'$ are palindrome. Since $\overline{1} \notin u_3$ and $\overline{p}$ is palindrome too, we can write the factor as $\overline{p} = u_3\overline{p}''\overline{1}\tilde{u}_3$ for a suitable $\overline{p}''$. From B, we obtain that $\widehat{w}1\widehat{v}\boldsymbol{\overline{1}}u_3\overline{p}''\boldsymbol{\overline{1}}\tilde{u}_3$ is a palindrome. This leads to a last contradiction since $\overline{1} \notin u_3$ and $|u_1| \leq |u_3|$ by hypothesis. □

Then, the only occurrence of $\overline{1}$ in u_1 is in the first or second half of X.

Corollary 1. *Since $\overline{1} \in u_1$ is unique and it is not the center of X, it follows that $|u_3| \neq |k|$.*

We continue the analysis of the position of $\overline{1} \in u_1$ in the BN-factor $X = k\tilde{u}_1u_3$. As final result, we will obtain that there are no available positions for it in X.

Lemma 6. *Let us assume that $u_1 = 1v\overline{1}w1$ is a factor of the boundary word P of a pds expressed as in Eq. (2), and $\overline{1} \in u_1$ is in the first half of X. Then, the BN-factor B is palindrome if and only if*

i) the center of B is $z''u_1k$, for a proper $z'' \in \Sigma^+$, or
ii) the center of B is $k\widehat{u}_1\overline{p}''$, for a proper $\overline{p}'' \in \Sigma^+$.

Proof. Since $\overline{1}$ is in the first half of $X = k\tilde{u}_1u_3$, we have that $|k| < |u_3|$, and $u_3 = x\overline{1}w1k$ for a proper non-empty x such that $\tilde{v}1x$ is palindrome. We now distinguish two cases.

1. $|v| < |x|$, and $x = z1v$ for a proper palindrome $z \in \Sigma^+$. Replacing in the factor, we get $u_3 = zu_1k$, where we remark that $z \neq \varepsilon$ since u_3 starts with 0 and u_1 with 1 (see Proposition 2). We now move to the palindrome

$$B = u_3\widehat{u}_1\overline{p} = zu_1k\widehat{u}_1\overline{p} = z1v\boldsymbol{\overline{1}}w1k\boldsymbol{\overline{1}}\widehat{w}1\widehat{v}\boldsymbol{\overline{1}}\overline{p}. \tag{3}$$

If $z = 0$, by the palindromicity of B and the property $\overline{1} \notin v$ we have that $\overline{p}$ has $\tilde{v}10$ as a suffix and then, being $\overline{p}$ palindrome, $01v$ as a prefix. Then in P we find the word $\widehat{v}\boldsymbol{\overline{1}}01v$, that intersects itself for any starting letter of v, contradiction. It follows that $|z| > 1$, and also $|z| \neq |\overline{p}|$ to have B palindrome. Again we have to distinguish two cases:

   i) $|\overline{p}| < |z|$. Now the palindrome z is written as $z = \overline{p}\overline{1}z'$ for a proper $z' \in \Sigma^+$, and B is palindrome if and only if $z'1v\boldsymbol{\overline{1}}w1k\boldsymbol{\overline{1}}\widehat{w}1\widehat{v}$ is palindrome, see Equation (3). If $z' = \overline{v}$, then the palindromicity can be obtained only with $|v| = |w| = 1$, in contradiction with Lemma 4. Then, being $\overline{1} \notin 1v$, we deduce that $\overline{v}1\overline{w}\boldsymbol{\overline{1}}$ is a prefix of z', so that $z' = \overline{v}1\overline{w}\overline{1}z''$ for a proper $z'' \in \Sigma^+$. Replacing in z, we get the palindrome $z = \overline{pu_1}z''$. Now, the study of the palindromicity of B is reduced to the study of its center, $z''u_1k$, and the thesis of case $i)$ is reached.

   ii) $|z| < |\overline{p}|$. We have $\overline{p} = \overline{p}'1z$ for a proper $\overline{p}' \in \Sigma^+$, and B is palindrome if and only if $v\boldsymbol{\overline{1}}w1k\boldsymbol{\overline{1}}\widehat{w}1\widehat{v}\boldsymbol{\overline{1}}\overline{p}'$ is. As seen in the previous case, $|\overline{p}'| \neq |v|$ and $\overline{1} \notin v, w$ allow to write $\overline{p}' = \overline{p}''1w\overline{1}\tilde{v}1z$, and then $\overline{p} = \overline{p}''u_1z$ for a proper non-empty word $\overline{p}''$. The center of B is now the palindrome $k\widehat{u}_1\overline{p}''$, and the thesis of case $ii)$ is reached.

