

Symbolic Sets for Proving Bounds on Rado Numbers

Tanbir Ahmed¹, Lamina Zaman¹ and Curtis Bright¹

¹School of Computer Science, University of Windsor, Canada

Abstract

Given a linear equation \mathcal{E} of the form $ax + by = cz$ where a, b, c are positive integers, the k -colour Rado number $R_k(\mathcal{E})$ is the smallest positive integer n , if it exists, such that every k -colouring of the positive integers $\{1, 2, \dots, n\}$ contains a monochromatic solution to \mathcal{E} . In this paper, we consider $k = 3$ and the linear equations $ax + by = bz$ and $ax + ay = bz$. Using SAT solvers, we compute a number of previously unknown Rado numbers corresponding to these equations. We prove new general bounds on Rado numbers inspired by the satisfying assignments discovered by the SAT solver. Our proofs require extensive case-based analyses that are difficult to check for correctness by hand, so we automate checking the correctness of our proofs via an approach which makes use of a new tool we developed with support for operations on symbolically-defined sets—e.g., unions or intersections of sets of the form $\{f(1), f(2), \dots, f(a)\}$ where a is a symbolic variable and f is a function possibly dependent on a . No computer algebra system that we are aware of currently has sufficiently capable support for symbolic sets, leading us to develop a tool supporting symbolic sets using the Python symbolic computation library SymPy coupled with the Satisfiability Modulo Theories solver Z3.

Keywords

Rado numbers, Ramsey Theory on the Integers, Satisfiability, SMT, Symbolic Computation

1. Introduction

The study of Rado numbers [1] and colouring problems lies at the heart of Ramsey theory and additive combinatorics. In this paper, we present a number of new values and bounds on Rado numbers, as well as an automated framework for proving lower bounds on Rado numbers using symbolic partitioning and satisfiability checking.

1.1. Combinatorial context

For integers a, b , let $[a, b]$ denote the set of integers $\{x : a \leq x \leq b\}$. For a positive integer $k \geq 2$, a system of linear equations is k -regular if there exists a positive integer n such that for every k -colouring of $[1, n]$, there exists a monochromatic solution to the system. A system is regular if it is k -regular for every integer $k \geq 2$. The following theorem by Rado [1] gives necessary and sufficient conditions for when a linear equation is regular.

Theorem 1.1 (Rado [1]). *For nonzero integers a_1, a_2, \dots, a_m and integer c , let $s = \sum_{i=1}^m a_i$. Then:*

- The equation $\sum_{i=1}^m a_i x_i = 0$ is regular if and only if there exists a nonempty set $D \subseteq [1, m]$ such that $\sum_{d \in D} a_d = 0$.
- The equation $\sum_{i=1}^m a_i x_i = c$ is regular if and only if one of the following two conditions holds: (a) $c/s \in \mathbb{Z}^+$; (b) $c/s \in \mathbb{Z}^-$ and $\sum_{i=1}^m a_i x_i = 0$ is regular.

For nonzero integers a_1, a_2, \dots, a_m and integer c , let the linear equation $\sum_{i=1}^{m-1} a_i x_i + c = a_m x_m$ be represented by $\mathcal{E}(m, c; a_1, a_2, \dots, a_m)$. For a linear equation $\mathcal{E}(m, c; a_1, a_2, \dots, a_m)$ and a positive integer k , the k -colour Rado number $R_k(\mathcal{E}(m, c; a_1, a_2, \dots, a_m))$ is defined as the smallest positive integer n , if it exists, from the definition of k -regularity demonstrating that the equation $\mathcal{E}(m, c; a_1, a_2, \dots, a_m)$ is k -regular. Otherwise, we say that $R_k(\mathcal{E}(m, c; a_1, a_2, \dots, a_m))$ is infinite if there is a k -colouring of

10th International Workshop on Satisfiability Checking and Symbolic Computation, August 2, 2025, Stuttgart, Germany

✉ tanbir@gmail.com (T. Ahmed); zamanl@uwindsor.ca (L. Zaman); cbright@uwindsor.ca (C. Bright)

🌐 <https://tanbir.github.io/> (T. Ahmed); <https://cs.curtisbright.com/> (C. Bright)

🆔 0000-0002-0462-625X (C. Bright)



© 2025 Copyright for this paper by its authors. Use permitted under Creative Commons License Attribution 4.0 International (CC BY 4.0).

the positive integers with no monochromatic solution to $\mathcal{E}(m, c; a_1, a_2, \dots, a_m)$. Surprisingly, not much is known about the properties of Rado numbers for $k = 3$. In 1995, Schaal [2] proved that the 3-colour Rado numbers $R_3(\mathcal{E}(3, c; 1, 1, 1))$ are always finite and $R_3(\mathcal{E}(3, c; 1, 1, 1)) = 13c + 14$ for $c \geq 0$. In 2015, Adhikari et al. [3] proved exact formulas for $R_3(\mathcal{E}(4, c; 1, 1, 1, 1))$ and $R_3(\mathcal{E}(5, c; 1, 1, 1, 1, 1))$ with $c \geq 0$. In 2022, Chang, De Loera, and Wesley [4] proved

- $R_3(\mathcal{E}(3, 0; 1, -1, a - 2)) = a^3 - a^2 - a - 1$ for $a \geq 3$ (previously conjectured by Myers [5]),
- $R_3(\mathcal{E}(3, 0; a, -a, (a - 1))) = a^3 + (a - 1)^2$ for $a \geq 3$, and
- $R_3(\mathcal{E}(3, 0; a, -a, b)) = a^3$ for $b \geq 1$, $a \geq b + 2$, and $\gcd(a, b) = 1$.

They also determined exact values of the Rado numbers

- $R_3(\mathcal{E}(3, 0; a, -a, b))$ for $1 \leq a, b \leq 15$,
- $R_3(\mathcal{E}(3, 0; a, a, b))$ for $1 \leq a, b \leq 10$,
- $R_3(\mathcal{E}(3, 0; a, b, c))$ for $1 \leq a, b, c \leq 6$,

as well as the following theorem.

Theorem 1.2. $R_3(\mathcal{E}(3, 0; 1, 1, a)) = \infty$ for $a \geq 4$ and $R_3(\mathcal{E}(3, 0; a, a, 1)) = \infty$ for $a \geq 2$.

1.2. Our contributions

In this paper, we have investigated values and bounds of Rado numbers for the equations $\mathcal{E}(3, 0; a, a, b)$ and $\mathcal{E}(3, 0; a, b, b)$. We have computed a number of previously unknown exact values using SAT solvers, extending the results presented by Chang, De Loera, and Wesley [4] to the values of

- $R_3(\mathcal{E}(3, 0; a, b, b))$ for $1 \leq b \leq 15$ and $1 \leq a \leq 30$, and
- $R_3(\mathcal{E}(3, 0; a, a, b))$ for $1 \leq a \leq 15$ and $1 \leq b \leq 25$.

Note that we may assume that a and b are coprime, since if a and b share a common factor g we may divide the coefficients of the equation \mathcal{E} by g without changing its solutions, e.g., $R_3(\mathcal{E}(3, 0; a, b, b)) = R_3(\mathcal{E}(3, 0; a/g, b/g, b/g))$. Guided by the satisfying assignments generated by the SAT solver in the process of computing these values, we proved the following two theorems.

Theorem 1.3. For coprime positive integers a, b with $a > b \geq 3$ and $a^2 + a + b > b^2 + ba$,

$$R_3(\mathcal{E}(3, 0; a, b, b)) \geq a^3 + a^2 + (2b + 1)a + 1.$$

Theorem 1.4. For odd integers $a \geq 7$,

$$R_3(\mathcal{E}(3, 0; a, a, a + 1)) \geq a^3(a + 1).$$

Based on the computational data we generated and the above lower bounds, we propose the following conjectures.

Conjecture 1.1. For coprime positive integers a, b with $a > b \geq 3$ and $a^2 + a + b > b^2 + ba$,

$$R_3(\mathcal{E}(3, 0; a, b, b)) = a^3 + a^2 + (2b + 1)a + 1.$$

Conjecture 1.2. For coprime positive integers a, b , with a odd, if $3 \leq a < b \leq 2a - 1$ then

$$R_3(\mathcal{E}(3, 0; a, a, b)) = a^3b.$$

The proofs of Theorems 1.3 and 1.4 are automatically verified using a symbolic analyzer we developed in Python called AutoCase.¹ The motivation has been to avoid errors in the intricate case-based analysis which is often routine and repetitive. For example, in order to verify Theorem 1.4, AutoCase takes as input three symbolically-defined subsets R, G, B of the integers $[1, a^3(a + 1) - 1]$ (where $a \geq 7$ is a symbolic odd integer). AutoCase verifies that the union of R, G , and B is the set $[1, a^3(a + 1) - 1]$ and that R, G, B are mutually disjoint—thereby confirming that R, G, B form a 3-colouring of the positive

¹github.com/laminazaman/RadoNumbers/tree/main/AutoCase

integers less than $a^3(a + 1)$. Finally, AutoCase confirms that there are no monochromatic solutions of the equation $ax + ay = (a + 1)z$, i.e., the set $\bigcup_{ax+ay=(a+1)z} \{(x, y, z)\}$ is disjoint with R^3 , G^3 , and B^3 . We stress that the sets defining the colouring are defined *symbolically*, e.g., $\bigcup_{i,j,k} \{f(i, j, k)\}$ where f is a function (possibly depending on a), the upper and lower bounds on the indices i, j, k may depend on a , and the elements of the set may be filtered by divisibility predicates such as specifying that all elements are divisible by a .

Although it would be easy to perform the necessary set operations in a typical computer algebra system like Maple, Mathematica, and SageMath if a was a known fixed integer, in our application a is a symbolic variable, and we are aware of no computer algebra system supporting the operations on symbolic sets we require in this paper. Maple and Mathematica seem to lack the ability to even express the set $[1, a]$ when a is symbolic. SageMath does have support for a “ConditionSet” allowing sets like $[1, a]$ to be formulated, but the operations supported by ConditionSets was too limited for our purposes. In AutoCase, operations on symbolic sets are performed by employing a combination of the Python symbolic computation library SymPy [6] and the SMT (SAT modulo theories) constraint solver Z3 [7].

Our work therefore fits into the “SC-Square” paradigm of combining satisfiability checking with symbolic computation [8, 9]. Over the past decade, the SC-Square community has combined the tools of satisfiability checking (e.g., SAT and SMT solvers) with the tools of symbolic computation (e.g., computer algebra systems and libraries) in order to make progress on problems benefiting from both the search and learning of SAT/SMT solvers and the mathematical sophistication of computer algebra. As just a single example, computer algebra libraries are able to detect if two mathematical objects are isomorphic. Augmenting a SAT solver with isomorphism detection can dramatically improve its efficiency on mathematical problems like proving the nonexistence of an order ten projective plane [10] or enumerating various kinds of graphs up to isomorphism, something that can be done by SAT modulo symmetries [11] or SAT+CAS [12] solvers.