2. $|x| < |v|$, and $v = v'1x$ for a proper $v' \in \Sigma^*$, in particular $\overline{1} \notin x$. Moving to B, we have $B = x\boldsymbol{\overline{1}}w1k\boldsymbol{\overline{1}}\widehat{w}1\widehat{v}\boldsymbol{\overline{1}}\overline{p}$ palindrome, and $|x| < |\overline{p}|$ since $\overline{1} \notin x$ and they can not have the same length, according to Lemma 4. Then, there exists $\overline{p}' \in \Sigma^+$ such that $\overline{p} = \overline{p}'\overline{1}\tilde{x}$, and

$$B = x\overline{\mathbf{1}}w1k\overline{\mathbf{1}}\widehat{w}1\widehat{v}\overline{\mathbf{1}}\overline{p}'\overline{\mathbf{1}}\tilde{x}. \tag{4}$$

Again, $\overline{1} \notin w$ guarantees that there exists a word $\overline{p}''$ to express $\overline{p} = \overline{p}''1\tilde{w}\overline{1}\tilde{x}$, and the center of B becomes $k\widehat{u}_1\overline{p}''$, as stated in case ii).

The case $x = v$ can be studied similarly to case 2, due to $\overline{1} \notin x$.
So all possible cases have been analyzed, and the thesis follows. □

We underline a relevant symmetrical result: the study of the BN-factor B, properly of its center, in both cases of Lemma 6 can be performed similarly to the study of the palindromicity of $X = k\tilde{u}_1u_3$ with $\overline{1} \in u_1$, where u_3 is replaced with z'' or $\overline{p}''$.

Lemma 7. *Let us assume that $u_1 = 1v\overline{1}w1$ is a factor of the boundary word P of a pds expressed as in Eq. (2), and $\overline{1} \in u_1$ is in the second half of X. Then*

i) the factor k can be expressed as $k = \tilde{u}_3u_1k''$ for a proper palindrome k'', or
ii) there exist $k', x \in \Sigma^+$ such that $k = \tilde{u}_31v\overline{1}k'$, $u_3 = x\overline{1}k'$ and $\tilde{x}1v$ palindrome.

Proof. Since $\overline{1} \in u_1$ is in the second half of X, we have $|u_3| < |k|$, and there exists $k' \in \Sigma^+$ such that $k = \tilde{u}_31v\overline{1}k'$ and $k'1\tilde{w}$ is palindrome. We distinguish two cases: if $|w| < |k'|$, we have $k' = w1k''$ for a proper palindrome k'' and $k = \tilde{u}_3u_1k''$. The thesis of case i) is obtained. On the other hand, let be $|k'| < |w|$. Then, $w = k'1w'$ for a proper $w' \in \Sigma^*$, and $\overline{1} \notin k'$. We have that $k = \tilde{u}_31v\overline{1}k'$ is palindrome.
If $k' = \tilde{v}1u_3$, then $|u_3| < |k'| < |w| < |u_1|$, in contradiction with the assumption on the mutual lengths of the factors u_i. So, $u_3 = x\overline{1}k'$ for a proper x such that $\tilde{x}1v$ is palindrome, and we get the thesis of case ii). The same occurs if $k' = w$. □

Again we underline the following symmetrical result: in both cases of Lemma 7 the study of the palindromicity of k can be performed as the study of $X = k\tilde{u}_1u_3$. In case i), just replacing k with k''; in case ii) we are in the same case of Lemma 6, where k' replaces the suffix $w1k$ of u_3 and $\overline{1} \in u_1$ is in the first half of X.
Such symmetrical results allow to iteratively apply Lemma 6 and 7 to the BN-factors X and B, thus crumbling them into two different common parts that alternate all over P.

Theorem 5. *If $u_1 = 1v\overline{1}w1$, there exist $\alpha, \beta, \gamma \geq 0$ and a palindrome $q \in \Sigma^+$ such that $k = (qu_1)^\alpha q$, $u_3 = (qu_1)^\beta q$ and $\overline{p} = (qu_1)^\gamma q$. Moreover, u_1 is palindrome.*

Proof. The palindromicity of the BN-factors X and B, and of the factor k, allows to iteratively apply Lemma 6 and Lemma 7, according to the length of the involved factors, and to finally obtain that $u_3, \overline{p}$ and k are concatenation of $u_1, \tilde{u}_1, q$ and $\tilde{q}$ for some $q \in \Sigma^+$. The palindromicity of u_1 and q is deduced by the fact that both X and k are palindrome at the same time, up to the parity of α.
We underline that $\overline{u}_1$ is not a factor of q, as shown in the proofs of Lemma 6 and Lemma 7. If Lemma 6 is never applied during the iterative proof, then $\overline{p} = u_3$. Indeed, in this case we have that u_1 is palindrome, $k = (qu_1)^\alpha q$ and $u_3 = (qu_1)^\beta q$ for a non-empty palindrome q and $\alpha, \beta \geq 0$. Since P is a pds, its BN-factor $B = u_3\widehat{u}_1\overline{p}$ is palindrome, that is $B = (qu_1)^\beta q\overline{u}_1\overline{p}$

palindrome. We can apply Lemma 3 with $x_1 = u_3$, $x_2 = \overline{u}_1$ and $x_3 = \overline{p}$. Since $\overline{u}_1$ is not a factor of q, the possible case is $x_1 = x_3$, i.e. $\overline{p} = u_3 = (qu_1)^\beta q$. □