2. Computational Results

One of the key challenges in Ramsey theory on the integers is the scarcity of data, as generating individual data points is an exceptionally difficult task. However, modern computational tools, including SAT solvers, have alleviated some of these computational challenges. During the last two decades, SAT solvers have been employed to compute Ramsey-type numbers at different scales and capacities. Some examples are the computation of the values and bounds of van der Waerden numbers (Kouril and Paul [13], Herwig et al. [14], Ahmed [15, 16, 17, 18], Ahmed et al. [19]) and Schur numbers (Ahmed and Schaal [20], Ahmed et al. [21]). A notable success of SAT solvers in this area of mathematics is the computation of the fifth Schur number by Heule [22].

Recently, Chang, De Loera, and Wesley [4] applied SAT solvers to compute Rado numbers. Building on this progress, we used SAT solvers to determine new exact values of Rado numbers and to uncover patterns in the colourings that avoid monochromatic solutions to the linear equation under investigation. These patterns provide general lower bounds on Rado numbers and we employed methods from symbolic computation and satisfiability checking to automate the verification of these lower bounds.

2.1. The satisfiability (SAT) problem

A *literal* is a Boolean variable (say x) or its negation (denoted \bar{x}). A *clause* is a logical disjunction of literals. A *formula* is in *Conjunctive Normal Form (CNF)* if it is a logical conjunction of clauses.

A *truth assignment* is a mapping of each variable in its domain to true or false. A truth assignment *satisfies a clause* if it maps at least one of its literals to true and the assignment *satisfies a formula* if it satisfies each of its clauses. A formula is called *satisfiable* if it is satisfied by at least one truth assignment and otherwise it is called *unsatisfiable*. The problem of recognizing satisfiable formulas is known as *the satisfiability problem*, or SAT for short.

2.2. Encoding Rado numbers as SAT problems

Given positive integers n and k and a linear equation \mathcal{E} , we now construct a formula in conjunctive normal form that is satisfiable if and only if there exists a k -colouring of $[1, n]$ avoiding monochromatic solutions of \mathcal{E} . Therefore, if our formula is satisfiable then $R_k(\mathcal{E}) > n$ and if our formula is unsatisfiable then $R_k(\mathcal{E}) \leq n$.

Variables: Variables are denoted by $v_{i,j}$ for $0 \leq i \leq k-1$ and $1 \leq j \leq n$, such that $v_{i,j}$ is true if and only if colour i is assigned to integer j . There are nk variables in the formula.

At least one colour is assigned to every integer: For each integer $j \in [1, n]$, the clause

$$(v_{0,j} \vee v_{1,j} \vee \cdots \vee v_{k-1,j})$$

ensures at least one colour $i \in [0, k-1]$ is assigned to j .

At most one colour is assigned to every integer: For each integer $j \in [1, n]$, the clauses

$$\bigwedge_{0 \leq i_1 < i_2 \leq k-1} (\bar{v}_{i_1,j} \vee \bar{v}_{i_2,j})$$

ensure at most one colour $i \in [0, k-1]$ is assigned to j .

There is no monochromatic solution to \mathcal{E} : For each colour $i \in [0, k-1]$ and for each solution (x_1, x_2, \dots, x_m) to \mathcal{E} , the clause

$$(\bar{v}_{i,x_1} \vee \bar{v}_{i,x_2} \vee \cdots \vee \bar{v}_{i,x_m})$$

ensures the solution (x_1, x_2, \dots, x_m) is not monochromatic in colour i . Let $\mathcal{S}_{\mathcal{E},n}$ denote the set of all solutions (x_1, x_2, \dots, x_m) in $[1, n]$ that satisfy the equation \mathcal{E} . There will be $k \cdot |\mathcal{S}_{\mathcal{E},n}|$ such clauses. We may compute $|\mathcal{S}_{\mathcal{E},n}|$ by summing over all possible values of $(x_1, x_2, \dots, x_{m-1})$ in the range $[1, n]$ via

$$|\mathcal{S}_{\mathcal{E},n}| = \sum_{x_1=1}^n \sum_{x_2=1}^n \cdots \sum_{x_{m-1}=1}^n \left[\frac{\sum_{i=1}^{m-1} a_i x_i}{a_m} \in [1, n] \right],$$

where summand uses Iverson bracket notation to ensure that the remaining variable x_m (determined by the equation) falls within $[1, n]$. This gives $|\mathcal{S}_{\mathcal{E},n}| \leq n^{m-1}$, since $[\cdot] \leq 1$.

Breaking permutation symmetry of colours: So far, no distinction has been made amongst the k colours. This artificially increases the size of the search space by a factor of $k!$, the number of permutations on k colours. To avoid having the SAT solver explore all $3!$ colour permutations on $\{0, 1, 2\}$ during solving, we ensure that colours are introduced in ascending order, i.e., colour 0 appears before colour 1 and colour 1 appears before colour 2. To ensure that colour 0 appears in the first position, we add the unit clause $v_{0,1}$. To ensure that colour 1 appears before colour 2, we add the clause $(\bigvee_{i=1}^{j-1} \bar{v}_{0,i}) \vee \bar{v}_{2,j}$ for $j = 2, \dots, R_1(\mathcal{E})$. These say that colour 2 cannot immediately follow an initial colouring consisting only of colour 0s, and thus the first colour other than 0 must be 1. One can work out the exact value of $R_1(\mathcal{E})$ by hand for the cases we consider in this paper, as in the following lemma whose proof we provide in appendix A.

Lemma 2.1. *If a and b are positive coprime integers then $R_1(\mathcal{E}(3, 0; a, b, b)) = \max(a + 1, b)$, $R_1(\mathcal{E}(3, 0; a, a, 1)) = 2a$, and $R_1(\mathcal{E}(3, 0; a, a, b)) = \max(a, \lceil b/2 \rceil)$ when $b > 1$.*

2.3. Efficient generation of integer solutions

To efficiently generate all integer solutions (x, y, z) within the interval $[1, n]$ for $ax + by = cz$, we iterate i from 1 to n and construct three linear Diophantine equations

$$\begin{aligned} ax &= cz - by & \text{where } x &= i, \\ by &= cz - ax & \text{where } y &= i, \\ cz &= ax + by & \text{where } z &= i. \end{aligned}$$

The goal is to independently find solutions for each equation.

- Step 1 (Solving a linear Diophantine equation). To begin, we solve one of the above equations (say $ax + by = ci$) for unknowns (x, y) using the Extended Euclidean Algorithm, providing a particular integer solution (x_0, y_0) . The general solution is

$$x = x_0 + \frac{bk}{\gcd(a, b)} \quad \text{and} \quad y = y_0 - \frac{ak}{\gcd(a, b)},$$

where k is a parameter that runs over all integers to generate all solutions.

- Step 2 (Restricting to the interval $[1, n]$). Since we are only interested in solutions within $[1, n]$, we impose the constraints

$$1 \leq x_0 + \frac{bk}{\gcd(a, b)} \leq n \quad \text{and} \quad 1 \leq y_0 - \frac{ak}{\gcd(a, b)} \leq n.$$

These inequalities provide upper and lower bounds on k , ensuring that both x and y remain in the valid range.

- Step 3 (Iterating over valid values of k). Once the feasible range for k is determined, we iterate over all valid values of k and solve for x and y at each step, adding (x, y, i) to our list of solutions.
- Finally, we repeat these steps for the remaining two equations $ai = cz - by$ and $bi = cz - ax$.

2.4. SAT solvers and computation resources

For this work, we used the SAT solvers CaDiCaL [23] and Kissat [24]. Initially, we used CaDiCaL integrated with PySAT [25] due to its incremental solving capabilities which enable the solver to reuse learned information across instances. Our approach starts with a counter $n = 1$. For each n , we add the corresponding clauses to the solver. If the instance is satisfiable, we increment n and continue incrementally. This process repeats until we encounter the first unsatisfiable instance, at which point the corresponding n is identified as the Rado number.

Once a Rado number was determined (or suspected to be known based on the patterns we observed), we transitioned to using a non-incremental solver and generated two instances: one for $n - 1$, the last satisfiable instance, and one for n , the first unsatisfiable instance. We generated clauses in DIMACS format, wrote them to two CNF files, and used Kissat to solve both instances. To be consistent in our timings, all instances were ultimately solved non-incrementally using Kissat. Our largest instance, the unsatisfiable instance showing $R_3(\mathcal{E}(3, 0; 15, 15, 8)) = 97875$, contained 293,625 variables and 255,884,401 clauses and took 58.8 hours to solve.

Altogether, generating the SAT instances took around 11 hours, the total CPU time used to solve the unsatisfiable instances (i.e., the instances that prove an upper bound) was around 251 hours, and the total CPU time used on satisfiable instances (i.e., the instances that prove a lower bound) was around 53 hours. Our computations were performed on the Compute Canada cluster Cedar, utilizing Intel E5-2683 v4 Broadwell processors running at 2.1 GHz.

2.5. Some new exact values of $R_3(\mathcal{E}(3, 0; a, b, b))$

We have computed $R_3(\mathcal{E}(3, 0; a, b, b))$ for $1 \leq b \leq 15$ and $1 \leq a \leq 30$ and given the results in Table 1. By Rado's theorem 1.1, all elements in Table 1 are finite. Altogether, the total CPU time on unsatisfiable

$a \setminus b$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	14	14	27	64	125	216	343	512	729	1000	1331	1728	2197	2744	3375
2	43	<u>14</u>	31	<u>14</u>	125	<u>27</u>	343	<u>64</u>	729	125	1331	<u>216</u>	2197	<u>343</u>	3375
3	94	61	<u>14</u>	73	125	<u>14</u>	343	512	<u>27</u>	1000	1331	<u>64</u>	2197	2744	<u>125</u>
4	173	<u>43</u>	109	<u>14</u>	141	<u>31</u>	343	<u>14</u>	729	<u>125</u>	1331	<u>27</u>	2197	<u>343</u>	3375
5	286	181	186	180	<u>14</u>	241	343	512	729	<u>14</u>	1331	1728	2197	2744	<u>27</u>
6	439	<u>94</u>	<u>43</u>	<u>61</u>	300	<u>14</u>	379	<u>73</u>	<u>31</u>	125	1331	<u>14</u>	2197	<u>343</u>	<u>125</u>
7	638	428	442	456	470	462	<u>14</u>	561	729	1000	1331	1728	2197	<u>14</u>	3375
8	889	<u>173</u>	633	<u>43</u>	665	<u>109</u>	644	<u>14</u>	793	<u>141</u>	1331	<u>31</u>	2197	<u>343</u>	3375
9	1198	856	<u>94</u>	892	910	<u>61</u>	896	896	<u>14</u>	1081	1331	<u>73</u>	2197	2744	<u>125</u>
10	1571	<u>286</u>	1171	<u>181</u>	<u>43</u>	<u>186</u>	1190	<u>180</u>	1206	<u>14</u>	1431	<u>241</u>	2197	<u>343</u>	<u>31</u>
11	2014	1508	1530	1552	1574	1596	1618	1584	1575	1580	<u>14</u>	1849	2197	2744	3375
12	2533	<u>439</u>	<u>173</u>	<u>94</u>	2005	<u>43</u>	2053	<u>61</u>	<u>109</u>	<u>300</u>	2024	<u>14</u>	2341	<u>379</u>	<u>141</u>
13	3134	2432	2458	2484	2510	2536	2562	2588	2574	2530	2541	2544	<u>14</u>	2913	3375
14	3823	<u>638</u>	3039	<u>428</u>	3095	<u>442</u>	43	<u>456</u>	3207	<u>470</u>	3113	<u>462</u>	3146	<u>14</u>	3571
15	4606	3676	<u>286</u>	3736	<u>94</u>	<u>181</u>	3826	3856	<u>186</u>	<u>61</u>	3795	<u>180</u>	3835	3836	<u>14</u>
16	5489	<u>889</u>	<u>4465</u>	<u>173</u>	4529	<u>633</u>	4593	<u>43</u>	4657	<u>665</u>	4576	<u>109</u>	4602	<u>644</u>	4620
17	6478	5288	5322	5356	5390	5424	5458	5492	5526	5560	5594	5424	5447	5474	5475
18	7579	<u>1198</u>	<u>439</u>	<u>856</u>	6355	<u>94</u>	6427	<u>892</u>	<u>43</u>	<u>910</u>	6571	<u>61</u>	6409	<u>896</u>	<u>300</u>
19	8798	7316	7354	7392	7430	7468	7506	7544	7582	7620	7658	7696	7488	7518	7530
20	10141	<u>1571</u>	8541	<u>286</u>	<u>173</u>	<u>1171</u>	8701	<u>181</u>	8781	<u>43</u>	8861	<u>186</u>	8941	<u>1190</u>	<u>109</u>
21	11614	9808	<u>638</u>	9892	9934	<u>428</u>	<u>94</u>	10060	<u>442</u>	10144	10186	456	10270	<u>61</u>	<u>470</u>
22	13223	<u>2014</u>	11287	<u>1508</u>	11375	<u>1530</u>	11463	<u>1552</u>	11551	<u>1574</u>	<u>43</u>	<u>1596</u>	11727	<u>1618</u>	11535
23	14974	12812	12858	12904	12950	12996	13042	13088	13134	13180	13226	13272	13318	13364	13110
24	16873	<u>2533</u>	<u>889</u>	<u>439</u>	14665	<u>173</u>	14761	<u>94</u>	<u>633</u>	<u>2005</u>	14953	<u>43</u>	15049	<u>2053</u>	<u>665</u>
25	18926	16376	16426	16476	<u>286</u>	16576	16626	16676	16726	<u>181</u>	16826	16876	16926	16976	186
26	21139	<u>3134</u>	18435	<u>2432</u>	18539	<u>2458</u>	18643	<u>2484</u>	18747	<u>2510</u>	18851	<u>2536</u>	<u>43</u>	<u>2562</u>	19059
27	23518	20548	<u>1198</u>	20656	20710	<u>856</u>	20818	20872	<u>94</u>	20980	21034	<u>892</u>	21142	21196	<u>910</u>
28	26069	<u>3823</u>	22933	<u>638</u>	23045	<u>3039</u>	<u>173</u>	<u>428</u>	23269	<u>3095</u>	23381	<u>442</u>	23493	<u>43</u>	23605
29	28798	25376	25434	25492	25550	25608	25666	25724	25782	25840	25898	25956	26014	26072	26130
30	31711	<u>4606</u>	<u>1571</u>	<u>3676</u>	<u>439</u>	<u>286</u>	28351	<u>3736</u>	<u>1171</u>	<u>94</u>	28591	<u>181</u>	28711	<u>3826</u>	<u>43</u>

Table 1

$R_3(\mathcal{E}(3, 0; a, b, b))$ for $1 \leq b \leq 15$ and $1 \leq a \leq 30$. The previously unknown values are presented in boldface, and the underlined entries correspond to equations whose coefficients are not coprime.

instances was 21,790 seconds, and the total CPU time on satisfiable instances was 13,516 seconds, amounting to 9.81 total hours. A maximum of 4.3 GiB of memory was used across all instances.

2.6. Some new exact values of $R_3(\mathcal{E}(3, 0; a, a, b))$

By Theorem 1.2, $R_3(\mathcal{E}(3, 0; a, a, 1))$ is infinite for $a \geq 2$ and $R_3(\mathcal{E}(3, 0; 1, 1, b))$ is infinite for $b \geq 4$. Also, by Chang et al. [4, Lemma 3.1], $R_3(\mathcal{E}(3, 0; a, a, b))$ is infinite if $2a \leq a^2/b$ or $2a \leq \sqrt{ab}$. They provided the values of $R_3(\mathcal{E}(3, 0; a, a, b))$ for $1 \leq a, b \leq 10$ and also for $3 \leq a \leq 6$ and $11 \leq b \leq 20$. We have extended them for $1 \leq a \leq 15$ and $1 \leq b \leq 25$ and reported these numbers in Table 2. The instances in this table proving upper bounds required a CPU time of 882,211 seconds to solve, while the instances proving lower bounds required 176,375 seconds to solve, amounting to 294.05 total hours. A maximum of 26.3 GiB of memory was used across all instances.

2.7. Some new patterns for $R_3(\mathcal{E}(3, 0; a, b, b))$

The data on $R_3(\mathcal{E}(3, 0; a, b, b))$ presented in Table 1 inspired us to make some general observations and a conjecture described below.

Observation 2.1. For coprime integers a and b , the data in Table 1 reveals the following.

$b \setminus a$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	14	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	1	<u>14</u>	243	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
3	54	54	<u>14</u>	384	2000	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
4	∞	<u>1</u>	108	<u>14</u>	875	<u>243</u>	4459	∞	∞	∞	∞	∞	∞	∞	∞
5	∞	105	135	180	<u>14</u>	864	3430	3072	12393	∞	∞	∞	∞	∞	∞
6	∞	<u>54</u>	<u>1</u>	54	750	14	3087	384	243	2000	27951	∞	∞	∞	∞
7	∞	455	336	308	875	756	<u>14</u>	1536	8748	7500	23958	10368	54925	∞	∞
8	∞	∞	<u>432</u>	<u>1</u>	1000	<u>108</u>	2744	<u>14</u>	8019	<u>875</u>	21296	<u>243</u>	48334	<u>4459</u>	97875
9	∞	∞	<u>54</u>	585	1125	<u>54</u>	3087	1224	<u>14</u>	6000	18634	<u>384</u>	41743	30184	<u>2000</u>
10	∞	∞	1125	<u>105</u>	<u>1</u>	<u>135</u>	3430	<u>180</u>	7290	<u>14</u>	17303	864	37349	<u>3430</u>	<u>243</u>
11	∞	∞	2019	847	1958	1188	3773	1672	8019	5500	<u>14</u>	6048	35152	24696	77625
12	∞	∞	∞	<u>54</u>	2400	<u>1</u>	4116	<u>54</u>	<u>108</u>	<u>750</u>	15972	<u>14</u>	32955	<u>3087</u>	<u>875</u>
13	∞	∞	∞	1710	3445	1963	4459	1456	9477	6500	17303	5616	<u>14</u>	21952	60750
14	∞	∞	∞	455	3675	<u>336</u>	<u>1</u>	<u>308</u>	10206	<u>875</u>	18634	<u>756</u>	30758	<u>14</u>	57375
15	∞	∞	∞	5408	<u>54</u>	<u>105</u>	6615	2760	<u>135</u>	<u>54</u>	19965	<u>180</u>	32955	20580	<u>14</u>
16	∞	∞	∞	∞	5725	<u>432</u>	7616	<u>1</u>	11664	<u>1000</u>	21296	<u>108</u>	35152	<u>2744</u>	54000
17	∞	∞	∞	∞	8330	4743	10064	3825	12393	8500	22627	7344	37349	23324	57375
18	∞	∞	∞	∞	12069	<u>54</u>	10962	<u>585</u>	<u>1</u>	<u>1125</u>	23958	<u>54</u>	39546	<u>3087</u>	<u>750</u>
19	∞	∞	∞	∞	16397	6726	14782	4332	16853	9500	25289	8208	41743	26068	60750
20	∞	∞	∞	∞	∞	<u>1125</u>	14700	<u>105</u>	19080	<u>1</u>	26620	<u>135</u>	43940	<u>3430</u>	<u>108</u>
21	∞	∞	∞	∞	∞	<u>455</u>	54	6699	336	12579	27951	308	46137	<u>54</u>	<u>875</u>
22	∞	∞	∞	∞	∞	2019	20580	<u>847</u>	25047	<u>1958</u>	<u>1</u>	1188	48334	3773	74250
23	∞	∞	∞	∞	∞	19056	27853	8556	32453	17250	35926	9936	50531	31556	77625
24	∞	∞	∞	∞	∞	∞	28956	<u>54</u>	<u>432</u>	<u>2400</u>	39600	<u>1</u>	52728	4116	<u>1000</u>
25	∞	∞	∞	∞	∞	∞	40163	10200	42975	<u>105</u>	47850	12850	54925	34300	<u>135</u>

Table 2

$R_3(\mathcal{E}(3, 0; a, a, b))$ for $1 \leq a \leq 15$ and $1 \leq b \leq 25$. The previously unknown values are presented in boldface, and the underlined entries correspond to equations whose coefficients are not coprime.

b	Range of a	$R_3(\mathcal{E}(3, 0; a, b, b))$	b	Range of a	$R_3(\mathcal{E}(3, 0; a, b, b))$
2	[7, 30]	$a^3 + a^2 + 5a + 1$	9	[14, 30]	$a^3 + a^2 + 19a + 1$
3	[4, 30]	$a^3 + a^2 + 7a + 1$	10	[17, 30]	$a^3 + a^2 + 21a + 1$
4	[7, 30]	$a^3 + a^2 + 9a + 1$	11	[17, 30]	$a^3 + a^2 + 23a + 1$
5	[7, 30]	$a^3 + a^2 + 11a + 1$	12	[19, 30]	$a^3 + a^2 + 25a + 1$
6	[11, 30]	$a^3 + a^2 + 13a + 1$	13	[20, 30]	$a^3 + a^2 + 27a + 1$
7	[11, 30]	$a^3 + a^2 + 15a + 1$	14	[23, 30]	$a^3 + a^2 + 29a + 1$
8	[13, 30]	$a^3 + a^2 + 17a + 1$	15	[26, 30]	$a^3 + a^2 + 31a + 1$

The formulas presented above were derived through curve fitting. Our analysis began by identifying patterns within each column of data. Some numbers exhibited a constant difference, suggesting a linear relationship, while others showed a constant third difference, indicating a cubic relationship. We constructed systems of linear and cubic equations based on these observations. We then applied Gaussian–Jordan elimination to solve these systems, obtaining the best-fitting formulas describing the sequence and ultimately arriving at the following conjecture.

Conjecture 2.1. For coprime positive integers a and b with $a^2 + a + b > b^2 + ba$ and $a > b \geq 3$, $R_3(\mathcal{E}(3, 0; a, b, b)) = a^3 + a^2 + (2b + 1)a + 1$.

Before proceeding, we note the equation $\mathcal{E}(3, 0; a, -a, b)$ can be transformed into $\mathcal{E}(3, 0; b, a, a)$ by permuting variables. This equivalence along with a theorem of Chang, De Loera, and Wesley [4] provides a closed-form expression for $R_3(\mathcal{E}(3, 0; a, 1, 1))$.