Hereafter we provide an example where one only iteration of Lemma 6, case $1, i)$ in the proof, is sufficient to express the BN-factors X and B in terms of u_1 and q. In this case we get $q = k = z'' = \overline{p}$ and, as a consequence,

$$u_3 = x\bar{1}w1k \overset{|v|\leq|x|}{=} zu_1k \overset{|\overline{p}|\leq|z|}{=} \overline{pu_1}z''u_1k,$$

and $z'' = \overline{p}$ since z is palindrome and, by hypothesis, the proof stops here after one single iteration. Then, the boundary word P can be obtained applying the non-trivial morphism $\varphi(0) = \overline{p}$, $\varphi(1) = u_1$ to the double square $Q = 1 \vdots 01 \vdots 0\bar{1}010 \vdots \bar{1}0|\bar{1} \vdots \overline{01} \vdots \bar{0}1\overline{010} \vdots 1\bar{0}$, and P is not prime.

Corollary 2. $u_1 = 1v\bar{1}w1$ *can not be a factor of the boundary word P of a pds expressed as in Eq. (2).*

Proof. If $u_1 = 1v\bar{1}w1$, from Theorem 5 we know that u_1 is palindrome, $k = (qu_1)^\alpha q$, $u_3 = (qu_1)^\beta q$ and $\overline{p} = (qu_1)^\gamma q$ for some palindrome q and $\alpha, \beta, \gamma \geq 0$. Then, it is possible to define an homologous morphism, $\varphi(0) = q$ and $\varphi(1) = u_1$, that maps the double square $Q = 1(01)^\alpha 01(01)^\beta 0\bar{1}(01)^\gamma 0|\ldots$ in P. We underline that: if $\alpha = \beta = \gamma = 0$, the morphism φ maps the cross in P, so it is not trivial. On the other hand, in case q is a single letter, u_1 always has length greater than five, so φ is not the identity. It follows that P is not prime. □

We analyzed all the possible positions of $\bar{1} \in u_1$ in the BN-factor X, each time reaching a contradiction. A similar argument can be used if more occurrences of $\bar{1}$ are present in u_1, so leading to the proof of Theorem 4, as desired. □

We conclude this section providing examples of the two possible contradictions reached when we include $\bar{1}$ in u_1: in the first, the boundary of the pds self intersects, while, in the former, the polyomino can be obtained from the unit square through the composition of more than one non-trivial homologous morphism.

Example 2. *Let us consider* $u_1 = 1010\bar{1}0101\bar{0}101\bar{0}1$. *We can choose* $k = 010$ *and* $u_3 = 0\bar{1}0101\bar{0}101\bar{0}1010$ *to construct the (palindrome) BN-factors* $A = u_1k\tilde{u}_1$ *and* $X = k\tilde{u}_1u_3$. *The boundary word we get is*

$$P_1 = 1010\bar{1}0101\bar{0}101\bar{0}1 \vdots 0101\bar{0}101\bar{0}1010\bar{1}0101 \vdots 0\bar{1}010\mathbf{\bar{1}0101\bar{0}1010\bar{1}\overline{0101}\overline{01}010}\ldots$$

and self intersects (boldface letters), so it does not define a polyomino.
On the other hand, choosing $u_1 = 1\bar{0}1\overline{010}1\bar{0}1$, $\overline{p} = 01010$ *and* $k = u_3 = \overline{p}u_1\overline{p}$ *we get the double square* $P_2 = (1\bar{0}1\overline{010}1\bar{0}101010)^4\bar{1}0\bar{1}010\bar{1}0\bar{1}01010|\ldots$.
The polyomino is well defined and the boundary does not self intersect. However, it is not prime since a non-trivial morphism can be easily defined.

Coming to an end, our results indicate a new way of investigating double square polyominoes starting from the basic class of prime ones. By a combinatorial approach, we started to characterize their boundary words in terms of one single letter's absence. The obtained results suggest the possibility of extending such property to the other factors of the boundary word, so providing a valuable tool (alternative to [5]) to characterize and successively generate and enumerate them.

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