Proposition 2.1. $R_3(\mathcal{E}(3, 0; a, 1, 1)) = a^3 + 5a^2 + 7a + 1$ for $a \geq 1$.

Proof. By permuting variables we have $R_3(\mathcal{E}(3, 0; a, 1, 1)) = R_3(\mathcal{E}(3, 0; 1, -1, a))$. By [4, Thm 1.2], for $m \geq 3$, $R_3(\mathcal{E}(3, 0; 1, -1, m - 2)) = m^3 - m^2 - m - 1$. Replacing a with $m - 2$, we get for $a \geq 1$, $R_3(\mathcal{E}(3, 0; 1, -1, a)) = a^3 + 5a^2 + 7a + 1$. Hence, $R_3(\mathcal{E}(3, 0; a, 1, 1)) = a^3 + 5a^2 + 7a + 1$. \square

2.8. Some new patterns for $R_3(\mathcal{E}(3, 0; a, a, b))$

The data on $R_3(\mathcal{E}(3, 0; a, a, b))$ presented in Table 2 has inspired the following conjecture and some general observations described below.

Conjecture 2.2. *For coprime positive integers a and b , if a is odd, then for $3 \leq a < b \leq 2a - 1$, $R_3(\mathcal{E}(3, 0; a, a, b)) = a^3b$.*

Observation 2.2. *For odd positive integers $7 \leq a \leq 15$, we have*

- $R_3(\mathcal{E}(3, 0; a, a, a - 1)) = R_3(\mathcal{E}(3, 0; a, a, a + 2)) = a^3(a + 2)$;
- if $\gcd(a, a + 3) = 1$, then $R_3(\mathcal{E}(3, 0; a, a, a - 2)) = R_3(\mathcal{E}(3, 0; a, a, a + 3)) = a^3(a + 3)$;
- $R_3(\mathcal{E}(3, 0; a, a, \frac{a+1}{2})) = R_3(\mathcal{E}(3, 0; a, a, 2a - 1)) = a^3(2a - 1)$;
- if $\gcd(a, \frac{a+3}{2}) = 1$, then $R_3(\mathcal{E}(3, 0; a, a, \frac{a+3}{2})) = R_3(\mathcal{E}(3, 0; a, a, 2a - 4)) = a^3(2a - 4)$.

Observation 2.3. *For coprime positive integers a and b ,*

$$R_3(\mathcal{E}(3, 0; a, a, b)) = \begin{cases} a^3b/2 & \text{if } a \in \{6, 10, 14\} \text{ and } a < b \leq 2a - 1, \\ a^3b/4 & \text{if } a \in \{12\} \text{ and } a < b \leq 2a - 1. \end{cases}$$

A general characterization of the cases with even a is not possible at the moment due to lack of further data.

3. General Results

In Section 2, we extended the known numerical values of Rado numbers corresponding to the equations $\mathcal{E}(3, 0; a, a, b)$ and $\mathcal{E}(3, 0; a, b, b)$ and observed several new bounds for these numbers. In this section, we discuss and prove some of those bounds using AutoCase. An overview of how AutoCase works is first provided in Section 3.1, and then as a demonstrative example we use AutoCase to prove Proposition 2.1 in Section 3.2. Finally, we use AutoCase to prove Theorems 1.3 and 1.4 in Sections 3.3 and 3.4.

3.1. Automating case-based analysis with SymPy and Z3

Shallit [26] proposed that traditional case-by-case analyses are better performed by automated search algorithms to enhance both efficiency and correctness. In that direction, we employ an automated verification approach (using our tool AutoCase) to verify the case-based proofs of Theorems 1.3 and 1.4. Our proof strategy relies on case-based analysis of solving linear equations whose variables lie in symbolic sets filtered through arithmetic and logical constraints. This section describes how we automate this process by symbolically encoding the problem structure and constraint system using SymPy and performing satisfiability checking using the SMT solver Z3.

The input provided to AutoCase is the linear equation to be considered along with a colouring of the integers $[1, N]$ specified by a collection of symbolic sets as described in Section 3.1.1. AutoCase verifies that the colouring is valid by ensuring the size of the symbolic sets sum to exactly N (see Section 3.1.2) and that the sets are pairwise disjoint (see Section 3.1.3). This verifies the provided symbolic sets provide a partition of $[1, N]$. Finally, AutoCase verifies that there are no solutions of the linear equation under consideration where all variables are from symbolic sets of the same colour (see Section 3.1.4). Thus, the symbolic sets define a colouring of the integers $[1, N]$ that have no monochromatic solutions to the linear equation, thereby proving a lower bound of $N + 1$ on the Rado number for that linear equation.

3.1.1. Input specification

The input to our automated system consists of:

- A symbolic linear equation \mathcal{E} , such as $ax + ay = (a + 1)z$, where coefficients and assumptions on the coefficients (e.g., $a \geq 7$, a odd) are symbolic.
- A symbolic N corresponding to the lower bound on $R_k(\mathcal{E})$, e.g., $N = a^3(a + 1) - 1$.

- A partition of the domain $[1, N]$ into named symbolic sets, each specified by
 - symbolic bounds (e.g., $[a, a^4 + a^3 - a]$),
 - a format expression generating integers in the set (e.g., $a(a + 1)i + aj + k$), and
 - optional divisibility filters (e.g., integers in the set must be divisible by a^2 but not a^3).
- A k -colouring of $[1, N]$ defined in terms of the above symbolic sets.

These inputs are defined using SymPy to facilitate symbolically:

- Building and manipulating (e.g., simplifying, expanding, combining, substituting symbolically) interval bounds, format expressions, and divisibility properties.
- Calculating size, disjointness, and covering conditions. The union $\bigcup_{i=1}^k S_i$ is considered a partition of $[1, N]$ if and only if $\bigcup_{i=1}^k S_i$ has size N , the sets S_i are pairwise disjoint, and $\bigcup_{i=1}^k S_i \subseteq [1, N]$.
- Simplifying and structurally transforming constraints before they are fed to Z3.

3.1.2. Symbolic size computation using SymPy

SymPy plays a central role in enabling symbolic reasoning, algebraic manipulation, and structured enumeration in the symbolic size analysis pipeline. The key uses of SymPy can be itemized as follows:

- **Size with divisibility filters:** Let the underlying interval be defined as $[L, H]$. Let D and ND denote the sets of required and excluded divisors, respectively, and consider

$$S = \{x \in [L, H] : \forall d \in D, d \mid x \text{ and } \forall n \in ND, n \nmid x\}.$$

For $T \subseteq ND$, define

$$S_T = \{x \in [L, H] : \text{lcm}(D \cup T) \mid x\}.$$

The number of elements $x \in [L, H]$ divisible by all divisors in D and none in ND is

$$|S| = \sum_{T \subseteq ND} (-1)^{|T|} |S_T| = \sum_{T \subseteq ND} (-1)^{|T|} \left(\left\lfloor \frac{H}{\text{lcm}(D \cup T)} \right\rfloor - \left\lfloor \frac{L-1}{\text{lcm}(D \cup T)} \right\rfloor \right).$$

This formula uses inclusion-exclusion to subtract overlapping exclusions in order to count elements satisfying the divisibility constraints. SymPy contributes with `floor`, `ceiling`, and `lcm` to enforce and combine arithmetic conditions. For example, consider the interval $[1, b^2a]$ with $D = \{b\}$ and $ND = \{b^2\}$. Then the size of the resulting set is

$$\left\lfloor \frac{b^2a}{\text{lcm}(b)} \right\rfloor - \left\lfloor \frac{b^2a}{\text{lcm}(b, b^2)} \right\rfloor = \left\lfloor \frac{b^2a}{b} \right\rfloor - \left\lfloor \frac{b^2a}{b^2} \right\rfloor = ba - a = a(b - 1).$$

- **Format-based symbolic summation:** Consider a symbolic set defined by a format expression $x = f(i_1, i_2, \dots, i_k)$ where $i_j \in [l_j, h_j]$. When f is injective, the total size of the set is given by

$$|\{x = f(i_1, \dots, i_k) \mid l_j \leq i_j \leq h_j\}| = \sum_{i_1=l_1}^{h_1} \sum_{i_2=l_2}^{h_2} \cdots \sum_{i_k=l_k}^{h_k} 1.$$

For example, the format expression $x = ai + j$ where $0 \leq i < m$ and $1 \leq j \leq a$ defines a set of size ma . If some variables are symbolic (e.g., a), the size remains symbolic and can be simplified using algebraic rules. For example, $\sum_{i=0}^a \sum_{j=1}^i 1 = a(a + 1)/2$.

- **Symbolic floor simplification:** Let a be a symbolic variable representing a positive integer, and consider the rational expression $f(a) = p(a)/q(a)$, where p and q are polynomials in a . We aim to simplify $\lfloor p(a)/q(a) \rfloor$ which can often be simplified using the degrees of the numerator and denominator polynomials.

Consider $\deg(p) < \deg(q)$. When the degree of the numerator $p(a)$ is strictly less than that of the denominator $q(a)$, the value of the rational expression $p(a)/q(a)$ has magnitude less than 1 for all

sufficiently large a , and is dominated by the leading terms of p and q . Its sign for large enough a is therefore determined by the sign of $\text{LC}(p)/\text{LC}(q)$ where $\text{LC}(\cdot)$ denotes the leading coefficient, so

$$\left\lfloor \frac{p(a)}{q(a)} \right\rfloor = \begin{cases} 0 & \text{if } \text{LC}(p)/\text{LC}(q) > 0, \\ -1 & \text{if } \text{LC}(p)/\text{LC}(q) < 0, \end{cases}$$

assuming that the bound on a is large enough for this simplification to occur. For example, $\lfloor -1/a \rfloor = -1$ if $a \geq 1$, $\lfloor 1/a \rfloor = 0$, $\lfloor (a+1)/a \rfloor = 1$, and $\lfloor (a^2+1)/a^3 \rfloor = 0$ if $a \geq 2$, and $\lfloor -(a+5)/a^2 \rfloor = -1$ if $a \geq 3$.

- **Unified symbolic simplification:** Applies `simplify` and `expand` to normalize symbolic size expressions such as converting $2a(a+1) + a(a^2+3a+1) + 4a$ to $a^3 + 5a^2 + 7a$.

3.1.3. Symbolic disjointness checking using SymPy and Z3

AutoCase constructs a symbolic system of constraints to determine whether two symbolic sets are disjoint. The key idea is to introduce a symbolic integer variable z that hypothetically belongs to both sets. Then, disjointness reduces to checking whether the conjunction of the constraints defining both sets is unsatisfiable. Z3 is used to test satisfiability.

- **Domain constraints (bounds):** For each interval, we require $z \in [\text{lower}, \text{upper}]$, encoded via `Ge(z, lower)` and `Le(z, upper)`. If sets A and B are defined over $[1, a^2 + 2a]$ and $[a^2 + 3a + 1, a^3 + 5a^2 + 7a]$, respectively, then

$$z \geq 1 \wedge z \leq a^2 + 2a \quad \text{and} \quad z \geq a^2 + 3a + 1 \wedge z \leq a^3 + 5a^2 + 7a.$$

These bounds are incompatible, so A and B are disjoint.

- **Divisibility and non-divisibility constraints:** Divisibility constraints are encoded as $d \mid z$ using `Mod(z, d) == 0`, and non-divisibility as $d \nmid z$ using `Mod(z, d) != 0`. Suppose set A contains all $z \in [1, ab]$ such that $b \mid z$, while set B contains those where $b \nmid z$. The constraint system $z \equiv 0 \pmod{b} \wedge z \not\equiv 0 \pmod{b}$ is unsatisfiable, so the sets are disjoint.
- **Format expression constraints:** To avoid symbol collisions in format expressions, all format variables are renamed using fresh symbols (e.g., `_fmt_i`). If a set is defined by a format expression $z = f(i_1, \dots, i_k)$, then we add `sp.Eq(z, format_expr)` along with symbolic bounds on the variables i_j (e.g., $0 \leq i < a^2$).

For instance, consider the set A defined by

$$z = a(a+1)i + aj + k \quad \text{where} \quad 0 \leq i < a^2, \quad 0 \leq j \leq a, \quad \text{and} \quad 2\lfloor j/2 \rfloor + 1 \leq k \leq a-1,$$

with constraint $z \not\equiv 0 \pmod{a}$. Also, let the set B be defined by

$$z = t(a^3 + a^2), \quad 1 \leq t \leq \lfloor (a-1)/2 \rfloor.$$

Now, assume $z \in A \cap B$. Then $z \in B$ implies $z \equiv 0 \pmod{a}$, and this contradicts the constraint from A . Therefore, $A \cap B = \emptyset$.

3.1.4. Monochromatic solution analysis using Z3

For a linear equation $\sum_{i=1}^{m-1} a_i x_i = a_m x_m$ and $t \geq 1$ sets (each given a colour class), the tool now generates all possible ways to assign the $m-1$ left-hand-side (LHS) variables across t sets using Cartesian products. AutoCase iterates over every way to assign the LHS variables to the same colour class and for every colour class it generates a list of cases for verification.

The prover systematically explores all combinations of sets assigned to the variables in the equation, where each variable is taken from the same colour class. For each such case, it verifies that no solution exists satisfying the equation under the imposed divisibility, format, and value constraints. If all such cases succeed this confirms the absence of monochromatic solutions.

- For each assignment of sets to the variables in the equation, the tool aggregates all associated symbolic constraints: interval bounds (e.g., $x \in [1, a^3]$), format constraints (e.g., $x = ia^2 + ja$), and arithmetic filters (e.g., $y \nmid a, z \mid a^2$).
- It then encodes the target equation (e.g., $ax + ay = (a + 1)z$) and all constraints into an SMT expression, along with assumptions (e.g., $a \geq 7, \text{prime}(a), \text{gcd}(a, b) = 1$). The translation process supports a broad range of symbolic constructs in SymPy, including integer inequalities, polynomial equalities, floor and ceiling operations, divisibility conditions, and logical connectives.
- The solver checks that the symbolic system is unsatisfiable. The contradiction analysis is exhaustive—it verifies that under all parameter values satisfying the assumptions, the given assignment cannot yield a valid solution.

If every such assignment leads to a contradiction, the tool concludes that the equation has no monochromatic solution in the constructed colouring, and thus the lower bound for the Rado number is symbolically proven.

Example 3.1. We illustrate a symbolic case analyzed by AutoCase, where each variable in the equation $ax_1 + ax_2 = (a + 1)x_3$ is assigned to distinct symbolic intervals with bounds, divisibility filters, and format expressions. The analyzer constructs the following constraint system:

- **Global Assumptions:** $a \in \mathbb{Z}$ and $a \geq 1$
- **Equation Constraint:** $ax_1 + ax_2 = (a + 1)x_3$
- **Bounds:** $x_1, x_2, x_3 \in [1, a^2] \Rightarrow x_i \geq 1, x_i \leq a^2$ for $i \in \{1, 2, 3\}$
- **Divisibility Filter on x_1 :** $a \mid x_1 \Rightarrow x_1 = a \cdot k_1, k_1 > 0$
- **Non-divisibility Filter on x_2 :** $a^2 \nmid x_2 \Rightarrow x_2 \neq a^2 \cdot k_2$ for all $k_2 > 0$
- **Format Expression for x_3 :**

$$x_3 = a(a + 1)i + aj + k \quad \text{with} \quad 0 \leq i < a^2, 0 \leq j \leq a, \text{ and } 2\lfloor j/2 \rfloor + 1 \leq k \leq a - 1$$

- **Total Constraint Set (passed to Z3):**

$$\begin{aligned} ax_1 + ax_2 &= (a + 1)x_3 \\ x_1 &= a \cdot k_1, \quad k_1 > 0 \\ \forall k_2 > 0, x_2 &\neq a^2 \cdot k_2 \\ x_3 &= a(a + 1)i + aj + k \\ 0 \leq i < a^2, \quad 0 \leq j \leq a, \quad 2\lfloor j/2 \rfloor + 1 &\leq k \leq a - 1 \\ x_1, x_2, x_3 &\in [1, a^2] \end{aligned}$$

The set of constraints in Z3 are given in Appendix D. If Z3 determines that this system is unsatisfiable for all admissible values of a , then this configuration leads to a contradiction—contributing to the overall proof that the equation has no monochromatic solution in the constructed colouring.

3.2. Proving with AutoCase: An example

A general technique to prove a lower bound is to construct a colouring avoiding forbidden patterns. Specifically, to show $R_k(\mathcal{E}) \geq N + 1$, constructing a k -colouring of the positive integers $[1, N]$ avoiding any monochromatic solution to equation \mathcal{E} would be sufficient. As an example, Proposition 2.1 states $R_3(\mathcal{E}(3, 0; a, 1, 1)) = a^3 + 5a^2 + 7a + 1$. Although we already showed this proposition is a consequence of a theorem of Chang, De Loera, and Wesley [4], as an example we now show how a form of Proposition 2.1 (with the equality replaced by a lower bound) can be proven using AutoCase.

3.2.1. Construction of the partition

Based on empirical observations of solutions to small instances of the problem, using certificates (satisfying assignments) generated by the SAT solver CaDiCaL, we identify a partition of $[1, a^3 + 5a^2 + 7a]$ into seven intervals P_0, \dots, P_6 along with a corresponding colouring function δ .

Consider the partition of $[1, a^3 + 5a^2 + 7a]$ using the intervals

$$\begin{aligned}
P_0 &= [1, a], \\
P_1 &= [a + 1, a^2 + 2a], \\
P_2 &= [a^2 + 2a + 1, a^2 + 3a], \\
P_3 &= [a^2 + 3a + 1, a^3 + 4a^2 + 4a], \\
P_4 &= [a^3 + 4a^2 + 4a + 1, a^3 + 4a^2 + 5a], \\
P_5 &= [a^3 + 4a^2 + 5a + 1, a^3 + 5a^2 + 6a], \\
P_6 &= [a^3 + 5a^2 + 6a + 1, a^3 + 5a^2 + 7a].
\end{aligned}$$

From these, we define the colouring function $\delta: [1, a^3 + 5a^2 + 7a] \rightarrow [0, 1, 2]$ as

$$\delta(i) = \begin{cases} 0 & \text{if } i \in P_0 \cup P_2 \cup P_4 \cup P_6, \\ 1 & \text{if } i \in P_1 \cup P_5, \\ 2 & \text{if } i \in P_3. \end{cases}$$

3.2.2. Input processing

The prover considers all possible pairs x, y drawn from the same colour class, and verifies that $z = ax + y$ cannot belong to any set in the same colour class. Based on the symbolic colouring provided by δ , the prover enumerates all ways of assigning x, y , and z to the same colour. Specifically, it generates $4^3 = 64$ cases for colour 0, $2^3 = 8$ cases for colour 1, and $1^3 = 1$ case for colour 2, yielding a total of 73 cases. These cases correspond to the combinations of source intervals for (x, y) and target intervals for z .

3.2.3. Monochromaticity analysis

Example 3.2. We check the cases where $x \in P_0$ and $y \in P_0$. The computed $z = ax + y$ must not belong to any of P_0, P_2, P_4, P_6 . The reasons why z cannot belong to any of P_0, P_2, P_4, P_6 are as follows:

- Lower bound of z : Since $x, y \in P_0 = [1, a]$, their smallest possible values are $x = 1$ and $y = 1$. This gives $z = ax + y \geq a(1) + 1 = a + 1$.
- Upper bound of z : The largest values in P_0 are $x = a$ and $y = a$. This gives $z = ax + y \leq a(a) + a = a^2 + a$.
- Now, we check whether z can belong to any forbidden partition:
 - $P_0 = [1, a]$: Since $z \geq a + 1$, $z \notin P_0$.
 - $P_2 = [a^2 + 2a + 1, a^2 + 3a]$: The maximum possible z is $a^2 + a$, which is less than $\min(P_2) = a^2 + 2a + 1$. Thus, $z \notin P_2$.
 - $P_4 = [a^3 + 4a^2 + 4a + 1, a^3 + 4a^2 + 5a]$: Clearly, $z \leq a^2 + a$ is much smaller than $\min(P_4)$, so $z \notin P_4$.
 - $P_6 = [a^3 + 5a^2 + 6a + 1, a^3 + 5a^2 + 7a]$: Again, z is far smaller than $\min(P_6)$, so $z \notin P_6$.

Since z is forced outside the monochromatic partitions, we conclude that no monochromatic solution exists in colour 0 when $x \in P_0$ and $y \in P_0$.

The same type of analysis applies to all other colour cases, ensuring that the generated constraints exhaustively eliminate any monochromatic solution. As a result, AutoCase has shown that $R_3(\mathcal{E}(3, 0; a, 1, 1)) \geq a^3 + 5a^2 + 7a + 1$ for all $a \geq 1$.

3.4. Proof of Theorem 1.4 (lower bound for $R_3(\mathcal{E}(3, 0; a, a, a + 1))$)

Observation 3.1. For odd positive integers $a \geq 7$, there exists a certificate (computed using a SAT solver) for $R_3(\mathcal{E}(3, 0; a, a, a + 1)) \geq a^3(a + 1)$ of length $a^3(a + 1) - 1$ where certain repetitive patterns appear as blocks of coloured integers of size $a(a + 1)$. Consider the following examples for $R_3(\mathcal{E}(3, 0; 7, 7, 8))$.

r1	r2	r3	r4	r5	r6	7
r8	r9	r10	r11	r12	r13	14
b15	b16	r17	r18	r19	r20	21
b22	b23	r24	r25	r26	r27	28
b29	b30	b31	b32	r33	r34	35
b36	b37	b38	b39	r40	r41	42
b43	b44	b45	b46	b47	b48	r49
b50	b51	b52	b53	b54	b55	56
Colour block 1 for $R_3(\mathcal{E}(3, 0; 7, 7, 8))$						

r57	r58	r59	r60	r61	r62	63
r64	r65	r66	r67	r68	r69	70
b71	b72	r73	r74	r75	r76	77
b78	b79	r80	r81	r82	r83	84
b85	b86	b87	b88	r89	r90	91
b92	b93	b94	b95	r96	r97	r98
b99	b100	b101	b102	b103	b104	105
b106	b107	b108	b109	b110	b111	112
Colour-block 2 for $R_3(\mathcal{E}(3, 0; 7, 7, 8))$						

r337	r338	r339	r340	r341	r342	b343
r344	r345	r346	r347	r348	r349	350
b351	b352	r353	r354	r355	r356	357
b358	b359	r360	r361	r362	r363	364
b365	b366	b367	b368	r369	r370	371
b372	b373	b374	b375	r376	r377	378
b379	b380	b381	b382	b383	b384	385
b386	b387	b388	b389	b390	b391	r392
Colour-block 7 for $R_3(\mathcal{E}(3, 0; 7, 7, 8))$						

r2689	r2690	r2691	r2692	r2693	r2694	b2695
r2696	r2697	r2698	r2699	r2700	r2701	2702
b2703	b2704	r2705	r2706	r2707	r2708	2709
b2710	b2711	r2712	r2713	r2714	r2715	2716
b2717	b2718	b2719	b2720	r2721	r2722	2723
b2724	b2725	b2726	b2727	r2728	r2729	2730
b2731	b2732	b2733	b2734	b2735	b2736	2737
b2738	b2739	b2740	b2741	b2742	b2743	
Colour-block 49 for $R_3(\mathcal{E}(3, 0; 7, 7, 8))$						

The integers that are not divisible by a get coloured in a consistent way in each block, while the integers that are divisible by a need to be carefully coloured in such a way (a different way in each block) to avoid monochromatic solutions to the equation $ax + ay = (a + 1)z$.

3.4.1. Construction of the partition

Before we prove the lower bound, we construct a partition of the set $[1, N]$ where $N = a^3(a + 1) - 1$ with $a \geq 7$ being an odd positive integer. Consider,

$$[1, N + 1] = \{i \cdot a(a + 1) + a \cdot j + k : 0 \leq i \leq a^2 - 1, 0 \leq j \leq a, 1 \leq k \leq a\}.$$

Let $S_d(N)$ and $\overline{S}_d(N)$ denote $\{x \in [1, N] : d \mid x\}$ and $\{x \in [1, N] : d \nmid x\}$, respectively. We have the partition $[1, N] = S_a(N) \cup \overline{S}_a(N)$ with

$$S_a(N) = \{i \cdot a(a + 1) + a \cdot j + a : 0 \leq i \leq a^2 - 1, 0 \leq j \leq a\} \setminus \{N + 1\}, \text{ and}$$

$$\overline{S}_a(N) = \{i \cdot a(a + 1) + a \cdot j + k : 0 \leq i \leq a^2 - 1, 0 \leq j \leq a, 1 \leq k \leq a - 1\}.$$

Let $\overline{S}_a(N) = R_\ell \cup B_\ell$ be a partition defined by

$$R_\ell = \{i \cdot a(a + 1) + a \cdot j + k : 0 \leq i \leq a^2 - 1, 0 \leq j \leq a, 2\lfloor j/2 \rfloor + 1 \leq k \leq a - 1\}, \text{ and}$$

$$B_\ell = \{i \cdot a(a + 1) + a \cdot j + k : 0 \leq i \leq a^2 - 1, 0 \leq j \leq a, 1 \leq k \leq 2\lfloor j/2 \rfloor\}.$$

Now, consider the set

$$S_{a^2}(N) = \{i \cdot a(a + 1) + a \cdot j + a : 0 \leq i \leq a^2 - 1, 0 \leq j \leq a, a \mid (i + j + 1)\} \setminus \{N + 1\}.$$

Let $S_{a^2}(N) = R_r \cup B_r$ be a partition defined by

$$B_r = \bigcup_{i=1}^{a-1} \{a^4 + ia^2\} \cup \bigcup_{i=1}^{a-1} \{ia^3 + ja^2 : 0 \leq j \leq i - 1\} \cup \bigcup_{i=(a+1)/2}^{a-1} \{ia^3 + ia^2\}, \text{ and}$$

$$R_r = \{a^4\} \cup \bigcup_{i=0}^{a-1} \{ia^3 + ja^2 : i + 1 \leq j \leq a - 1\} \cup \bigcup_{i=1}^{(a-1)/2} \{ia^3 + ia^2\}.$$

Our colouring $\delta: [1, N] \rightarrow [0, 1, 2]$ will be defined by

$$\delta(x) = \begin{cases} 2 & \text{if } x \in B_\ell \cup B_r, \\ 1 & \text{if } x \in R_\ell \cup R_r, \\ 0 & \text{if } x \in S_a(N) \setminus S_{a^2}(N). \end{cases}$$

Proof of Theorem 1.4. A written proof of Theorem 1.4 is provided in Appendix C. The symbolic correctness of the partition and the verification of the cases can be found in the AutoCase repository. \square

4. Conclusion

In this work, we explored the computation of 3-colour Rado numbers for certain 3-term linear equations. Our SAT-based approach enabled us to extend the known values of $R_3(\mathcal{E}(3, 0; a, b, b))$ and $R_3(\mathcal{E}(3, 0; a, a, b))$, computing previously unknown Rado numbers and formulating new conjectures based on observed patterns. We proved the lower bounds $R_3(\mathcal{E}(3, 0; a, b, b)) \geq a^3 + a^2 + (2b + 1)a + 1$ and $R_3(\mathcal{E}(3, 0; a, a, a + 1)) \geq a^3(a + 1)$ for certain ranges of a and b . The case-based proofs were automatically verified using a combination of symbolic computation and SMT solvers in a tool we call AutoCase.

We did not attempt to compute Rado numbers involving more than three colours due to the rapid growth in the SAT instances, and for simplicity we focused on linear equations with a constant term of zero. In principle, our methods should be applicable to nonhomogeneous linear equations and to Rado numbers involving four or more colours, though they may require more computational resources or more advanced combinatorial optimization techniques. We leave such cases for future work.

Declaration on Generative AI

No generative AI was used.

References

- [1] R. Rado, Studien zur Kombinatorik, *Mathematische Zeitschrift* 36 (1933) 242–280. doi:10.1007/BF01188632.
- [2] D. Schaal, A family of 3-color Rado numbers, *Congressus Numerantium* 111 (1995) 150–160.
- [3] S. D. Adhikari, L. Boza, S. Eliahou, J. M. Marin, M. P. Revuelta, M. I. Sanz, On the n -color Rado number for the equation $x_1 + x_2 + \dots + x_k + c = x_{k+1}$, *Mathematics of Computation* 85 (2016) 2047–2064. doi:10.1090/mcom3034.
- [4] Y. Chang, J. A. De Loera, W. J. Wesley, Rado numbers and SAT computations, in: *Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation, ISSAC '22*, ACM, 2022, p. 333–342. doi:10.1145/3476446.3535494.
- [5] K. Myers, Computational advances in Rado numbers, Ph.D. thesis, Rutgers The State University of New Jersey - New Brunswick, 2015. doi:10.7282/T3GH9KTT.
- [6] A. Meurer, C. P. Smith, M. Paprocki, O. Čertík, S. B. Kirpichev, M. Rocklin, A. Kumar, S. Ivanov, J. K. Moore, S. Singh, T. Rathnayake, S. Vig, B. E. Granger, R. P. Muller, F. Bonazzi, H. Gupta, S. Vats, F. Johansson, F. Pedregosa, M. J. Curry, A. R. Terrel, v. Roučka, A. Saboo, I. Fernando, S. Kulal, R. Cimrman, A. Scopatz, *SymPy: Symbolic computing in Python*, *PeerJ Computer Science* 3 (2017) e103. doi:10.7717/peerj-cs.103.
- [7] L. de Moura, N. Bjørner, *Z3: An Efficient SMT Solver*, Springer Berlin Heidelberg, 2008, p. 337–340. doi:10.1007/978-3-540-78800-3_24.
- [8] E. Ábrahám, J. Abbott, B. Becker, A. M. Bigatti, M. Brain, B. Buchberger, A. Cimatti, J. H. Davenport, M. England, P. Fontaine, S. Forrest, A. Griggio, D. Kroening, W. M. Seiler, T. Sturm, *SC²: Satisfia-*

- bility Checking Meets Symbolic Computation (Project Paper), Springer International Publishing, 2016, p. 28–43. doi:10.1007/978-3-319-42547-4_3.
- [9] C. Bright, I. Kotsireas, V. Ganesh, When satisfiability solving meets symbolic computation, *Communications of the ACM* 65 (2022) 64–72. doi:10.1145/3500921.
- [10] C. Bright, K. K. H. Cheung, B. Stevens, I. Kotsireas, V. Ganesh, A SAT-based resolution of Lam’s problem, *Proceedings of the AAAI Conference on Artificial Intelligence* 35 (2021) 3669–3676. doi:10.1609/aaai.v35i5.16483.
- [11] M. Kirchweger, T. Peitl, S. Szeider, Co-certificate learning with SAT modulo symmetries, in: *Proceedings of the Thirty-Second International Joint Conference on Artificial Intelligence, IJCAI-2023*, International Joint Conferences on Artificial Intelligence Organization, 2023, p. 1944–1953. doi:10.24963/ijcai.2023/216.
- [12] Z. Li, C. Bright, V. Ganesh, A SAT solver + computer algebra attack on the minimum Kochen–Specker problem, in: *Proceedings of the Thirty-Third International Joint Conference on Artificial Intelligence, IJCAI-2024*, International Joint Conferences on Artificial Intelligence Organization, 2024, p. 1898–1906. doi:10.24963/ijcai.2024/210.
- [13] M. Kouril, J. L. Paul, The van der Waerden number $w(2, 6)$ is 1132, *Experimental Mathematics* 17 (2008) 53–61. doi:10.1080/10586458.2008.10129025.
- [14] P. R. Herwig, M. J. H. Heule, P. M. van Lambalgen, H. van Maaren, A new method to construct lower bounds for van der Waerden numbers, *The Electronic Journal of Combinatorics* 14 (2007). doi:10.37236/925.
- [15] T. Ahmed, Some new van der Waerden numbers and some van der Waerden-type numbers, *Integers* 9 (2009) 65–76. doi:10.1515/INTEG.2009.007.
- [16] T. Ahmed, Two new van der Waerden numbers: $w(2; 3, 17)$ and $w(2; 3, 18)$, *Integers* 10 (2010) 369–377. doi:10.1515/integ.2010.032.
- [17] T. Ahmed, On computation of exact van der Waerden numbers, *Integers* 12 (2012) 417–425. doi:10.1515/integ.2011.112.
- [18] T. Ahmed, Some more van der Waerden numbers, *Journal of Integer Sequences* 16 (2013). Article 13.4.4.
- [19] T. Ahmed, O. Kullmann, H. Snevily, On the van der Waerden numbers $w(2; 3, t)$, *Discrete Applied Mathematics* 174 (2014) 27–51. doi:10.1016/j.dam.2014.05.007.
- [20] T. Ahmed, D. J. Schaal, On generalized Schur numbers, *Experimental Mathematics* 25 (2016) 213–218. doi:10.1080/10586458.2015.1070776.
- [21] T. Ahmed, L. Boza, M. P. Revuelta, M. I. Sanz, Exact values and lower bounds on the n -color weak Schur numbers for $n = 2, 3$, *The Ramanujan Journal* 62 (2023) 347–363. doi:10.1007/s11139-023-00760-y.
- [22] M. J. H. Heule, Schur number five, in: *Proceedings of AAAI-18*, 2018, pp. 6598–6606. doi:10.1609/aaai.v32i1.12209.
- [23] A. Biere, T. Faller, K. Fazekas, M. Fleury, N. Froleys, F. Pollitt, CaDiCaL 2.0, in: A. Gurfinkel, V. Ganesh (Eds.), *Computer Aided Verification - 36th International Conference, CAV 2024*, Montreal, QC, Canada, July 24–27, 2024, *Proceedings, Part I*, volume 14681 of *Lecture Notes in Computer Science*, Springer, 2024, pp. 133–152. doi:10.1007/978-3-031-65627-9_7.
- [24] A. Biere, T. Faller, K. Fazekas, M. Fleury, N. Froleys, F. Pollitt, CaDiCaL, Gimsatul, IsaSAT and Kissat entering the SAT Competition 2024, in: M. Heule, M. Iser, M. Järvisalo, M. Suda (Eds.), *Proc. of SAT Competition 2024 – Solver, Benchmark and Proof Checker Descriptions*, volume B-2024-1 of *Department of Computer Science Report Series B*, University of Helsinki, 2024, pp. 8–10.
- [25] A. Ignatiev, A. Morgado, J. Marques-Silva, PySAT: A Python toolkit for prototyping with SAT oracles, in: *SAT, 2018*, pp. 428–437. doi:10.1007/978-3-319-94144-8_26.
- [26] J. Shallit, Say no to case analysis: Automating the drudgery of case-based proofs, in: *CIAA 2021*, volume 12803 of *LNCS*, 2021, pp. 14–24. doi:10.1007/978-3-030-79121-6_2.

A. Proof of Lemma 2.1

In this section we assume a and b are positive coprime integers. Lemma 2.1 then says

$$R_1(\mathcal{E}(3, 0; a, b, b)) = \max(a + 1, b) \quad \text{and} \quad R_1(\mathcal{E}(3, 0; a, a, b)) = \begin{cases} 2a & \text{if } b = 1, \\ \max(a, \lceil b/2 \rceil) & \text{if } b > 1. \end{cases} \quad (1)$$

For the first equation of (1), we want to find the solution (x, y, z) of $ax + by = bz$ that minimizes $\max(x, y, z)$; then $R_1(\mathcal{E}(3, 0; a, b, b)) = \max(x, y, z)$. Since a and b are coprime, x must be divisible by b , so $(b, 1, a + 1)$ is the solution of $ax + by = bz$ with smallest possible x and y values. Since $z = (ax + by)/b$, the solution with the smallest possible x and y will also minimize z . Thus, the smallest possible value of $\max(x, y, z)$ is $\max(b, 1, a + 1) = \max(a + 1, b)$.

For the second equation of (1), we want to find the solution (x, y, z) of $ax + ay = bz$ that minimizes $\max(x, y, z)$; then $R_1(\mathcal{E}(3, 0; a, a, b)) = \max(x, y, z)$. Since a and b are coprime, z must be divisible by a , so say $z = ak$, giving $x + y = bk$. Then $\max(x, y, z)$ is minimized when x and y are as close to equal as possible (i.e., $x \approx y \approx bk/2$).

If b is even, $\max(x, y, z)$ is minimized when $k = 1$ and $x = y = b/2$, giving $\max(x, y, z) = \max(a, b/2)$, so $R_1(\mathcal{E}(3, 0; a, a, b)) = \max(a, b/2)$ when b is even. If $b > 1$ is odd, then $\max(x, y, z)$ is minimized when $k = 1$ and $\{x, y\} = \{\lfloor b/2 \rfloor, \lceil b/2 \rceil\}$, so $R_1(\mathcal{E}(3, 0; a, a, b)) = \max(a, \lceil b/2 \rceil)$ when $b > 1$ is odd. If $b = 1$, then $\max(x, y, z)$ is minimized when $k = 2$ and $x = y = 1$ (note $x + y = k$ has no solutions in positive integers when $k = 1$) and then $R_1(\mathcal{E}(3, 0; a, a, b)) = \max(1, 1, 2a) = 2a$.

B. Written proof of Theorem 1.3

Suppose there is a solution to $ax + by = bz$ monochromatic in colour 0 (Dark Gray). Since $\gcd(a, b) = 1$ and there is no multiple of b in D_1 , the variable x must be in D_2 . Taking $x = ba^2 + ba + b$ and $y = \min(D_1) = ba + 1$ as smallest possible values, we get $z = a^3 + a^2 + (b + 1)a + 1$, but $\delta(z) = 1$ (Red) for $z \geq a^3 + a^2 + (b + 1)a + 1$, a contradiction.

Suppose there is a solution to $ax + by = bz$ monochromatic in colour 1 (Red). Since R_1 does not contain a multiple of b , variable x cannot be in R_1 . Also, if $x, y \in R_3$, then clearly $z > a^3 + a^2 + (2b + 1)a$. So, x and y both cannot be in R_3 . Now we consider the following cases:

(a) $x \in R_2$ and $y \in R_1$: We have $z = \frac{ax}{b} + y = baw + y$ for $1 \leq w \leq a$. Since, $y \not\equiv 0 \pmod{b}$, we have $baw + y \not\equiv 0 \pmod{b}$ which implies $baw + y \not\equiv 0 \pmod{b^2}$. Hence $baw + y \notin R_2$. If $baw + y \in R_1$, then since $w \geq 1$ and $y \geq 1$, we have $baw + y > ba = \max(R_1)$, a contradiction.

(b) $x \in R_2$ and $y \in R_2$: We have $z \notin R_3$ since

$$\begin{aligned} z &\leq a(b^2a)/b + b^2a = ba^2 + b^2a \\ &\leq a^3 + a^2 + (b + 1)a < \min(R_3) \end{aligned}$$

when $a^2 + a + b > b^2 + ba$. Suppose $z \in R_2$. In that case, $ax_1 + bx_2 = bx_3$ with $1 \leq x_1, x_2, x_3 \leq ab^2$ implies $ak_1 + bk_2 = bk_3$ with $1 \leq k_1, k_2, k_3 \leq a$. Then, $k_1 = (b/a)(k_3 - k_2)$ has no integer solution since $\gcd(a, b) = 1$ and $0 \leq k_3 - k_2 < a$. Hence, $\delta(z) \neq 1$.

(c) $x \in R_3$ and $y \in R_1$: Consider the condition, with $a^3 + a^2 + (b + 1)a + c$ being the smallest integer in R_3 which is divisible by b . Then, we have $a^2 + a + b > ba + b^2$ which can be re-written as $\frac{a}{b}(a^3 + a^2 + (b + 1)a + c) + 1 > \frac{a}{b}(ba^2 + ab^2 + a + c) + 1$, which implies

$$z > (a^3 + a^2b + a^2/b + ac/b) > \max(R_3).$$

Hence, $\delta(z) \neq 1$, a contradiction.

- (d) $x \in R_3$ and $y \in R_2$: As in the previous case, with $a^3 + a^2 + (b+1)a + c$ being the smallest integer in R_3 which is divisible by b , and b^2 being the smallest integer in R_2 , we get $z > \max(R_3)$ implying $z \notin R_3$ implying $\delta(z) \neq 1$, a contradiction.

Suppose there is a solution to $ax + by = bz$ monochromatic in colour 2 (Blue). Then we have the following cases:

- (i) If $x, y \in B_2$, then

$$\begin{aligned} z &= \frac{ax}{b} + y \geq \frac{a(b^2a + b)}{b} + (b^2a + b) \\ &= ba^2 + b^2a + a + b > \max(B_2), \end{aligned}$$

which results a contradiction since integers beyond $\max(B_2)$ are coloured in either colour 0 or in colour 1.

- (ii) If $x, y \in B_1$, then in the equation $z = \frac{ax}{b} + y$, we have $\frac{x}{b} \equiv 1, 2, \dots, b-1 \pmod{b}$. Since $\gcd(a, b) = 1$, we have $\frac{ax}{b} \not\equiv 0 \pmod{b}$. Hence, $z = \frac{ax}{b} + y \not\equiv b, 2b, \dots, b(b-1) \pmod{b^2}$ and $z \not\equiv 0 \pmod{b}$ implying $\delta(z) \neq 2$, a contradiction.
- (iii) If $x \in B_1$ and $y \in B_2$, then as in the previous case, we have $\frac{ax}{b} \not\equiv 0 \pmod{b}$. Since $y \equiv 0 \pmod{b}$, we have $\frac{ax}{b} + y \not\equiv 0 \pmod{b}$, that is, $z \notin B_2$. Hence, $\delta(z) \neq 2$, a contradiction.
- (iv) If $x \in B_2$ and $y \in B_1$, then $z = \frac{ax}{b} + y \geq a(b^2a + b)/b + b = a^2b + a + b > \max(B_1)$. Also, since $z \not\equiv 0 \pmod{b}$, we have $z \notin B_2$. Hence, $\delta(z) \neq 2$, a contradiction.

Hence,

$$R_3(\mathcal{E}(3, 0; a, b, b)) \geq a^3 + a^2 + (2b+1)a + 1$$

if $a^2 + a + b > b^2 + ba$.

C. Written proof of Theorem 1.4

C.1. Integrality properties

Lemma C.1. *Let a be a positive integer. There is no integer solution x_1, x_2, x_3 to the equation $ax_1 + ax_2 = (a+1)x_3$ if any of the following is true:*

- a. $x_1, x_2, x_3 \not\equiv 0 \pmod{a}$;
- b. $x_1, x_2, x_3 \equiv 0 \pmod{a}$ and $x_1, x_2, x_3 \not\equiv 0 \pmod{a^2}$;
- c. $x_1, x_2 \equiv 0 \pmod{a^2}$ and $x_3 \not\equiv 0 \pmod{a}$;
- d. $x_1, x_3 \equiv 0 \pmod{a^2}$ and $x_2 \not\equiv 0 \pmod{a}$;
- e. $x_2, x_3 \equiv 0 \pmod{a^2}$ and $x_1 \not\equiv 0 \pmod{a}$;
- f. $x_1 \equiv 0 \pmod{a^2}$ and $x_2, x_3 \not\equiv 0 \pmod{a}$;
- g. $x_2 \equiv 0 \pmod{a^2}$ and $x_1, x_3 \not\equiv 0 \pmod{a}$;

Proof. In each of the cases, assume that an integer solution x_1, x_2, x_3 exists to $ax_1 + ax_2 = (a+1)x_3$.

- a. Simplifying the equation, we get $x_1 + x_2 = x_3 + x_3/a$, which due to the integrality assumption, implies $a|x_3$, that is, $x_3 \equiv 0 \pmod{a}$, a contradiction.
- b. For $i = 1, 2, 3$, since $x_i \equiv 0 \pmod{a}$ and $x_i \not\equiv 0 \pmod{a^2}$, let $x_i = a \cdot k_i$ such that $k_i \not\equiv 0 \pmod{a}$. Upon substitution, we obtain $a^2 \cdot (k_1 + k_2 - k_3) = a \cdot k_3$ which implies $k_3 \equiv 0 \pmod{a}$, a contradiction.
- c. Let $x_1 = a^2k_1$ and $x_2 = a^2k_2$. Then after substitution and simplification, we obtain $a^2k_1 + a^2k_2 = x_3 + x_3/a$, which due to the integrality assumption, implies $a|x_3$, that is, $x_3 \equiv 0 \pmod{a}$, a contradiction.

- d. Let $x_1 = a^2k_1$ and $x_3 = a^2k_3$. Then after substitution and simplification, we obtain $ak_1 + x_2/a = ak_3 + k_3$, which due to the integrality assumption, implies $a|x_2$, that is, $x_2 \equiv 0 \pmod{a}$, a contradiction.
- e. Let $x_2 = a^2k_2$ and $x_3 = a^2k_3$. Then after substitution and simplification, we obtain $x_1/a + ak_2 = ak_3 + k_3$, which due to the integrality assumption, implies $a|x_1$, that is, $x_1 \equiv 0 \pmod{a}$, a contradiction.
- f. Let $x_1 = a^2k_1$. Then after substitution and simplification, we obtain $a^2k_1 + x_2 = x_3 + x_3/a$, which due to the integrality assumption, implies $a|x_3$, that is, $x_3 \equiv 0 \pmod{a}$, a contradiction.
- g. Let $x_2 = a^2k_2$. Then after substitution and simplification, we obtain $x_1 + a^2k_2 = x_3 + x_3/a$, which due to the integrality assumption, implies $a|x_3$, that is, $x_3 \equiv 0 \pmod{a}$, a contradiction. \square

C.2. Proof of good colouring

Lemma C.2. *For odd positive integer $a \geq 7$, there exists no monochromatic solution to $ax_1 + ax_2 = (a+1)x_3$ in colour 0.*

Proof. If $x \in S_a(N) \setminus \overline{S_{a^2}}(N)$, then by Lemma C.1(b), there exists no solution to $ax_1 + ax_2 = (a+1)x_3$. Hence there is no solution to $ax_1 + ax_2 = (a+1)x_3$ monochromatic in colour 0. \square

Lemma C.3. *There exists no monochromatic solution to $ax_1 + ax_2 = (a+1)x_3$ if $x_1, x_2, x_3 \in R_r$.*

Proof. Given $x_1, x_2 \in R_r$, assume that $x_3 \in R_r$ where $x_3 = (ax_1 + ax_2)/(a+1)$. Let $x_1 = k_1a^2$ and $x_2 = k_2a^2$ for positive integers k_1 and k_2 . Substituting for x_3 , we get the numerator to be $a^3(k_1 + k_2)$. Since a^3 shares no factor with $a+1$, if $a+1$ does not divide $k_1 + k_2$, then $x_3 \notin \mathbb{Z}$, a contradiction. Assume that $k_1 + k_2 = m(a+1)$ for some positive integer m .

Assume that $k_1 + k_2 = m(a+1)$ for some positive integer m . Then, $z = a^3m$, but for $1 \leq m \leq a-1$, by definition of the sets B_r and R_r , we have $a^3m \in B_r$ and since $B_r \cap R_r = \emptyset$, we have $a^3m \notin R_r$. If $z = a^4$, then $m = a$ implying $k_1 + k_2 = a(a+1)$ which is impossible since $k_1 + k_2 \leq ai + j \leq a^2 - 1 < a(a+1)$ given $0 \leq i \leq a-1, i+1 \leq j \leq a-1$. \square

Lemma C.4. *For odd positive integer $a \geq 7$, there exists no monochromatic solution to $ax_1 + ax_2 = (a+1)x_3$ if $x_1, x_2 \in R_\ell$ and $x_3 \in R_r$.*

Proof. Since $x_1, x_2 \in R_\ell$, let $x_1 = i_1a(a+1) + r_1$ and $x_2 = i_2a(a+1) + r_2$ with $0 \leq i_1, i_2 \leq a^2 - 1, 1 \leq r_1, r_2 < a(a+1)$. Also, let $x_3 = a^2k_3$. Therefore,

$$k_3 = (i_1 + i_2) + \frac{r_1 + r_2}{a(a+1)}.$$

If $r_1 + r_2 = a(a+1)$, then by construction of B_ℓ and R_ℓ , the integers x_1 and x_2 both not in the same set, a contradiction. \square

Lemma C.5. *For odd positive integer $a \geq 7$, there exists no monochromatic solution to $ax_1 + ax_2 = (a+1)x_3$ in colour 1.*

Proof. Let $R_r = R_1 \cup R_2 \cup R_3$ with $R_1 = \{a^4\}$, $R_2 = \bigcup_{i=1}^{a-1} \{ia^3 + ja^2 : i+1 \leq j \leq a-1\}$, and $R_3 = \bigcup_{i=1}^{(a-1)/2} \{ia^3 + ia^2\}$. There are $4^3 = 64$ ways to select an integer for the variables x_1, x_2, x_3 from the 4 sets R_ℓ, R_1, R_2, R_3 . Most of these cases can be analyzed directly using Lemma C.1 as follows:

- $x_1, x_2, x_3 \in R_\ell$: No monochromatic solution by Lemma C.1 (a) covering 1 case.
- $x_1 \in R_\ell, x_2, x_3 \in R_r$: No monochromatic solution by Lemma C.1 (e) covering 9 cases.
- $x_2 \in R_\ell, x_1, x_3 \in R_r$: No monochromatic solution by Lemma C.1 (d) covering 9 cases.
- $x_3 \in R_\ell, x_1, x_2 \in R_r$: No monochromatic solution by Lemma C.1 (c) covering 9 cases.
- $x_2, x_3 \in R_\ell, x_1 \in R_r$: No monochromatic solution by Lemma C.1 (f) covering 3 cases.
- $x_1, x_3 \in R_\ell, x_2 \in R_r$: No monochromatic solution by Lemma C.1 (g) covering 3 cases.
- $x_1, x_2 \in R_\ell, x_3 \in R_r$: No monochromatic solution by Lemma C.4 covering 3 cases.

- $x_1, x_2, x_3 \in R_r$: No monochromatic solution by Lemma C.3 covering 27 cases.

Hence, there exists no monochromatic solution to $ax_1 + ax_2 = (a + 1)x_3$ in colour 1. \square

Lemma C.6. *For an odd positive integer $a \geq 7$, the set B_r contains no solution to $\mathcal{E}(3, 0; a, a, a + 1)$.*

Proof. Given $x_1, x_2 \in B_r$, assume that $x_3 = \frac{ax_1 + ax_2}{a+1}$ and $x_3 \in B_r$. Let $x_1 = k_1a^2$ and $x_2 = k_2a^2$ for positive integers k_1 and k_2 . Substituting for x_3 , we get the numerator to be $a^3(k_1 + k_2)$. Since a^3 shares no factor with $a + 1$, if $a + 1$ does not divide $k_1 + k_2$, then $z \notin \mathbb{Z}$, a contradiction. Assume that $k_1 + k_2 = m(a + 1)$ for some positive integer m . Then, $z = a^3m$ and assume $z \in B_r$. For $1 \leq i \leq a - 1$, $a^4 + ia^2$ is not an integer multiple of a^3 . Since $\min(B_r) = a^3$, we have $k_1 + k_2 \geq 2a$ and the smallest integer value of m is obtained when $k_1 + k_2 = (a + 1)a$. This implies $z \geq a^4$, but $\max(B_2 \cup B_3) < a^4$. Hence, $z \notin B_r$.

Therefore, the set B_r contains no solution to $ax_1 + ax_2 = (a + 1)x_3$. \square

Lemma C.7. *For odd positive integer $a \geq 7$, there exists no monochromatic solution to $ax_1 + ax_2 = (a + 1)x_3$ if $x_1, x_2 \in B_\ell$ and $x_3 \in B_r$.*

Proof. Similar to the proof of Lemma C.4. \square

Lemma C.8. *For odd positive integer $a \geq 7$, there exists no monochromatic solution to $ax_1 + ax_2 = (a + 1)x_3$ in colour 2.*

Proof. Let $B_r = B_1 \cup B_2 \cup B_3$ with partition blocks $B_1 = \bigcup_{i=1}^{a-1} \{a^4 + ia^2\}$, $B_2 = \bigcup_{i=1}^{a-1} \{ia^3 + ja^2 : 0 \leq j \leq i - 1\}$, and $B_3 = \bigcup_{i=(a+1)/2}^{a-1} \{ia^3 + ia^2\}$. There are $4^3 = 64$ ways to select an integer for the variables x_1, x_2, x_3 from the 4 sets B_ℓ, B_1, B_2, B_3 . Most of these cases can be analyzed directly using Lemma C.1 as follows:

- $x_1, x_2, x_3 \in B_\ell$: No monochromatic solution by Lemma C.1 (a) covering 1 case.
- $x_1 \in B_\ell, x_2, x_3 \in B_r$: No monochromatic solution by Lemma C.1 (e) covering 9 cases.
- $x_2 \in B_\ell, x_1, x_3 \in B_r$: No monochromatic solution by Lemma C.1 (d) covering 9 cases.
- $x_3 \in B_\ell, x_1, x_2 \in B_r$: No monochromatic solution by Lemma C.1 (c) covering 9 cases.
- $x_2, x_3 \in B_\ell, x_1 \in B_r$: No monochromatic solution by Lemma C.1 (f) covering 3 cases.
- $x_1, x_3 \in B_\ell, x_2 \in B_r$: No monochromatic solution by Lemma C.1 (g) covering 3 cases.
- $x_1, x_2 \in B_\ell, x_3 \in B_r$: No monochromatic solution by Lemma C.7 covering 3 cases.
- $x_1, x_2, x_3 \in B_r$: No monochromatic solution by Lemma C.6 covering 27 cases.

Hence, there exists no monochromatic solution to $ax_1 + ax_2 = (a + 1)x_3$ in colour 2. \square

D. SMT Instance via the Z3 Python API for Example 3.1

```
from z3 import *

# Declare symbolic variables
a = Int('a')
x1, x2, x3 = Ints('x_1 x_2 x_3')
k1, k2 = Ints('k_1 k_2')
i, j, k = Ints('i j k')

# Create the solver
s = Solver()

# Global assumptions
s.add(a >= 1)

# Equation constraint
```

```
s.add(a * x1 + a * x2 == (a + 1) * x3)

# Divisibility: a | x1
s.add(And(k1>0, x1 == a * k1))

# Non-divisibility: not(a^2 | x2)
s.add(ForAll([k2], And(k2>0, x2 != a**2 * k2)))

# Format expression for x3
s.add(x3 == a*(a + 1)*i + a*j + k)

# Symbolic format bounds
s.add(i >= 0, i < a**2)
s.add(j >= 0, j <= a)
s.add(k >= 2*(j // 2) + 1, k <= a - 1)

# Bounds for x1, x2, x3
s.add(x1 >= 1, x1 <= a**2)
s.add(x2 >= 1, x2 <= a**2)
s.add(x3 >= 1, x3 <= a**2)
```